Preliminary and Incomplete.
Comments Welcome.

NOTES ON THE EXISTENCE OF EQUILIBRIUM WITH DISTORTIONS IN INFINITE HORIZON ECONOMICS

by

Larry E. Jones*

and

Rodolfo E. Manuelli**

First Version: October 1989
This Version: March 1999

---

* Department of Economics, University of Minnesota, 1035 Heller Hall, Minneapolis, MN 55455.

**Department of Economics, University of Wisconsin, Madison, 1160 Observatory Dr., Madison, WI 53706.

1. Introduction

A variety of recent papers have studied the properties of infinite horizon models of economies with distortions. Particular examples include the recent work on models of endogenous growth. These papers have concentrated on two different types of models. The first type has emphasized the possibility of externalities in some aspect of the capital accumulations process as a possible engine of growth. Examples of this include Romer (1986), Lucas (1988) and (1989), and Stokey (1990). Yet another strand of this literature has concentrated on models that are convex on the technological side but include the presence of distortionary tax and spending policies as possible explanations for the heterogeneity across countries of growth and development histories. Examples of this literature include Rebelo (1987), Barro (1988), Jones and Manuelli (1990) and (1990).

The purpose of these notes is to present a preliminary discussion of the general matehmatical properties of this class of models. In particular, we will be interested in providing a general result on the existence of Walrasian equilibrium in these settings. This exercise can be viewed simultaneously as a generalization of two distinct strands of the theoretical literature.

First, it provides a generalization of the recent literature on the existence of Walrasian equilibrium in infinite dimensional settings to include distortions. The most relevant work in this area from the perspective of these notes is the early work of Bewley (1972) on the case of L_∞. Other recent work in this area includes the paeprs by Mas-Colell (1986), Jones (1987), Aliprantis, Brown and Burkinshaw (1989), Richard (1989), and Zame (1987). The recent paper by Mas-Colell and Zame (1990) provides a survey of the results in this area.

Similarly, it provides a generalization of the work on the existence of equilibrium in the presence of distortions to the infinie dimensional setting. Examples of this branch of the literature include Shafer and Sonnenschein (1976) and McKenzie (1955). Indded, we will follow the structure developed in Shafer and Sonnenschein quite closely.

The difficulty in providing an existence result for these models arises because of the failure of standard infinite dimensional techniques in environments with distortions. That is, the usual technique for proving existence in infinite dimensional models (since Mas Colell (1986)) is to search the Pareto frontier for allocations which, at supporting prices, the value of consumption equals the value of initial endowments. Since the equilibria in the presence of distortions are not efficient in general, this proof technique will no longer work. For this reason, we combine a generalization of the limiting argument developed in Bewley's original paper with the existence proof given in Arrow and Debreu (1954).

The remainder of these notes is organized as follows. Section 2 contains notation, assumptions, definitions, and the statement of an existence result in tax distorted economies. Section 3 contains the proof of the theorem. Section 4 discusses a variaety of desireable extensions of the main result of the notes and the difficultes they pose. These include the incorporation of externalities, more general infinite dimensional spaces and a characterization of equilibrium prices. Finally, section 5 discusses some relevant examples.

2. Notation and Assumptions

We will follow Shafer and Sonnenschein's notation as closely as possible. Let $(\Omega,\mathscr{F},\mu)$ be a measure space where μ is assumed σ-finite. This includes infinite horizon discrete time models with certainty and uncertainty, and finite and infinite horizon continuous time models as special cases. See Bewley for details. Let $L = L_\infty(\Omega,\mathscr{F},\mu)$, $L' = ba(\Omega,\mathscr{F},\mu)$ the norm dual of L.

There are I consumers indexed by i and J firms indexed by j. Market prices will be denoted by p and are located in L'. A **state** of the economy, s, is a point $s = (x,y) \in L'^{I+J} = S$. This is interpreted as a complete description of the actions of both firms and households.

For each i, there is a set of feasible consumption vectors given by $X_i \subset L$. Similarly, $Y_j \subset L$ is firm j's production set.

Consumer i has preferences which may depend on the state and are defined on X_i. Denote these by $\succeq_i(s) \subset X_i \times X_i$. We assume that preferences are complete, transitive, reflexive and convex for all s. Endowments are given by e_i and initial firm shares are denoted by θ_{ij}.

