Efficiency With Endogenous Population Growth
Appendix

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In this Appendix we provide additional details on some of the ideas developed in the paper. The first section elaborates on the sequential $\mathcal{P}$-efficiency concept, which is a refinement of $\mathcal{P}$-efficiency. The second section provides notation and proofs for an environment in which fertility is a continuous variable. The last section contains details and proofs concerning the Barro Becker fertility model.

1 Sequential Efficiency

1.1 Relation between $\mathcal{P}$-efficiency and sequential $\mathcal{P}$-efficiency

**Proposition 1** Assume $(z,y)$ is sequentially $\mathcal{P}$-efficient. Then $(z,y)$ is $\mathcal{P}$-efficient.

**Proof.** This will be proved by contradiction. Suppose $(z,y)$ is sequentially $\mathcal{P}$-efficient but not $\mathcal{P}$-efficient. Then there exists a feasible $(\hat{z},\hat{y})$, which is weakly preferred by all $i \in \mathcal{P}$, and strictly preferred by at least one $j \in \mathcal{P}$. If there are several such $j$, then pick the one that is born earliest, and w.o.l.g. assume she is from generation $T$. Then there exists a $(\hat{z},\hat{y})$ such that

1. $(\hat{z},\hat{y})$ is feasible.
2. $u_i(\hat{z}) \geq u_i(z)$ for all $i \in \bigcup_{t \leq T} \mathcal{P}_t \subseteq \mathcal{P}$.
3. $u_j(\hat{z}) > u_j(z)$ for some $j \in \mathcal{P}_T$. 

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But then \((\hat{z}, \hat{y})\) is sequentially \(\mathcal{P}\)-superior to \((z, y)\). A contradiction.

The converse is not true however. Typically, sequential \(\mathcal{P}\)-efficiency is a strict subset of \(\mathcal{P}\)-efficiency. To illustrate in what sense the sequential criterion reduces the set of efficient allocations, we consider a simple 2-period example of fertility choice only. There is one initial parent, who can have at most \(\bar{f}\) children. Thus the set of potential people is \(\mathcal{P} = \{1, (1,1), (1,2), \ldots, (1,\bar{f})\}\).

Let \(f = \sum_{i=1}^{\bar{f}} f^i\) be the number of children the parent chooses. The utility of the parent is \(u_1(f) = -(f - f^*)^2\), where \(f^*\) is a satiation point in the number of children. The utility of child \((1,i)\) is \(u_{(1,i)}(f^i) = f^i\), that is a child cares solely about whether she is alive or not. In this example, any allocation with \(f \geq f^*\) is \(\mathcal{P}\)-efficient, while only \(f = f^*\) is sequentially \(\mathcal{P}\)-efficient.

This makes the sequential concept look like the solution to a social welfare function giving a unique solution rather than the entire contract curve. This is true within a dynasty, however, it is not true across dynasties, as the next example illustrates. Suppose there are two dynasties of the above type:
\(\mathcal{P} = \{1, (1,1), (1,2), \ldots, (1,\bar{f}), 2, (2,1), (2,2), \ldots, (2,\bar{f})\}\).

Suppose there is a resource constraint in this economy such that \(f_1 + f_2 \leq af^*\), where \(1 < a < 2\). Now there is a whole continuum of sequentially \(\mathcal{P}\)-efficient allocations. Any \((f_1, f_2)\) with \(f_1 + f_2 = af^*\) and \(f_1, f_2 \leq f^*\) is sequentially \(\mathcal{P}\)-efficient. This shows that sequential \(\mathcal{P}\)-efficiency is not simply a social welfare concept that picks one particular point on the contract curve. But it is still a strict subset of \(\mathcal{P}\)-efficiency, as \(\mathcal{P}\)-efficiency does not require the second constraint: \(f_1, f_2 \leq f^*\).

Note that even in environments with exogenous fertility, the set of sequentially \(\mathcal{P}\)-efficient allocations is typically smaller than the set of \(\mathcal{P}\)-efficient allocations. This implies that sequential \(\mathcal{P}\)-efficiency does not coincide with Pareto efficiency in environments where Pareto efficiency is well-defined. Sequential \(\mathcal{P}\)-efficiency is a small subset of the Pareto efficient allocations, in the sense that it is inherently biased towards earlier generations. So a Pareto efficient allocation that gives a lot of consumption to later generations, or has a large number of children being born even though the costs are high would
typically not be sequentially $\mathcal{P}$-efficient, except when there is perfect altruism in the environment.

