Complements versus Substitutes and Trends in Fertility Choice in Dynastic Models
Additional Appendix

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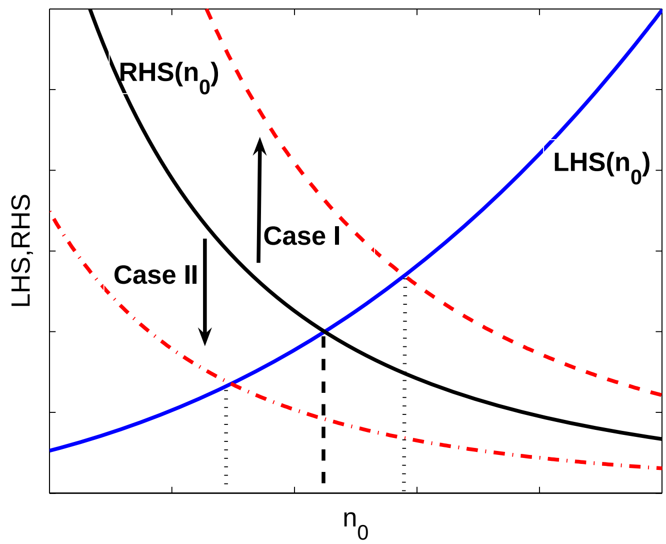
Contents

B Theoretical Additions

B.1 Illustration of equation (1)

Figure B.1: Comparative Statics with respect to U_1



B.2 The log utility case

Here we show that the utility function used in Lucas (2002), a special case of Razin and Ben-Zion (1975)'s utility function using log utility, is also a special case of the Becker and Barro (1988) and Barro and Becker (1989) utility function.[1] Define ψ such that $\psi \equiv \frac{\eta}{1-\sigma}$, where $\psi \geq 1$ under both, AI and AII. Then, we can rewrite the aggregate utility function in equation (3) as:

(1) $$U_0 = \sum_{t=0}^{\infty} \beta^t g(N_t) u\left[\frac{C_t}{N_t}\right] = \sum_{t=0}^{\infty} \beta^t N_t^{\psi(1-\sigma)} \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t \frac{(N_t^{\psi} c_t)^{1-\sigma}}{1-\sigma}$$

Next, consider the monotone transformation given by:

(2) $$\hat{U}_0 = \sum_{t=0}^{\infty} \beta^t \frac{(N_t^{\psi} c_t)^{1-\sigma} - 1}{1-\sigma}$$

[1] For the AI parameter configuration with $\pi = 0$, proofs are also available in Lucas (2002) and Bar and Leukhina (2007).

Taking the limit as $\sigma \to 1$ gives:

$$\begin{aligned}\lim_{\sigma\to 1} \hat{U}_0 &= \lim_{\sigma\to 1} \sum_{t=0}^{\infty} \beta^t \frac{(N_t^{\psi} c_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t \lim_{\sigma\to 1} \frac{(N_t^{\psi} c_t)^{1-\sigma} - 1}{1-\sigma} \\ &= \sum_{t=0}^{\infty} \beta^t \log(N_t^{\psi} c_t) = \sum_{t=0}^{\infty} \beta^t (\log c_t + \psi \log N_t).\end{aligned}$$

Recall that $N_t = N_0 \Pi_{k=0}^{t-1}(\pi + \pi_s n_{b,k})$ and we normalize $N_0 = 1$. Thus:

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t \log N_t &= \beta \sum_{t=0}^{\infty} \beta^t \log(\pi + \pi_s n_{b,0}) + \beta^2 \sum_{t=0}^{\infty} \beta^t \log(\pi + \pi_s n_{b,1}) + \beta^3 \sum_{t=0}^{\infty} \beta^t \log(\pi + \pi_s n_{b,2}) + ... \\ &= \frac{\beta}{1-\beta} \log(\pi + \pi_s n_{b,0}) + \frac{\beta^2}{1-\beta} \log(\pi + \pi_s n_{b,1}) + \frac{\beta^3}{1-\beta} \log(\pi + \pi_s n_{b,2}) + ... \\ &= \frac{\beta}{1-\beta} \sum_{t=0}^{\infty} \beta^t \log(\pi + \pi_s n_{b,t})\end{aligned}$$

Hence, we can rewrite the limit above as:

$$\lim_{\sigma\to 1} \hat{U}_0 = \sum_{t=0}^{\infty} \beta^t \left(\log c_t + \frac{\beta\psi}{1-\beta} \log(\pi + \pi_s n_{b,t})\right).$$

Let $\tilde{U}_0 \equiv \lim_{\sigma\to 1} \hat{U}_0$ and $\phi \equiv \frac{\beta\psi}{1-\beta}\left(= \frac{\beta\eta}{(1-\beta)(1-\sigma)}\right)$. Then

$$\tilde{U}_t = \log c_t + \phi \log(\pi + \pi_s n_{b,t}) + \beta \tilde{U}_{t+1}$$

and for $\pi = 0$ and $\pi_s = 1$

$$\tilde{U}_t = \log c_t + \phi \log n_t + \beta \tilde{U}_{t+1}$$

The utility function $\tilde{U}$ can then be interpreted as the period t household's utility function and is the one used in Lucas's book.