The next step is to specify the basics of government policy. The difficulty here is that in the constructions used in the proof, since prices are endogenous, we will need to have a specification of the actions of the government for all possible states of the economy. Here, then, a **government policy** is a collection of functions, $\phi_i(s,p)$, $\mu_i(s,p)$, and $\psi_j(s,p)$ mapping from $S \times L'$ to L', $\mathbb{R}$, and L' respectively. The interpretation of this is: Given a state $s \in S$, and a $p \in L'$, $\phi_i(s,p) \in L'$ represents the after tax prices faced by consumer i; Similarly, the after tax prices faced by producer j are $\psi_j(s,p) \in L'$; Finally, the state contingent lump sum transfer from the government to consumer i is $\mu_i(s,p)$.

Note that in this formalization, we have allowed the description of preferences and the levels of the relevant government policy variables for household i to depend on i's own actions. This is purely a formal convenience for some of the examples that we will present below. There is no loss in interpretation if you assume, for example that ϕ_i depends only on s_{-i}. A similar comment holds for producers.

An <u>equilibrium</u> is then a state $s^* = (x^*,y^*)$ and a price vector $p^* \in L'$ such that:

(i) $x_i^* \in X_i \; \forall$ i.

(ii) $y_j^* \in Y_j \; \forall$ j.

(iii) $\sum x_i^* \leq \sum e_i + \sum y_j^*$.

(iv) $\forall$ i, x_i^* maximizes $\succeq_i (s^*)$ over $x \in L$ subject to the constraints $x \in X_i$ and $\phi_i(s^*,p^*)(x - e_i) \leq \sum_j \theta_{ij}\, \psi_j(s^*,p^*)y_j^* + \mu_i(s^*,p^*)$.

(v) $\forall$ j, y_j^* maximizes $\psi_j(s^*,p^*)y$ subject to $y \in Y_j$.

(vi) $\sum_i \mu_i(s^*,p^*) = \sum_i (\phi_i(s^*,p^*) - p^*)(x_i^* - e_i) + \sum_j (p^* - \psi_j(s^*,p^*))y_j^*$.

(vii) $p^*(\sum_i e_i + \sum_j y_j^* - \sum_i x_i^*) = 0$.

(i)-(v) are standard. (vi) says that, in equilibrium, the government's budget must balance in present value terms. Finally, condition (vii) requires that if there is any excess supply in equilibrium, it must be concentrated on a set of goods where market prices are zero.

There are two features of this definition that should be noted. First, as can be seen from the form of the government budget constraint (i.e., the right hand side of (vi)), taxes are only paid on **net** transactions in the market by both firms and consumers. Second, note that both firms and consumers are a trifle schizophrenic. Although they affect the prices that they face (through the dependence of both ϕ_i and ψ_j on x_i and y_j), they ignore this fact when making their private decisions. This is a formal convenience that will allow us to treat certain types of non-linear tax systems and aggregate externalities without further changes. Alternatively, this can be justified by interpreting the model as one with an infinite number of each of the types of firms and consumers, where we are looking for symmetric equilibria.

We will make the following assumptions:

H1. For all i, $X_i \subset L_+$ is convex and $0 \in X_i$.

H2. For all i, X_i is $\sigma(L_\infty, L_1)$ closed.

H3. For all i, there exists $A_i \subset \Omega$ such that $\mu(A_i) > 0$ and for all $B \subset A_i$ with $\mu(B) > 0$ and all $\alpha > 0$, $x + \alpha\chi_B \succ_i (s)\, x$ for all s and $x \in X_i$.

H4. For all i, $\{(x_1,x_2,s) \in L^{I+J+2} \mid x_1 \succ_i(s)\, x_2\}$ is $\|\ \|_\infty \times (\sigma(L_\infty, L_1))^{I+J+1}$ open.

F1. For all j, Y_j is convex and $-L_+ \subset Y_j$.

F2. For all j, Y_j is $\sigma(L_\infty, L_1)$ closed.

G1. For all s and p, $\phi_i(s,p)$ and $\psi_j(s,p)$ are non-negative and not zero.

G2. (1) For all i, ϕ_i is $(\sigma(L_\infty, L_1))^{I+J} \times \sigma(ba, L_\infty)$ to $\sigma(ba, L_\infty)$ continuous.

(2) For all i, μ_i is $(\sigma(L_\infty, L_1))^{I+J} \times \sigma(ba, L_\infty)$ to $\mathbb{R}$ continuous.

(3) For all j, ψ_j is $(\sigma(L_\infty, L_1))^{I+J} \times \sigma(ba, L_\infty)$ to $\sigma(ba, L_\infty)$ continuous.

G3. For all s and p,

$$\sum_i \mu_i(s,p) = \sum_i (\phi_i(s,p) - p)(x_i - e_i) + \sum_j (p - \psi_j(s,p))y_j.$$

J1. There is an $M > 0$ such that, if $x_i \in X_i$, $y_j \in Y_j$ and $\sum_i x_i \leq \sum_i e_i + \sum_j y_j$, then $\|x_i\| < M$ for all i and $\|y_j\| < M$ for all j.