### 1.2 Characterization and Existence

Next, we will give a planner’s algorithm to help identify sequentially $\mathcal{P}$-efficient allocations. Pick any set of weights $\{a_i\}_{i \in P}$ such that $a_i \geq 0$ for all $i$ and $\sum_i a_i = 1$. Define a sequence of sets of allocations $\{W_t\}_{t=0}^\infty$ recursively as follows

\[
W_0 = \arg\max_{(z,y)} \sum_{i \in P_0} a_i u_i(z) \quad \text{s.t. feasibility}
\]

\[
W_t = \arg\max_{(z,y)} \sum_{i \in P_t} a_i u_i(z) \quad \text{s.t. } (z,y) \in W_{t-1}
\]

Define the set of allocations that solves all the problems above as $W_\infty = \lim_{t \to \infty} W_t$. To be precise, $W_\infty$ is defined by

\[
W_\infty = \lim_{T \to \infty} \left\{ \arg\max_{(z,y)} \sum_{i \in P_T} a_i u_i(z) \quad \text{s.t. } (z,y) \in W_{T-1} \right\}
\]

**Proposition 2** If $a_i > 0 \ \forall i \in P$ and $z \in W_\infty$, then $z$ is sequentially $\mathcal{P}$-efficient.

**Proof.** Again, we prove this by contradiction. Suppose $(z, y) \in W_\infty$, but $(z, y)$ is not sequentially efficient. Then there exists a $T$ and $(\hat{z}, \hat{y})$ such that $(\hat{z}, \hat{y})$ if feasible, $\hat{z}$ is weakly preferred over $z$ for all $i \in \bigcup_{t \leq T} P_t$ and strictly preferred for at least one $i \in P_T$. Since all $a_i$ are strictly positive, it follows that $\sum_{i \in P_T} a_i u_i(\hat{z}) > \sum_{i \in P_T} a_i u_i(z)$. Note that the sum is over a finite set of people, hence the strict inequality is preserved. Further, since $\hat{z}$ is weakly preferred over $z$ for all $i \in \bigcup_{t \leq T} P_t$, it follows that $(\hat{z}, \hat{y}) \in W_{T-1}$. But this implies that $(z, y)$ does not solve the planner’s problem at stage $T$. And therefore $(z, y) \notin W_\infty$. A contradiction. $\square$

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**Proposition 3** Assume utility functions are continuous and uniformly bounded above and below, that $\tilde{Z} \subset \mathbb{R}^k \times \mathcal{F}$ is closed, that $Y \subset \mathbb{R}^{k\infty}$ is closed in the product topology and that the set of feasible allocations is bounded period by period, then the set of sequentially $\mathcal{P}$-efficient allocations is non-empty.

**Proof.** This will be proved by showing that the solution to the planning algorithm above is non-empty. Since the objective function is continuous in the product topology and the feasible set is compact in the product topology it follows from the Theorem of the Maximum that $W_0$ is compact in the product topology and non-empty. Assuming that $W_{t-1}$ is non-empty and compact in the product topology, it follows immediately from this same argument that $W_t$ is also non-empty and compact. By induction then, this hold for every $W_t$. By construction, the $W_t$ is a decreasing sequence of compact sets, $W_{t+1} \subseteq W_t$. It satisfies the finite intersection property since $W_t \neq \emptyset$. Thus, it follows that $W_{\infty}$ is non-empty (and compact). Thus, by Proposition 2 above, it follows immediately that a sequentially $\mathcal{P}$-efficient allocation always exists. 

\[ \Box \]

### 1.3 First Welfare Theorem

An analogue first welfare theorem also holds for our second notion of efficiency. The details follow below.

**Definition 1** Given $(p, y, z_{-r})$, a dynastic allocation $z_r = \{z_i\}i \in D_r \in Z_r$ is said to be dynastically sequentially $\mathcal{P}$-maximizing if $\forall T \exists \tilde{z} = (\tilde{z}_r, z_{-r})$ s.t.

1. $\tilde{z}_i \in Z_i(\tilde{z}_{-i})$ for all $i \in D_r$.

2. $u_i(\tilde{z}_i, \tilde{z}_{-i}) \geq u_i(z_i, z_{-i})$ for all $i \in \bigcup_{t \leq T} D_r \cap \mathcal{P}_t$.

3. $u_i(\tilde{z}_i, \tilde{z}_{-i}) > u_i(z_i, z_{-i})$ for at least one $i \in D_r \cap \mathcal{P}_T$.

4. $\sum_t p_t \sum_{i \in D_r \cap \mathcal{P}_t} (\tilde{x}_i + c(\tilde{f}_i)) \leq \Pi_T + \sum_t p_t \sum_{i \in D_r \cap \mathcal{P}_t} c_i$

**Definition 2** $(p^*, z^*, y^*)$ is a Dynastic, sequential $\mathcal{P}$-equilibrium if
1. for all dynasties, given \((p^*, y^*, z_{-\tau}^*)\), \(z_{\tau}^*\) is dynastically sequentially \(\mathcal{P}\)-maximizing.

2. \((z^*, y^*)\) is feasible.

3. Given \(p^*, y^*\) maximizes profits, i.e. \(p^*y \leq p^*y^*\ \forall y \in Y\).

**Lemma 1** Assume the dynastic head of dynasty \(D_\tau\) has strictly monotone preferences. Let \(z_{\tau}^*\) be dynastically sequentially \(\mathcal{P}\)-maximizing for dynasty \(D_\tau\) given \((p, y, z_{-\tau})\). Take any \(z_\tau\) such that for some \(\hat{T}\) \(u_i(z) \geq u_i(z^*)\) for all \(i \in \mathcal{P}_t \cap D_\tau, t \leq \hat{T}\). Then the following must be true

\[
\sum_t p_t \sum_{i \in D_\tau \cap \mathcal{P}_t} (x_i + c(f_i)) \geq \Pi_\tau + \sum_t p_t \sum_{i \in D_\tau \cap \mathcal{P}_t} e_i.
\]

**Proof.** The proof mirrors that given for \(\mathcal{P}\)-efficiency and is not included \(\square\)

**Proposition 4** Suppose \(u_i(x_i, f_i, z_{-i})\) is strictly monotone in \(x_i\) for all \(i \in \mathcal{P}_0\). If \((p^*, z^*, y^*)\) is a Dynastic Sequential \(\mathcal{P}\)-equilibrium, then \(\sum_t p_t(\sum_{i \in \mathcal{P}_t} e_i + y_i^*) < \infty\), and \((z^*, y^*)\) is sequentially \(\mathcal{P}\)-efficient.