Notice that since $\psi \geq 1$, we need

$$\phi(1-\beta) \geq \beta$$

if we are working with the log case.

B.3 Altruism toward unborn children

This section concerns the utility of unborn children and altruism toward them. That is, one interpretation of the fact that children that are not born do not enter the calculation of time-t utility is that they are assigned $U = 0$. This interpretation is fine when $\sigma < 1$, but causes difficulty when $\sigma > 1$. When utility is negative we can assume that unborn children also get negative utility, and even less than that received by born children ($\bar{u}_{unborn} < \bar{u}_{born}$) and that parents are altruistic towards these children too. To see this, let the utility of the parent be given by:

$$U_t = u(c_t) + g(n_t)\bar{u}_{born} + h(n_p - n_t)\bar{u}_{unborn}$$

where n_p is the number of potential children and n_t is the number of children born. As can be seen from this, one interpretation of the preferences we use is that $h = 0$, not that $\bar{u}_{unborn} = 0$. Under this interpretation, parents are only weakly altruistic toward their children however.

Strict altruism with respect to the level of utilities holding the number of births fixed requires $g(\cdot) > 0$ and $h(\cdot) > 0$. Since $\bar{u}_{unborn} < \bar{u}_{born}$, strict altruism also requires that increasing n_t strictly increases U_t. This can be written as:

$$\tfrac{d}{dn}\left[g(n)\bar{u}_{born} + h(n_p - n)\bar{u}_{unborn}\right] > 0.$$

This condition is not necessarily satisfied. In particular, it is important that the marginal utility from the unborn increases more slowly than the marginal utility from born children decreases as children move from the unborn to the born state. To gain some intuition, consider the case where utilities are isoelastic and the same—$g(n) = h(n) = n^{\eta}$. Then the condition above becomes:

$$1 < \left(\tfrac{n_p}{n_t} - 1\right)\left[\tfrac{\bar{u}_{unborn}}{\bar{u}_{born}}\right]^{1/(\eta-1)}.$$

One simple way to satisfy this condition is to assume that the number of children that can feasibly be born, $\bar{n}_t = \frac{w_t}{\theta_t}$, is small relative to the number of potential children, n_p, i.e., as $\frac{\bar{n}_t}{n_p} \to 0$, the additive term representing the unborn in parent's utility is more or less independent of parent's choices. In this case, the decisions made are approximately the same as the ones made with the utility functions used throughout the paper.

B.4 Solution of the model in Section 6 (with physical capital)

The representative dynasty problem we are interested in is given by

$$Max_{\{C_t,N_t,K_t\}} \qquad U_0(\{C_t,N_t,K_t\}) = \textstyle\sum_{t=0}^{\infty} \beta^t N_t^{\eta+\sigma-1}\frac{C_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t.} \quad C_t + \theta_{st}N_{st} + X_t \le w_t N_t + r_t K_t$$

$$K_{t+1} \le (1-\delta)K_t + X_t,$$

$$N_{t+1} \le \pi N_t + N_{st}$$

$$(N_0, K_0) \text{ given.}$$

Note that this problem is well defined under both assumption AI and AII derived in Section 2 as long as $\eta \neq 1-\sigma$. The first order condition with respect to K_{t+1} and N_{t+1} together with the budget constraint, boil down to the following system of equations governing the solution to this (partial equilibrium) problem:

$$\gamma_{ct}^{\sigma}\gamma_{Nt}^{1-\eta} = \beta(r_{t+1} + 1 - \delta)$$

$$\theta_t(r_{t+1} + 1 - \delta) = \left[\tfrac{(\eta+\sigma-1)}{(1-\sigma)}\tfrac{C_{t+1}}{N_{t+1}} + [w_{t+1} + \theta_{t+1}\pi]\right]$$

$$\tfrac{C_t}{N_t} + \theta_t\gamma_{Nt} + \tfrac{K_{t+1}}{N_{t+1}}\gamma_{Nt} = [w_t + \pi\theta_t] + (r_t + 1 - \delta)\tfrac{K_t}{N_t}.$$