J2. For all s and p, $\phi_i(s,p)e_i + \mu_i(s,p) > 0$.

Most of these assumptions are straightforward generalizations of the standard treatment. Assumption H4 says that preferences are continuous both as a function of the households' own choice and the actions of other households and firms. Assumption H3 is a (Bewley's) weak monotonicity condition. Note that it implicitly places some restrictions on X_i.

Assumption G3 states the government plans to balance the budget even out of equilibrium. This coupled with the usual maximization arguments will imply that Walras' Law holds in the aggregate.

Assumption J1 implies that the activities that are feasible are uniformly bounded. Thus, the relevant part of the firm's production setes are bounded even if outputs of other firms are used as inputs. Assumption J2 is a standard minimum wealth condition. Typically, it will imply that $e_i(\omega) > 0$ a.e., $d\mu$ for all i.

Notice that we have allowed different after tax prices for different firms and consumers. This will allow for the interpretation of the model as one in which distinct consumers (or firms) are located in different regions with different tax rules. Notice that although we have followed the Arrow and Debreu tradition in terms of the specifications of firms (i.e., it is exogenous), this makes even less sense here than usual -- the presence of distinct taxes for different firms may result in natural tax arbitrage opportunities if the ψ_j are truly different. This can be handled by assuming that the Y_j are CRS and identical, or more weakly, every firm type is represented in every distinct location (i.e., for every distinct ψ_j).

Theorem: Under the assumptions listed above, an equilibrium exists.

3. Proof of the Theorem

Proof: Let $\mathscr{F}$ be the collection of finite dimensional subspaces of (L,L') of the form (F_1,F_2) where $F_1 \subset L$ and $F_2 \subset L'$, each of the F_i is finite dimensional, $e_i \in F_1$ for all i, $\chi_{Ai} \in F_1$ for all i, and $F_2 \cap P \neq \phi$, where $P = \{p \in L'_+ \mid \| p \| = 1\}$. Order $\mathscr{F}$ by component wise set inclusion.

For fixed $F \in \mathscr{F}$, consider the following pseudogame, $\Gamma(F)$:

There are I + J + 1 players--players 1, ..., I represent the consumers, players I+1, ...I+J represent the firms, and player I+J+1 is an 'auctioneer.'

Strategy arrays lie in $L^I \times L^J \times L' \equiv Z$. Thus, a strategy array is a $z = (x_1,...,x_I;y_1,...,y_J;\ p) = (x;y;p)$.

It follows from our assumptions that we can restrict attention to arrays such that

$\|x_i\| \leq M$, $\|y_j\| \leq M$, and $\|p\| = 1$, where M is as determined in assumption J1.

Henceforth, this restriction will be implicit.

Given an array z, the choice sets for the players are:

For the consumers (k = 1,...,I)-- $\gamma_k(z) = \{x \in F_1 | x \in X_i \text{ and } \phi_i(z)(x - e_i) \leq \mu_i + \sum_j \theta_{ij} \max(0,\psi_j(z)y_j\}$

For the firms (k = I + 1,...,I + J)-- $\gamma_k(z) = \{y \in F_1 | y \in Y_j\}$,

and for the auctioneer, $\gamma_k(z) = P \cap F_2$ (k = I + J + 2), where P is defined above.

Preferences for consumers (given z) and firms are defined in the obvious way. For the auctioneer, preferences are defined by $p(\sum_i x_i - \sum_i e_i - \sum_j y_j)$.

The first thing to check is that this game has an equilibrium for each $F \in \mathscr{F}$. Define δ_k to be the best reply correspondence for player k. We need to know that δ_k is non-empty, compact and convex valued and upper hemi continuous for all k. That δ_k is non-empty, compact and convex valued are straightforward to check.

That the δ_k are u.h.c., is a more delicate argument. As an example, consider the case of the firms. Note that since F is finite dimensional, it is metrizeable. Take a sequence, $(s_n,p_n) \to (s^*,p^*)$. Since ψ_j is continuous, it follows that $\psi_j(s_n,p_n) \to \psi_j(s^*,p^*)$ in the $\sigma(ba, L_\infty)$ topology. (Note that neither $\psi_j(s_n,p_n)$ nor $\psi_j(s^*,p^*)$ will necessarily lie in F_2.) Further, since $y_j^n \in F_1$, and F_1 is finite dimenisonal, $y_j^n \to y^*$ implies that $\|y_j^n - y^*\| \to 0$. Thus, it follows that $\psi_j(s_n,p_n)\, y_j^n \to \psi_j(s^*,p^*)y^*$. That is, preferences of the firms are continuous. That the best response correspondences of the firms are u.h.c. follows immediately from this fact. A similar argument can be used to see that the best response correspondences of the consumers are also u.h.c. (Of course, assumption J2 plays a crucial role in this argument.)