**Proof.** The proof mirrors that given for \(\mathcal{P}\)-efficiency and is not included \(\square\)

## 2 Non-Integer Case

In this Appendix we present a version of our environment that can be used to show that the BB model is efficient. There are two reasons why the BB model doesn’t quite fit into the framework of Sections 2 and 3 in the paper. The first is that there is no integer constraint, i.e. the number of children can be anything in \(\mathbb{R}_+\), while our framework constrains children to be natural numbers. Secondly, the BB model imposes symmetry, i.e. only allocations where all siblings do the same thing are considered feasible.
2.1 Notation and Definitions

Assume that there is a finite number of dynasties, each associated with one dynastic head $\tau \in \mathcal{P}_0 = \{1, \ldots, k\}$. The maximal number of children per person is $\bar{f}$. The measure of children actually born to person $i$ is $n_i \in \mathbb{R}$. Then $\bar{f} - f_i$ is the measure of children of person $i$ who are not born. Let $\mathcal{F} = [0, \bar{f}]$. Then we can define the set of potential people recursively as $\mathcal{P}_1 = \mathcal{P}_0 \times \mathcal{F}$, and $\mathcal{P}_t = \mathcal{P}_{t-1} \times \mathcal{F}$. As before, let $\mathcal{P} = \cup_t \mathcal{P}_t$ be the set of all potential people in this economy. A person $i \in \mathcal{P}_t$ can be written as $i^t = (i^{t-1}, i_t)$ where $i^{t-1}$ is $i^t$’s parent and $i_t$ specifies $i^t$’s position in the sibling order. Then $i_t \leq f_{i^{t-1}}$ means that $i^t$ is born and $i_t > f_{i^{t-1}}$ implies that $i^t$ is not born.

We assume that there are $k$ goods available in each period. There is one representative firm, which behaves competitively. The technology is characterized by a production set: $Y \subset \mathbb{R}^{k\infty}$. In other words, an element of the production set is an infinite sequence of $k$-tuples, that describes feasible input/output combinations. Note that goods are defined in a broad sense here, it can include labor, leisure, capital stock, etc. An element of the production set will be denoted by $y \in Y$. We can write $y = \{y_t\}_{t=0}^\infty$, where $y_t = (y^1_t, \ldots, y^k_t)$ is the projection of the production plan onto time $t$.

An allocation is $(z, y)$ with $z_i = (x_i, f_i)$ for all $i \in \mathcal{P}$.

Define the consumption set as follows:

$$z_i \in Z_i(z_{-i}) = \begin{cases} (e_i, 0) & \text{if } j > f_{i^{t-1}} \\ \mathcal{Z} & \text{if } j \leq f_{i^{t-1}} \end{cases}$$

As before, define dynasties as sets of people that are linked through external effects. A dynastic structure is a partition of the set of potential people into a finite number of dynasties, $\mathcal{P} = \cup_{\tau} \mathcal{D}_\tau$. Let $\mathcal{D}_\tau^t$ be the subset of dynasty $\tau$ that is alive at time $t$.

Definition 3 An allocation is feasible if

1. $z_i \in Z_i(z_{-i})$ for all $i$
2. $\sum_t \left( \int_{P \cap D_t} x_i di + \int_{P \cap D_t} c(f_i) di \right) \leq \sum_t \int_{P \cap D_t} e_i di + y_t$ for all $t \geq 1$

3. $y \in Y$

Note this formulation allows for different children of the same parent to be treated differently. This is more general than the Barro-Becker formulation where all children of the same parent receive the same allocation by assumption.

**Definition 4** An allocation $(z, y) = \{(x_i, f_i)\}_{i \in P}$ is $\mathcal{P}$-efficient if it is feasible and there is no other feasible allocation $(\tilde{z}, \tilde{y})$ s.t.

1. $u_i(\tilde{x}_i, \tilde{f}_i, \tilde{z}_i \rangle \geq u_i(x_i, f_i, z_i)$ for all $i \in P$

2. There exists $S \subset P$ with positive measure s.t. $u_i(\tilde{x}_i, \tilde{f}_i, \tilde{z}_i) > u_i(x_i, f_i, z_i)$ for all $i \in S$.

**Definition 5** Given $(p, y, z_{\sim r})$, a dynastic allocation $z_r = \{z_i\}_{i \in D_r}$ is said to be Dynastically $\mathcal{P}$-maximizing if $\exists \hat{z}_r$ s.t.

1. $\hat{z}_i \in Z_i(\hat{z}_{\sim i})$ for all $i \in D_r$.