To ensure interiority in partial equilibrium, we have to (1) either rule out $\eta = 1-\sigma$ or, (2) if $\eta = 1-\sigma$, make the necessary parameter assumptions so that rates of returns to children and capital are equalized. In general equilibrium, prices will adjust to achieve this. To close the model, wages and interest rates are determined in equilibrium by a firm hiring labor and capital to maximize profits with a constant returns to scale—Cobb-Douglas—production function, $F(K_t, \gamma^t N_t) = AK_t^{\alpha}(\gamma^t N_t)^{1-\alpha}$. That is,

$$r_t = F_K(K_t, \gamma^t N_t)$$

$$w_t = F_N(K_t, \gamma^t N_t).$$

On a balanced growth path, we have $\gamma_{c,t} = \gamma_c = \gamma$, $\gamma_{N,t} = \gamma_N$, $\gamma_C = \gamma_K = \gamma\gamma_N$, wages grow at γ and interest rates are constant. Denoting detrended variables by $\widehat{x}$ the above equations become:

$$\gamma_N = \tfrac{[\beta(r+1-\delta)]^{\frac{1}{1-\eta}}}{\gamma^{\frac{\sigma}{1-\eta}}}$$

$$\widehat{c}^* = [\widehat{w} + \pi\theta_s] + \gamma\,(r + 1 - \delta)\,\widehat{k}^* - \theta_s\gamma_N - \widehat{k}^*\gamma_N\gamma$$

$$r + 1 - \delta = \tfrac{\gamma}{\theta_s}\tfrac{(\eta+\sigma-1)}{(1-\sigma)}\widehat{c}^* + \tfrac{\gamma}{\theta_s}\widehat{w} + \pi\gamma$$

$$r = \alpha A\widehat{k}^{*\alpha-1}$$

$$\widehat{w} = (1-\alpha)A\widehat{k}^{*\alpha}.$$

These five equations together with initial conditions completely characterize the equilibrium path.

C Quantitative Additions

C.1 Auxiliary figures

C.1.1 Historical experiment in Section 5 (labor income only)

Here we give figures—like the one given in the main text for CBR in Section 5—for the other measures of fertility we have discussed, $CTFR$ and γ_N. They show a similar pattern overall with the model capturing significant fractions of the overall changes seen in the data—about half (see also Table 2).

Figure B.2: The U.S. experience from 1800 to 1990, $CTFR$ and TFR

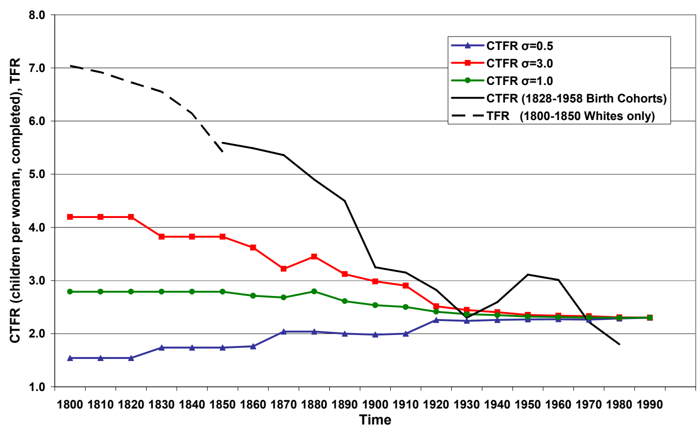
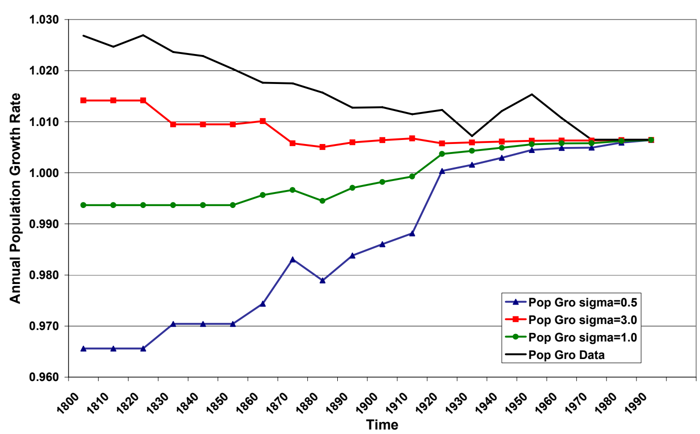


Figure B.3: The U.S. experience from 1800 to 1990, annual population growth rate



Next, we plot the decomposition exercise in Section 5 of the overall changes in these three measures into the three components, changes in γ only, changes in π only and changes in π_s only (see also Table 3).