It follows that there is an equilibrium for each $F \in \mathscr{F}$. Denote this equilibrium by x_i^F, y_j^F, p^F. Let $s^F=(x^F,y^F)$. It follows from the definition of equilibrium that:

(1) x_i^F maximizes $\succeq_i(s^F)$ on

$\beta_i^F(s^F,p^F) = \{x \in X_i \cap F_1 | \, \phi_i(s^F,p^F)(x - e_i) \leq \mu_i(s^F,p^F) + \sum_j \theta_{ij} \psi_j(s^F,p^F)y_j^F\}$.

(2) y_j^F maximizes $\psi_j(s^F,p^F)y$ over $y \in Y_j \cap F_1$.

(3) p^F maximizes $p(\sum_i x_i^F - \sum_i e_i - \sum_j y_j^F)$ over $p \in P \cap F_2$.

(Note that since $\psi_j(s^F,p^F)y_j^F \geq \psi_j(s^F,p^F)\, 0 = 0$, it is $\psi_j(s^F,p^F)y_j^F$ that enters the consumers budget constraints in equilibrium.)

Adding up the individual budget constraints, we have, $\sum_i \phi_i^F(x_i^F - e_i) \leq \sum_i \mu_i^F + \sum_i\sum_j \theta_{ij}\, \psi_j^F\, y_j^F$. Hence, using assumption G3, we have

$\sum_i \phi_i^F(x_i^F - e_i) \leq \sum_i (\phi_i^F - p^F)(x_i^F - e_i) + \sum_j (p^F - \psi_j^F)y_j^F + \sum_i\sum_j \theta_{ij}\, \psi_j^F\, y_j^F$.

Hence, $p^F \sum_i x_i^F \leq p^F \sum_i e_i + p^F \sum_j y_j^F$, and so, $p^F(\sum_i x_i^F - \sum_i e_i - \sum_j y_j^F) \leq 0$. (It need not be true that $(\sum_i x_i^F - \sum_i e_i - \sum_j y_j^F) \leq 0$, however.)

By construction, it follows that the x_i^F, y_j^F and p^F lie in $\sigma(L_\infty,L_1)$, $\sigma(L_\infty,L_1)$, and $\sigma(ba,L_\infty)$ compact subsets, respectively. It follows that there is a directed set, $(D,\geq)$ and a subnet, $F(d)$, such that $x_i^{F(d)} \to x_i^*$ in the $\sigma(L_\infty,L_1)$ topology, $y_j^{F(d)} \to y_j^*$ in the $\sigma(L_\infty,L_1)$ topology, and $p^{F(d)} \to p^*$ in the $\sigma(ba,L_\infty)$ topology.

Since $p^F \geq 0$ and $\| p^F \|=1$, it follows that $p^*\chi_\Omega = \lim p^F \chi_\Omega = 1$, and hence, $p^* \neq 0$. (In fact, $\| p^* \|=1$.)

Let $s^* = (x_1^*,...,x_I^*;y_1^*,...,y_J^*)$, $p_i^* = \phi_i(s^*,p^*)$, and $p_j^* = \psi_j(s^*,p^*)$. It follows from assumption G2 that $\phi_i(s^{F(d)},p^{F(d)}) \to p_i^*$ in the $\sigma(ba,L_\infty)$ topology, $\psi_j(s^{F(d)},p^{F(d)}) \to p_j^*$ in the $\sigma(ba,L_\infty)$ topology and $\mu_i(s^{F(d)},p^{F(d)}) \to \mu_i^*$.

We will show that the x_i^*, y_j^* and p^* make up an equilibrium.

First, note that from assumptions H1 and F2, it follows that $x_i^* \in X_i$ and $y_j^* \in Y_j$.