2. $u_i(\hat{z}_i, \hat{z}_{\sim i}) \geq u_i(z_i, z_{\sim i})$ for all $i \in D_r$.

3. There exists $S \subset D_r$ s.t. $u_i(\hat{z}_i, \hat{z}_{\sim i}) > u_i(z_i, z_{\sim i})$ for all $i \in S$ and $S$ has a positive measure.

4. $\sum_t \int_{D_r \cap \mathcal{P}_t} (\hat{x}_i + c(\hat{f}_i)) di \leq \Pi_r + \sum_t p_t \int_{D_r \cap \mathcal{P}_t} e_i di$

Next we define the analogue of a competitive equilibrium among the dynasties in the partition (exactly the same as before).

**Definition 6** $(p^*, z^*, y^*)$ is an $\mathcal{P}$-dynastic Walrasian equilibrium if

1. for all dynasties, given $(p^*, y^*, z_{\sim r}^*)$,

   $z_r^*$ is dynastically $\mathcal{P}$-maximizing.

2. $(z^*, y^*)$ is feasible.

3. Given $p^*$, $y^*$ maximizes profits, i.e. $p^*y \leq p^*y^* \ \forall y \in Y$.  

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2.2 First Welfare Theorem and Proof

Assumption 1 (No negative externalities)
We assume that $u_i$ is monotone increasing in $x_j$, that is each agent is weakly better off when consumption is increased for a set of agents of positive measure. Thus, there are no negative external effects in consumption.

Assumption 2 (Positive externalities only within a Dynasty)
We assume that if $z$ and $z'$ are two allocations such that $z_i = z'_i$ for almost every $i \in D$, then, $u_j(z) = u_j(z')$ and $Z_j(z) = Z_j(z')$ for all $j \in D$.

Lemma 2 Assume there exists at least one $i \in P_0 \cap D$ with strictly monotone preferences. Let $z^\ast$ be dynastically $P$-maximizing for dynasty $D$, given prices $p$ and production $y$. Then $u_i(z^\ast_r, z^\ast_r) \geq u_i(z_i, z^\ast_r)$ for all $i \in D$ implies that $\sum t_i \int_{D^r \cap P_i} (x_i + c(f_i))di \geq \Pi_r + \sum t_i \int_{D^r \cap P_i} e_idi$.

Proof. This will be proved by contradiction. Suppose not. Then there exists a $z_r$ such that $u_i(z_r, z^\ast_r) \geq u_i(z^\ast)$ for all $i \in D$ and $\sum t_i \int_{D^r \cap P_i} (x_i + c(f_i))di < \Pi_r + \sum t_i \int_{D^r \cap P_i} e_idi$. Then construct a new allocation $\tilde{z}$ as follows: $\tilde{z}_j = (z_j + \epsilon, f_j)$ for $j \in P_0 \cap D$ with strictly monotone preferences and $\tilde{z}_i = z_i \forall$ other $i \in D$. Then $\exists \epsilon > 0$ such that the dynastic allocation $\tilde{z}$ does not violate the dynastic budget constraint. Moreover, by Assumption 1 $\tilde{z}$ is weakly preferred over $z$ by all $i$ in the dynasty, and hence also over $z^\ast$. Finally, by strict monotonicity, $u_j(\tilde{z}) > u_j(z^\ast)$, but this contradicts the assumption that $z^\ast$ was dynastically $P$-maximizing.

Proposition 5 Suppose $u_i(x_i, f_i, z_i)$ is strictly monotone in $x_i$ for all $i \in P_0$. If $(p^\ast, z^\ast, y^\ast)$ is an $P$-dynamic Walrasian equilibrium, then $\sum [y_i^\ast + \int_{P_i} p_e d_i] < \infty$, and $(z^\ast, y^\ast)$ is $P$-efficient.

Proof. Suppose $(z^\ast, y^\ast, p^\ast)$ is an $P$-dynamic Walrasian equilibrium and by way of contradiction, assume that it is not $P$-efficient. Then there exists an alternative feasible allocation $(z, y)$ that is $P$-superior to $(z^\ast, y^\ast)$. That is,
\[ u_i(z) \geq u_i(z^*) \] for all \( i \in P \) and there exists \( S \subset P \) s.t. \( \forall i \in S \) \( u_i(z) > u_i(z^*) \). W.l.o.g. assume \( S \subset D_\tau \) for some \( \tau \). Then, since \( z^*_\tau \) was dynastically \( P \)-maximizing, and since there are no external effects across dynasties (Assumption 2), for dynasty \( D_\tau \) it must be that \( z_\tau \) was not affordable, i.e.

\[ \sum_t p_t^* \int_{D_\tau \cap P_t} (x_i + c(f_i))di > \Pi_\tau + \sum_t p_t^* \int_{D_\tau \cap P_t} e_idi \]

Moreover, by Lemma 2, we know that for all other dynasties the following must hold.

\[ \sum_t p_t^* \int_{D_\tau \cap P_t} (x_i + c(f_i))di \geq \Pi_\tau + \sum_t p_t^* \int_{D_\tau \cap P_t} e_idi \]