Figure B.4: Decomposition, $CTFR$

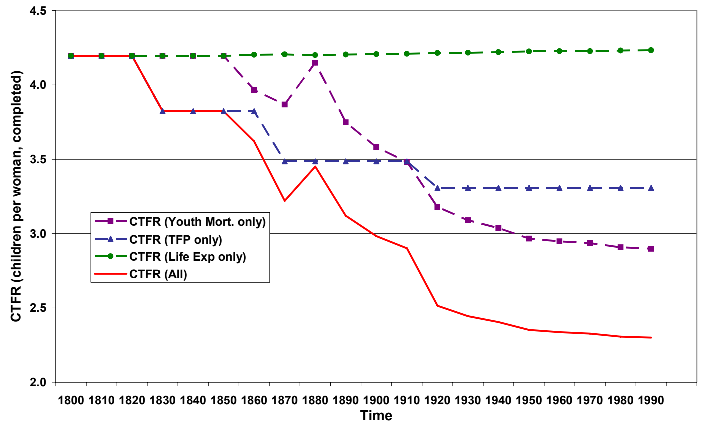


Figure B.5: Decomposition, CBR

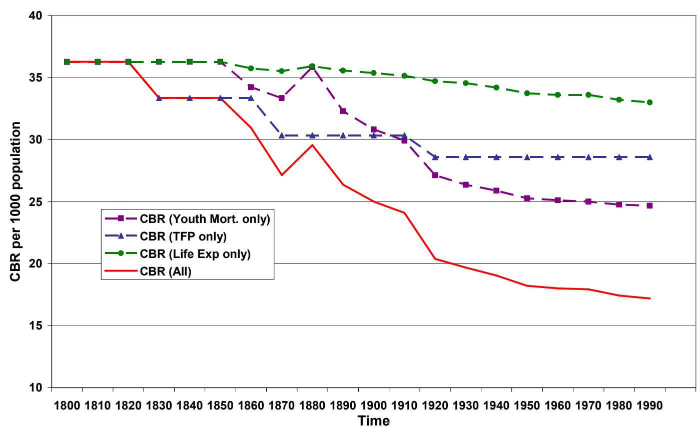
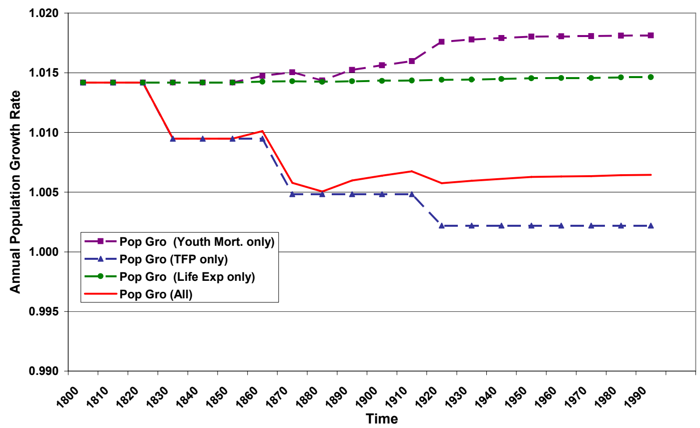


Figure B.6: Decomposition, annual population growth rate



C.1.2 Historical experiment in Section 6 (with physical capital)

Here, we show figures of the calculations for the history of U.S. fertility using the version of the model including physical capital discussed in Section 6.2. We include only the calculations for $\sigma = 3.0$ and $\sigma = 0.5$. We find that the results are very similar as described in the main text. The decomposition confirms our previous finding about timing of events.