Second, we will show that $\sum_i x_i^* \leq \sum_i e_i + \sum_j y_j^*$. Suppose that this is not true. Let h^+ and h^- denote the positive and negative parts of $\sum_i x_i^* - \sum_i e_i - \sum_j y_j^*$, respectively. Choose $p^\sim \in P$ so that $p^\sim h^+ > 0$ and $p^\sim h^- = 0$ and d^* so that $d \geq d^*$ implies that $p^\sim \in F_2(d)$. For example, let $A_\epsilon = \{\omega \,|\, h^+(\omega) > \epsilon\}$. By assumption, $\mu(A_\epsilon) > 0$ for some $\epsilon > 0$. Let $p^\sim = \chi_{A\epsilon} / \mu(A_\epsilon)$ for this ϵ. Since $p^F(\sum_i x_i^F - \sum_i e_i - \sum_j y_j^F) \leq 0$ for all F, $p^\sim \in F_2(d)$ for $d \geq d^*$ and $p^{F(d)}$ is maximal, it follows that

$$0 \geq p^{F(d)}(\textstyle\sum_i x_i^{F(d)} - \sum_i e_i - \sum_j y_j^{F(d)}) \geq p^\sim (\sum_i x_i^{F(d)} - \sum_i e_i - \sum_j y_j^{F(d)}).$$

Now, the right hand side of this inequality converges to $p^\sim(\sum_i x_i^* - \sum_i e_i - \sum_j y_j^*)$ which by construction is $p^\sim h^+ > 0$. This contradiction establishes the result.

Note that it follows from this and assumption J2 that $\| x_i^* \| < M$ for all i and $\| y_j^* \| < M$ for all j.

Let $b_j^d = \psi_j^{F(d)} y_j^{F(d)}$, and $a_i^d = \phi_i^{F(d)} x_i^{F(d)}$. We can, without loss of generality, assume that $a_i^d \to a_i^*$ and $b_j^d \to b_j^*$.

Claim 1. $p_j^* y_j^* \leq b_j^*$.

Suppose that the claim is false. Choose d^* so that $d \geq d^*$ implies that $y_j^* \in F_1(d)$. It follows that for $d \geq d^*$, $\psi_j^{F(d)} y_j^{F(d)} \geq \psi_j^{F(d)} y_j^*$. But, $\psi_j^{F(d)} y_j^{F(d)} \to b_j^*$ and $\psi_j^{F(d)} y_j^* \to p_j^* y_j^* > b_j^*$. This contradiction establishes the claim.

Claim 2. $p_i^* x_i^* \geq a_i^*$.

Suppose to the contrary that $p_i^* x_i^* < a_i^*$. It follows that $a_i^* > 0$ since $p_i^* \geq 0$, and $x_i^* \geq 0$. Further, since $\sum_i x_i^* \leq \sum_i e_i + \sum_j y_j^*$, it follows that $\|x_i^*\| < M$. By assumption H3, there is an $x_i \in X_i$ with $x_i \succ_i (s^*)\, x_i^*$ and $p_i^* x_i < a_i^*$ and $\|x_i\| \leq M$. Choose d^* large enough so that $d \geq d^*$ implies that $x_i \in F_1(d)$, and let $x_i^{\sim F(d)} = \alpha^d x_i$ where $\alpha^d = \min((\alpha_i^d)/(\phi_i^{F(d)} x_i), 1)$ and $\alpha_i^d = \phi_i^{F(d)} e_i + \mu_i^{F(d)} + \Sigma_j \theta_{ij} \psi_j^{F(d)} y_j^{F(d)}$. Then, $\phi_i^{F(d)} x_i^{\sim F(d)} = \alpha^d \phi_i^{F(d)} x_i = \min(\alpha_i^d, \phi_i^{F(d)} x_i) \leq \alpha_i^d$ so that $x_i^{\sim F(d)}$ is affordable at prices $\phi_i^{F(d)}$. Moreover, since $X_i \cap F_1$ is convex and $x_i^{\sim F(d)}$ is a convex combination of 0 and x_i both of which are in X_i, it is in $X_i \cap F_1$. Finally, by construction, $\|x_i^{\sim F(d)}\| < M$. Thus, $x_i^{\sim F(d)}$ is a feasible choice for i for each d.

Note that $\phi_i^{F(d)} x_i \to p_i^* x_i$ and $\alpha_i^d \to \alpha_i^* \geq a_i^* > 0$, so that $\alpha^d \to 1$. Thus, it follows that $\|x_i^{\sim F(d)} - x_i\| \to 0$. Since $x_i^{F(d)} \to x_i^*$ in the $\sigma(L_\infty, L_1)$ topology and $s^{F(d)} \to s^*$ in the $(\sigma(L_\infty, L_1))^{I+J}$ topology and $x_i \succ_i (s^*)\, x_i^*$, it follows from assumption H4 that for large d, $x_i^{\sim F(d)} \succ_i (s^{F(d)})\, x_i^{F(d)}$. This contradicts the maximality of $x_i^{F(d)}$, and completes the proof of the claim.

Claim 3. $p_j^* y_j^* = b_j^*$ for all j, and $p_i^* x_i^* = a_i^*$ for all i.