Summing up over all dynasties, we get

\[ \sum_t p_t^* \int_{P_t} (x_i + c(f_i))di > \sum_t p_t^* [y_t + \int_{P_t} e_idi] \] (1)

Note that the right hand side is finite by assumption, and hence the strict inequality is preserved. Profit maximization implies that \( p^*y^* \geq p^*y \) for all other production plans \( y \in Y \). Using this, we can rewrite equation 1 as

\[ \sum_t p_t^* \int_{P_t} (x_i + c(f_i))di > \sum_t p_t^* [y_t + \int_{P_t} e_idi] \] (2)

Finally, feasibility of \((x,y)\) implies that

\[ \int_{P_t} (x_i + c(f_i))di \leq y_t + \int_{P_t} e_idi \] for all \( t \)

Multiplying the above by \( p_t^* \) and summing up over all \( t \) gives

\[ \sum_t p_t^* \int_{P_t} (x_i + c(f_i))di \leq \sum_t p_t^* [y_t + \int_{P_t} e_idi] \]

But this contradicts equation 2. This completes the proof. \( \square \)
3 Finite Horizon BB Game

A $T$ horizon B&B game is a game in $T+1$ stages. The stages will be denoted by $t = 0, 1, \ldots, T$. In period 0, there is one player, player 0. His actions and preferences are denoted with 0 subscripts. In period $t$, $t \geq 1$, there are a continuum of players indexed by $i^t$, $i^t \in \mathcal{P}^t = [0, \bar{f}]^t$.

The strategy sets are as follows. In period 0, player 0 must choose

$$s^0 \in S^0 = \{(x_0, f_0, b_0(\cdot)) \mid p_0x_0 + p_0c_0(f_0) + \int_0^f b_0(i)di \leq p_0e_0\},$$

where $S^0 \subset R^+ \times [0, \bar{f}] \times L_\infty([0, \bar{f}])$. Recursively, let $h^{t-1}$ denote the history up to and including period $t-1$. In period $t$, $T > t \geq 1$, player $i^t$ must choose

$$s^t \in S^t(h^{t-1}) = \begin{cases} A_t(h^{t-1}) & \text{if } i_t \leq f(t^{t-1}) \\ \{(0, 0, 0)\} & \text{if } i_t > f(t^{t-1}). \end{cases}$$

Where $A_t(h^{t-1}) = \{(x_t(i^t), f_t(i^t), b_t(\cdot; i^t)) \mid p_t x_t(i^t) + c_t(f_t(i^t)) + \int_0^f b_t(i; i^t)di \leq p_t e_t(i^t) + b_{t-1}(i_t; t^{t-1})\}$.

That is, if $i^t$ is not ‘born’ he has no choices to make. In the case where $i^t$ is born, $S^t(h^{t-1}) \subset R^+ \times [0, \bar{f}] \times L_\infty([0, \bar{f}])$.

Finally, a player in period $T$ makes similar choices except that he is constrained to choose $f_T(i^T) = 0$, and $b_T(\cdot; i^T) \equiv 0$.

Note that $c_t(f)$ is childbirth costs in terms of the $k$ goods.

Period 0 utility is given by:

$$U_0 = u_0(x_0) + \beta g_1(f_0) \int_0^{f_0} [u_1(x_1(i^1)) + \beta g_2(f_1(i^1)) \int_0^{f_1(i^1)} [u_2(x_2(i^2)) + \ldots] di_T di_{T-1} \ldots di_1$$

Period $t$ utility for player $i^t$ is given by:

$$U_{i^t} = u_t(x_t(i^t)) + \beta g_{t+1}(f_t(i^t)) \int_0^{f_t} [u_{t+1}(x_{t+1}(i^{t+1}))$$

$$+ \beta g_{t+2}(f_{t+1}(i^{t+1})) \int_0^{f_{t+1}(i^{t+1})} [u_{t+2}(x_{t+2}(i^{t+2})) + \ldots] di_T di_{T-1} \ldots di_{t+1}$$

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Assumption 3 $u(\cdot)$ is continuous, strictly increasing, strictly concave and $u(0) = 0$.

Assumption 4 Assume $c(f) = \theta f$.

Assumption 4 implies that $f_1 c(f_2) = c(f_1 f_2)$ which will be used below.

Assumption 5 Assume that $g(\cdot)$ satisfies $g(x)g(y) = g(xy)$.

Assumption 6 Assume that $H(F,X) = g(F)Fu(X/F)$ is strictly concave in $(F,X)$.

Proposition 6 For every $T$, there is a unique SPE outcome to the game. This equilibrium outcome maximizes the utility of the time 0 player, and hence is as if the time 0 player was choosing the actions for all players at all dates and nodes of the game tree. It has the property that every born child in every generation is treated symmetrically, and is as if the time 0 player faced a single budget constraint aggregating expenditures over all allocations in all periods.

The proof of this proposition is by induction starting at $T = 1$. That is, with a 2 stage game. First, we characterize the SPE outcomes of the game and show that it is unique. To simplify what comes later, we will index these games by $b_0$ in what follows.

The proof for $T = 1$ proceeds in 3 steps. In the first step, we characterize the SPE outcomes of the sequential player game, show that they have the property that all descendants are treated symmetrically – each receives the same bequest – and all choose the same consumption. We also show that because of this they can be characterized as the solution to the maximization problem:

$$\max_{x_1,f_1,x_2} u(x_1) + \beta g(f_1)u(x_2)$$

$$s.t. \quad p_1 x_1 + p_1 c(f_1) + f_1 p_2 x_2 \leq p_1 e_1 + f_1 p_2 e_2 + b_0$$
In Step 2, we characterize the solution to the planner’s problem in which the period 1 player chooses everything, both in period one, and in period 2, subject to one overall budget constraint. We show that the solution to this problem is also symmetric, all period 2 descendants receive the same consumption level. We also show that the solution to this problem can be obtained by solving the same maximization problem as above.

Finally, in Step 3, we show that the solution to the planner’s problem is unique. This implies that the SPE outcome and the solution to the planner’s problem are the same.

Step 1:

As usual, we solve the game backwards. Since players in the last period cannot have children by assumption, it follows that any period 2 agent will optimally choose his consumption level from the budget constraint he faces given prices, his endowment and his bequest. That is, he solves:

$$\max_{x_2} U \equiv u(x_2)$$

$$s.t. \ p_2 x_2 \leq p_2 e_2 + b_1$$

Denote the solution to this problem by $x_2(b_1)$. Note that standard arguments imply that this maximized utility is concave in $b_1$. That is, $u(x_2(b_1))$ is a strictly concave function of $b_1$.

Knowing this, the problem of a period 1 agent is the following (given the bequest she receives from her parent $b_0$).

$$\max_{x_1, f_1, b_1(i^2), x_2(i^2)} U_1 \equiv u(x_1) + \beta g(f_1) \int_{0}^{f_1} u(x_2(i^2)) di_2$$

$$s.t. \ p_1 c(f_1) + p_1 x_1 + \int_{0}^{f_1} b_1(i^2) di_2 \leq p_1 e_1 + b_0$$

$$x_2(i^2) = x_2(b_1(i^2)) \text{ for all } i^2$$

The last constraint (the IC constraint) captures the fact that descendants will behave optimally given the choice of bequests, and can be substituted into the objective function.
Claim 1  The solution to this problem has $b_1(i^2) = b_1$ for some $b_1$. It follows that $x_2(b_1(i^2)) = x_2(b_1)$, that is, all period 2 descendants of any given period 1 player get the same bequest and choose the same consumption.

Proof: To see this, suppose that $x_1^*, f_1^*$, and $b_1^*(i^2)$, is the optimal choice for the period 1 player, and assume to the contrary that $b_1^*$ is not constant. Note that an alternative strategy that is feasible is $(x_1^*, f_1^*, b_1)$ where $b_1 = \frac{1}{f_1^*} \int_0^{F_1} b_1^*(i^2)di_2$. This alternative strategy fixes $(x_1, f_1) = (x_1^*, f_1^*)$ but makes bequests equal to all children. Under this alternative strategy, the payoff he receives is:

$$U_1 = u(x_1^*) + \beta g(f_1^*)u(x_2(b_1^*)) > u(x_1^*) + \beta g(f_1^*) \int_0^{F_1} u(x_2(b_1^*(i^2)))di_2 = U_1^*,$$

the payoff of the supposed optimal strategy. The inequality is strict because $u(x_2(b_1^*))$ is a strictly concave function of $b_1$ and $b_1^*$ is assumed non-constant. This contradiction establishes the claim.

Given this result, it follows that we can rewrite the problem of a period 1 player receiving bequest $b_0$ as:

$$\max_{x_1, f_1, b_1, x_2} \quad u(x_1) + \beta g(f_1)f_1u(x_2)$$
$$\text{s.t.} \quad p_1c_1(f_1) + p_1x_1 + f_1b_1 \leq p_1e_1 + b_0$$
$$x_2 = x_2(b_1).$$

That the solution to this problem satisfies $p_2x_2 = p_2e_2 + b_1$ follows immediately from the monotonicity of $u$ in conjunction with the fact that $p_2x_2(b_1) \leq p_2e_2 + b_1$ for all $b_1$.

Using this, it follows from the assumption that preferences are time consistent (i.e., the preferences over $x_2$ for time 1 and 2 players are the same) that this problem is equivalent to:

$$\max_{x_1, f_1, x_2} \quad u(x_1) + \beta g(f_1)f_1u(x_2)$$
$$\text{s.t.} \quad p_1c_1(f_1) + p_1x_1 + f_1p_2x_2 \leq p_1e_1 + f_1p_2x_2 + b_0.$$
That is, because of time consistency of preferences, if \((x_1^*, f_1^*, x_2^*)\) is the solution to this problem, then \((x_1^*, f_1^*, b_1^*, x_2^*)\) is the solution to the previous one where \(b_1^* = p_2 x_2^* - p_2 e_2\). (And hence, \(x_2(p_2 x_2^* - p_2 e_2) = x_2^*\).)

This completes Step 1. Let \((x_1(b_0), f_1(b_0), b_1(b_0))\) denote the solution to this problem.