We will show that $p_i^* x_i^* = a_i^*$ for all i, the proof that $p_j^* y_j^* = b_j^*$ for all j is similar. Assume to the contrary that $p_i^* x_i^* > a_i^*$ for some i, and note that since $\|x_i^*\| < M$, it follows (use assumption H3 for this) that for large d that

$$\phi_i^{F(d)} x_i^{F(d)} = a_i^{F(d)} = \mu_i^{F(d)} + \phi_i^{F(d)} e_i + \Sigma_j \theta_{ij} \psi_j^{F(d)} y_j^{F(d)} \text{ for all i.}$$

Thus,

$$\Sigma_i p_i^* x_i^* > \Sigma_i a_i^* = \lim \Sigma_i a_i^d = \lim [\, \Sigma_i [\mu_i^{F(d)} + \phi_i^{F(d)} e_i + \Sigma_j \theta_{ij} \psi_j^{F(d)} y_j^{F(d)}]]$$

$$\geq \Sigma_i \mu_i^* + \Sigma_i p_i^* e_i + \Sigma_j p_j^* y_j^*$$

where we have used assumption G2 and claim 1.

Thus, $\Sigma_i p_i^* x_i^* - \Sigma_i p_i^* e_i - \Sigma_j p_j^* y_j^* > \Sigma_i \mu_i^*$.

Hence,

$$p^*(\Sigma_i x_i^* - \Sigma_i e_i - \Sigma_j y_j^*) =$$

$$\Sigma_i (p^* - p_i^*) x_i^* + \Sigma_i p_i^* x_i^* - \Sigma_i (p^* - p_i^*) e_i - \Sigma_i p_i^* e_i - \Sigma_j (p^* - p_j^*) y_j^* - \Sigma_j p_j^* y_j^* =$$

$$\Sigma_i (p^* - p_i^*)(x_i^* - e_i) + \Sigma_j (p_j^* - p^*) y_j^* + \Sigma_i p_i^* (x_i^* - e_i) - \Sigma_j p_j^* y_j^* =$$

$$- \Sigma_i \mu_i^* + \Sigma_i p_i^* (x_i^* - e_i) - \Sigma_j p_j^* y_j^* > 0, \text{ where we have used assumption G3.}$$

This is impossible however, since $p^* \geq 0$, and $\Sigma_i x_i^* - \Sigma_i e_i - \Sigma_j y_j^* \leq 0$. This contradiction establishes the result.

It follows that $p_i^* x_i^* = a_i^* = \lim a_i^{F(d)} = \lim \phi_i^{F(d)} x_i^{F(d)} \leq$

$\lim [\, \phi_i^{F(d)} e_i + \mu_i^{F(d)} + \Sigma_j \theta_{ij} \psi_j^{F(d)} y_j^{F(d)}] = p_i^* e_i + \mu_i^* + \Sigma_j \theta_{ij} p_j^* y_j^*$.

That is, x_i^* is affordable at prices p_i^* for i. Suppose that $x_i \succ_i (s^*) x_i^*$. An argument similar to that in claim 2 shows that it must be that $p_i^* x_i > p_i^* e_i + \mu_i^* + \Sigma_j \theta_{ij} p_j^* y_j^*$ and hence it follows that x_i^* is utility maximizing for i. Similarly, the fact that y_j^* maximizes profits for j follows from an argument similar to that used to prove claim 1. That the government's budget balances follows immediately from assumption G3. Finally, note that, as above, $p_i^* x_i = p_i^* e_i + \mu_i^* + \Sigma_j \theta_{ij} p_j^* y_j^*$, so that, $p^* [\, \Sigma_i\ x_i^* - \Sigma_i e_i - \Sigma_j y_j^*] = \Sigma_i p_i^* x_i^* - \Sigma_i p_i^* e_i - \Sigma_j p_j^* y_j^* - \Sigma \mu_i^* =$ 0.

This completes the proof.

4. <u>Possible Extensions</u>

In this section, we will briefly discuss some possible extensions of the result and some of the difficulties that they pose.

A. <u>Externalities</u> The most obvious extension is to include externalities in the formulation of the model. There are now many examples in the literature which feature infinite horizon models with externalities. Examples include Romer (1986), Lucas (1988) and (1989), Barro (1988) and Stokey (1988) and (1990).

Formally, externalities require allowing consumption sets, X_i and production sets, Y_j to depend on the state, s. If this dependence is weak to norm lower hemi continuous and weak to weak upper hemi continuous, it is straightforward to check that most the proof can be adapted.