Step 2:

Claim 2 For all \(b_0\), the solution to the maximization problem:

\[
\max_{x_1, f_1, x_2(i^2)} U_1 \equiv u(x_1) + \beta g(f_1) \int_0^{f_1} u(x_2(i^2)) di_2
\]

\[s.t.\] \(p_1 c(f_1) + p_1 x_1 + p_2 \int_0^{f_1} x_2(i^2) di_2 \leq p_1 e_1 + f_1 p_2 e_2 + b_0.\)

is given by \((x_1(b_0), f_1(b_0), x_2(b_1(b_0)))\).

Proof. First we show that at the solution to this problem, \(x_2(i^2) = x_2\) for some choice of \(x_2\). To see this, suppose that the solution is given by \((x_1^*, f_1^*, x_2^*(i^2))\) where \(x_2^*(i^2)\) is not constant. Consider the alternative plan \((x_1^*, f_1^*, \hat{x}_2(i^2))\) where \(\hat{x}_2(i^2) = \frac{1}{f_1} \int_0^{f_1} x_2^*(i^2) di_2\). This clearly satisfies the constraints and gives strictly higher utility since \(u\) is strictly concave, and \(x_2^*\) is assumed non-constant. Using this fact, and substituting gives:

\[
\max_{x_1, f_1, x_2} U_1 \equiv u(x_1) + \beta g(f_1) f_1 u(x_2)
\]

\[s.t.\] \(p_1 c(f_1) + p_1 x_1 + p_2 f_1 x_2 \leq p_1 e_1 + f_1 p_2 e_2 + b_0.\)

This proves the claim and completes Step 2.

Claim 3 For any \(b_0\), the SPE outcome and the solution to the planner’s problems are unique.

Proof. Here, we follow Alvarez (1999). Define \(X_2 \equiv x_2 f_1\) and \(H(f, X) \equiv g(f) fu(\frac{X}{f})\). Then we can rewrite the problem above as:

\[
\max_{x_1, f_1, X_2} u(x_1) + \beta H(f_1, X_2)
\]

\[s.t.\] \(p_1 c(f_1) + p_1 x_1 + p_2 f_1 X_2 \leq p_2 f_1 e_2 + p_1 e_1 + b_0\)
Since the objective function is strictly concave and the constraint set is convex due to linearity, it follows that the maximizer is unique.

This completes Step 3, and shows that for $T = 1$, the SPE outcome is unique, symmetric and characterized by either, the solution to (3), or, equivalently the solution to the planner’s problem of the time 1 player.

What remains is to show that the argument extends by induction. The proof that if the statement in the proposition holds for $T$ then it holds for $T + 1$ is exactly the same as the proof that it holds for $T = 2$ given that it holds for $T = 1$, but is much more burdensome in terms of notation. Because of this, we will outline the extensions needed to go from $T = 1$ to $T = 2$. The induction argument for going from $T$ to $T + 1$ being completed with the obvious changes.

From the argument given in Step 1 above, it follows that in any SPE of the $T = 2$ game, the play from period $T = 1$ onward for player $i^1$, assumed to receive the bequest $b_0(i^1)$, is obtained from the solution to the following problem:

$$\max_{x_1(i^2), f_1(i^1), x_2(i^1)} u(x_1(i^1)) + \beta g(f_1(i^1))f_1(i^1)u(x_2(i^1))$$

$$\text{s.t.} \quad p_1c_1(f_1(i^1)) + p_1x_1(i^1) + f_1(i^1)p_2x_2(i^1) \leq p_1e_1 + f_1p_2x_2 + b_0(i^1).$$

As above, the solution to this problem is unique, and depends only on $b_0$. Accordingly, denote the solution by $(x_1(b_0), f_1(b_0), x_2(b_0))$. It follows that in any SPE, the period 0 player must solve:

$$\max_{x_0, f_0, b_0(i^1), x_1(i^2), f_1(i^1), x_2(i^2)} u(x_0) + \beta g(f_0) \int_0^{f_0} [u(x_1(i^1)) + \beta g(f_1(i^1))f_1(i^1)u(x_2(i^1))]di_1$$

$$\text{s.t.} \quad p_0x_0 + p_0c(f_0) + \int_0^{f_0} b_0(i^1)di_1 \leq p_0e_0$$

$$(x_1(i^1), f_1(i^1), x_2(i^1)) = (x_1(b_0(i^1)), f_1(b_0(i^1)), x_2(b_0(i^1))).$$

Next, we proceed as above to show that the solution to this problem must be symmetric in its treatment of time 1 children. That is $b_0(i^1) = b_0$ for some $b_0$ and hence, $(x_1(i^1), f_1(i^1), x_2(i^1))$ are also constant (and equal to $(x_1(b_0), f_1(b_0), x_2(b_0))$).
Using the notation above, the problem of the period 0 agent can be written as follows.

\[
\max_{x_0, f_0, b_0, x_1, f_1, X_2} U_0 = u_0(x_0) + \beta g(f_0) \int_0^{f_0} [u_1(x_1(i)) + \beta H(f_1(i), X_2(i))] di
\]

\[
s.t. \quad p_0 c(f_0) + p_0 x_0 + \int_0^{f_0} b_0(i) di \leq p_0 e_0
\]

\[
p_1 c(f_1(i)) + p_1 x_1(i) + p_2 X_2(i) \leq p_2 f_1(i)e_2 + p_1 e_1 + b_0(i) \quad \forall i \in [0, f_0] \quad (4)
\]