The difficulty is in the construction of the approximating pseudogames. The way the construction is currently done is to take the choice set in the infinite dimensional model and use the intersection of this with the finite dimensional subspaces as the choice sets in the approximation. The problem with this is that the intersection of an lhc correspondence with a fixed set is no longer necessarily lhc. This loses the existence result in the finite dimensional approximation and hence the proof breaks down.

An alternative possibility is to use projections of the infinite dimensional choice sets onto the finite dimensional approximations. This restores the lhc property of the finite dimensional choice sets and gives existence in the approximating games. The difficulty is then guaranteeing that the projections have sufficiently strong properties so that the limiting arguments are still valid. We have not accomplished this as of yet.

A more serious problem lies in the fact that in many of the examples of interest, the choice sets are not norm to weak lhc in any case. Because of this , a different approach may need to be adopted.

B. <u>More General Spaces</u> A natural extension of this work is to include more general choice spaces as in the work of Mas Colell (1986) and Aliprantis, Brown and Burkinshaw (1989). In particular, it is of interest to have an existence result which would allow an infinite horizon model with a continuum of goods at each date as in Stokey (1988) and (1990).

As can be checked, much of the argument presented here goes through in a straightforward way to the lattice theoretic framework now used in this literature. The difficulty lies with one step--showing tha the limiting price vector is non-trivial (i.e., not zero). It is well known that this problem cannot be solved without further complications (since it won't work in even the undistorted case). This is due to the non-compactness (weak) of the surface of the unit sphere in general linear spaces. The solution to this problem should be in the spirit of Mas Colell's properness restriction on preferences.

Thus, rather that having the auctioneer make choices from $F_2 \cap P$, where $P = \{ x' \in L'_+ \mid \| x \| = 1\}$, we should have him make choices in $F_2 \cap C$, where $C = \{ x' \in C^* \mid \| x \| = 1\}$ and C^* is the polar of the properness cone. This guarantees that the equilibrium prices have a non-trivial limit. The difficulty will then be in showing that C^* is large enough to guarantee that the limiting allocation is feasible. Presumably, this will involve some additional restrictions on consumption sets as well as the restriction to proper preferences (as is common in this literature.)

C. Price Representations The result given here gives the existence of an equilibrium in which prices lie in $ba(\Omega,\mathscr{F},\mu)$, the norm dual of L_∞. It is natural to ask, as in Bewley (1972) and Prescott and Lucas (1972), that prices lie in L_1 instead. Presumably, the arguments given by those authors can be used in this setting as well. Note, however, that this will only give that the individual prices lie in L_1. Some additional assumption (restricting the forms of the ϕ_i and ψ_j) will presumably have to be introduced to ensure that the market prices themselves will lie in L_1.

D. Correspondences for the Government Decision Variables It is quite likely that the result can be extended to allow for the government choice variables to be correspondences. Although the current formulation allows for some forms of non-linear taxes (since ϕ_i is allowed to depend on x_i -- see the next section) it does not allow for 'kinks' in the budget sets of consumers. These arise naturally in applications when the schedule of marginal tax rates is not continuous as a function of the action of the agents. For this reason, it would be useful to extend the result to situations in which the government choice variables are convex, compact-valued and continuous correspondences.

Section 5: Examples

A. Linear and Non-Linear Taxes.

B. Government Spending as private firm that is subsidized.

C. Preference formulation as preferences over states-- with $\succ_i = \succ_i (x_i | s_{-i})$.

References

Aliprantis, C. and D. Brown, (1983), "Equilibria in Markets with a Reisz Space of Commodities," *Journal of Mathematical Economics*, 11, 189-207.

Aliprantis, C., D. Brown and O. Burkinshaw, (1987), "Edgeworth Equilibria," *Econometrica*, 55, 1109-1137.

Aliprantis, C., D. Brown and O. Burkinshaw, (198), *Existence and Optimality of Competitive Equilibria*, New York: Springer-Verlag.

Araujo, A., (1985), "Lack of Equilibria in Economies with Infinitely Many commodities: The Need for Impatience," *Econometrica*, 53, 455-462.

Arrow, K. and G. Debreu, (1954), "Existence of Equilibrium for a Competitive Economy," *Econometrica*, 22, 265-290.

Barro, R., (1988), "Government Spending in a Simple Model of Endogenous Growth," ****

Bewley, T., (1972), "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory*, 43, 514-540.

Brown, D. and L. Lewis, (1981), "Myopic Economic Agents," *Econometrica*, 49, 359-368.

Chamley, C., (1986), "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54, 607-622.

Debreu, G., (1954), "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences*, 40, 588-592.

Duffie, D., (1986), "Competitive Equilibria in General Choice Spaces," *Journal of Mathematical Economics*, 15, 1-25.

Dunford, N. and J. Schwartz, (1988), *Linear Operators Part I: General Theory*, New York: Wiley.

Jones, L., (1983), "Special Problems Arising in the Study of Economies with Infinitely Many Commodities," in *Models of Economic Dynamics*, H. Sonnenschein, Ed., New York: Springer-Verlag.

Jones, L., (1984), "A Competitive Model of Commodity Differentiation," *Econometrica*, 52, 507-530.

Jones, L., (1987), "The Efficiency of Monopolistic Competition in Large Economies: Commodity Differentiation with Gross Substitutes," *Journal of Economic Theory*, 41, 356-391.

Jones, L., (1987), "Existence of Equilibria with Infinitely Many Commodities: Banach Lattices Revisited," *Journal of Mathematical Econoimcs*, 16, 89-104.

Jones, L. and R. Manuelli, (1990), "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy*, 98, 1008-1038.

Jones, L. and R. Manuelli, (1992), "Finite Lifetimes and Growth," *Journal of Economic Theory*, Vol. 58, 171-197.

Jones, L., R. Manuelli and P. Rossi, (1993), " Optimal Taxation in Models of Endogenous Growth," *Journal of Political Economy*, Vol. 101, 485-517.

Kehoe, T., D. Levine, and P. Romer, (1989), "Determinacy of Equilibria in Dynamic Models with Finitely Many Consumers," *Journal of Economic Theory*, Vol. 50, 1-21.

Kehoe, T., D. Levine, and P. Romer, (1988), "Characterizing Equilbria of Models with Externalities and Taxes as Solutions to Optimization Problems," University of Minnesota working paper.

King, R. and S. Rebelo, (1990), "Public Policy and Economic Growth: Developing Neoclassical Implications," *Journal of Political Economy*, Vol. 98, S126-S150.

Lucas, R., (1988), "On the Mechanics of Development," *Journal of Monetary Economics*, 22, 3-42.

Lucas, R., (1990), "Supply Side Economics: An Analytical Review," *Oxford Economics Papers*, Vol. 42, 293-316.

Magill, M., (1981), "An Equilibrium Existence Theorem," *Journal of Mathematical Analysis and Applications*, 84, 162-169.

Mas-Colell, A., (1975), "A Model of Equilibrium with Differentiated Commodities," *Journal of Mathematical Economics*, 2, 263-296.

Mas-Colell, A., (1986), "The Price Equilibrium Existence Problem in Topological Vector Lattices," *Econometrica*, 54, 1039-1054.

Mas-Colell, A. and S. Richard, (1991), "A New Approach to the Existence of Equilibria in Vector Lattices," *Journal of Economic Theory*, Vol. 53, 1-11.

Mas-Colell, A. and W. Zame, (1990), "Equilibrium Theory in Infinite Dimensional Spaces," in *The Handbook of Mathematical Economics, Vol. IV*, W. Hildenbrand and H. Sonnenschein, Eds., Amsterdam: North Holland.

McKenzie, L., (1955), "Competitive Equilibrium with Dependent Consumer Preferences," in National Bureau of Standards and Department or the Air Force, *The Second Symposium on Linear*

Programming, Washington, D.C.

Prescott, E. and R. Lucas, (1972), "A Note on Price Systems in Infinite Dimensional Spaces," *International Economic Review*, 13, 416-422.

Rebelo, S., (1991), "Long Run Policy Analysis and Long Run Growth," *Journal of Political Economy*, Vol. 99, 500-521.

Richard, S., (1989), "A New Approach to Production Equilibria in Vector Lattices," *Journal of Mathematical Economics*, 18, 41-56.

Romer, P., (1986), "Increasing Returns and Long Run Growth," *Journal of Political Economy*, Vol. 94, 1002-1037.

Shafer, W. and H. Sonnenschein, (1976), "Equilibrium with Externalities, Commodity Taxation and Lump Sum Transfers," *International Economic Review*, 17, 601-611.

Stokey, N., (1988), "Learning By Doing and the Introduction of New Goods," *Journal of Political Economy*, 96, 701-717.

Stokey, N., (1991), "Human Capital, Product Quality and Growth," *Quarterly Journal of Economics*, Vol. CVI, 587-616.

Toussaint, S., (1985), "On the Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory*, 13, 98-115.

Yannelis, N. C., (1985), "On a Market Equilibrium Theorem with an Infinite Number of Commodities," *Journal of Mathematical Analysis and Applications*, 108, 595-599.

Zame, W., (1987), "Competitive Equilibria in Production Economies with an Infinite Dimensional Commodity Space," *Econometrica*, 55, 1075-1108.