Claim: Holding \(x_0, f_0\) fixed, the optimal solution has the form \(x_1(i) = x_1, f_1(i) = f_1, X_2(i) = X_2,\) and \(b_0(i) = b_0.\)

Proof: Suppose not. Take the solution \(x_1(i), f_1(i), X_2(i).\) Define \(\hat{x}_1 = \frac{1}{f_0} \int_0^{f_0} f_1(i) di,\)
\(\hat{f}_1 = \frac{1}{f_0} \int_0^{f_0} f_1(i) di,\)
\(\hat{X}_2 = \frac{1}{f_0} \int_0^{f_0} X_2(i) di,\) and \(\hat{b}_0 = \frac{1}{f_0} \int_0^{f_0} b_0(i) di.\) Then \(U_0\) is strictly higher due to the strict concavity of \(H()\) and the strict concavity of \(u().\) It is also budget feasible. Suppose not, then

\[
p_{1} c(\hat{f}_1) + p_1 \hat{x}_1 + p_2 \hat{X}_2 > p_1 e_1 + p_2 e_2 \hat{f}_1 + \hat{b}_0.
\]

But this means that

\[
p_{1} c \left( \frac{1}{f_0} \int_0^{f_0} f_1^*(i) di \right) + p_1 \frac{1}{f_0} \int_0^{f_0} x_1^*(i) di + p_2 \frac{1}{f_0} \int_0^{f_0} X_2^*(i) di
\]

\[
> p_1 e_1 + p_2 e_2 \frac{1}{f_0} \int_0^{f_0} f_1^*(i) di + p_1 \frac{1}{f_0} \int_0^{f_0} b_0^*(i) di
\]

A contradiction to the budget constraint (4) above. \(\Box\)

Because of this, it follows that the time zero player’s problem can be rewritten as solving:

\[
\max_{x_0, f_0, b_0, x_1, f_1, x_2} u(x_0) + \beta g(f_0) f_0 [u(x_1) + \beta g(f_1) f_1 u(x_2)]
\]

\[
s.t. \quad p_0 x_0 + p_0 c(f_0) + f_0 b_0 \leq p_0 e_0
\]

\[
(x_1, f_1, x_2) = (x_1(b_0), f_1(b_0), x_2(b_0)).
\]

Using the time consistency of preferences as in the argument above, this problem can be rewritten as:

\[
\max_{x_0, f_0, b_0, x_1, f_1, x_2} u(x_0) + \beta g(f_0) f_0 [u(x_1) + \beta g(f_1) f_1 u(x_2)]
\]

\[
s.t. p_0 x_0 + p_0 c(f_0) + f_0 [p_1 c(f_1) + p_1 x_1 + f_1 p_2 x_2] \leq p_0 e_0 + f_0 [p_1 e_1 + f_1 p_2 e_2].
\]
Or equivalently:

\[
\max_{x_0, f_0, x_1, f_1, X_2} U_0 \equiv u_0(x_0) + \beta g(f_0) f_0 u(x_1) + \beta^2 g(f_0) f_0 H(f_1, X_2) \\
\text{s.t. } \rho c(f_0) + \rho x_0 + p_1 f_0 c(f_1) + p_1 f_0 x_1 + p_2 f_0 X_2 \leq \rho e_0 + f_0 p_1 e_1 + f_0 f_1 p_2 e_2
\]

The proof that the solution to the problem in which the time zero player chooses everything also solves this problem (i.e., Step 2) is similar to the argument given above and is omitted.

Finally, as in Alvarez (1999), the transformation of variables can be used to make this a convex problem. Let \( X_1 = f_0 x_1, \bar{X}_2 = f_0 X_2, \) and \( F_1 = f_0, F_2 = f_0 f_1. \) Using this together with Assumptions 4 and 5 and recalling the definition of \( H(f, X) = g(f) f u(X_f), \) the problem can be written as

\[
\max_{x_0, F_1, F_2, X_1, X_2} \quad u(x_0) + \beta H(F_1, X_1) + \beta^2 H(F_1, \bar{X}_2) \\
\text{s.t. } \rho c(F_1) + \rho x_0 + p_1 c(F_2) + p_1 X_1 + p_2 \bar{X}_2 \leq \rho e_0 + p_1 e_1 F_1 + p_2 e_2 F_2
\]

This is a strictly concave objective function due to the concavity of \( u(\ ) \) and \( H(\ ) \). The constraint set is convex due to linearity. Therefore, the maximizer is unique. This completes the induction step, establishing the Proposition.

Thus, we see that every SPE outcome of this game has the property that it is exactly the same as that outcome induced when the period 0 player chooses everything in every period. It follows that there is a unique SPE outcome to the finite horizon truncation.

Since the limit of the finite horizon truncations of the infinite horizon game converges to a SPE of the infinite horizon game, it follows that there is a unique equilibrium outcome of the infinite horizon game that is the limit of the finite horizon truncations. Moreover, this equilibrium outcome is the same as if the time 0 player was allowed to choose everything. Note that this is the choice of the time 0 player under a single budget constraint.