Figure B.7: CBR: Model with K vs. Data $\sigma = 3$

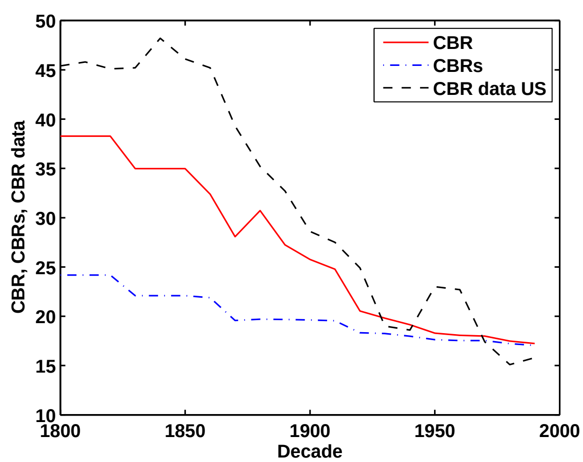


Figure B.8: CBR: Model with K vs. Data $\sigma = 0.5$

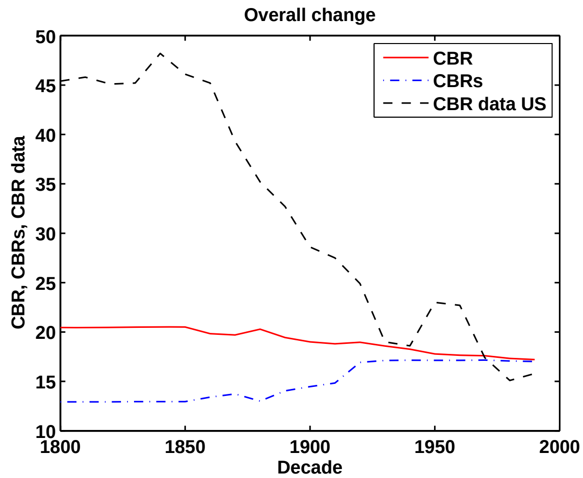


Table B.1: Decomposition: Productivity (γ) vs. Mortality (π_s) and Longevity (π) for the model with physical capital (Section 6)

$\sigma = 3$	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.027	1.016	1.006	45.4	35.2	15.8	7.04	4.90	1.97
Productivity (γ)	1.018	1.007	1.004	38.3	31.5	29.5	4.48	3.62	3.40
Mortality (π_s)	1.018	1.018	1.020	38.3	37.8	25.4	4.48	4.43	3.02
Longevity (π)	1.018	1.018	1.018	38.3	37.9	35.2	4.48	4.48	4.50

Figure B.9: CBR: Decomposition, Model with K, $\sigma = 3$

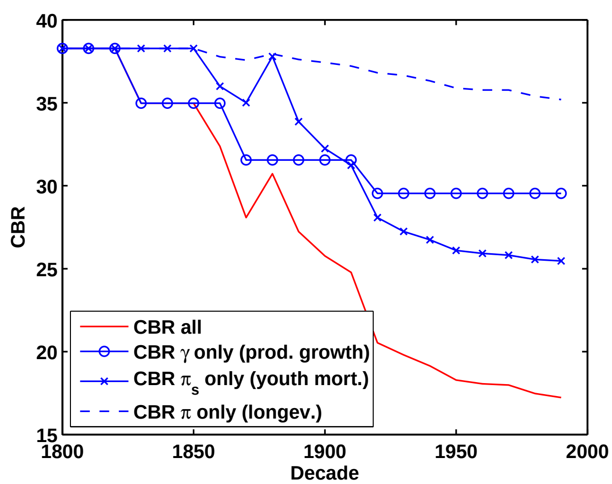
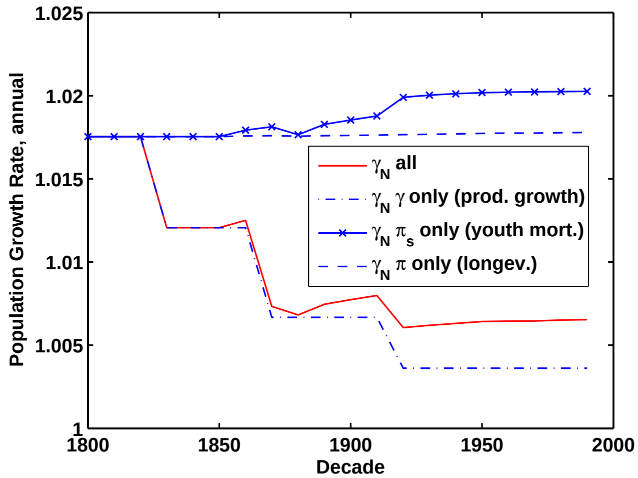


Figure B.10: Population Growth Rate: Decomposition, Model with K, $\sigma = 3$



C.2 Sensitivity analysis

C.2.1 Simple quantitative comparative statics

Here we give some simple comparative statics of our measures of fertility for various values of σ for the range of relevant parameter values. We calibrate θ_s/w to match $\gamma_N = 1$ using $\gamma = 1.02$, $\pi = 1.0$, $\pi_s = 1.0$. Table B.2 shows the results of changing only γ from $\gamma = 1.00$ to $\gamma = 1.02$ while Table B.3 does the same for $\pi_s = 0.6$ to $\pi_s = 1$. Finally, Table B.4 examines changes in π corresponding to an expected lifetime at age twenty ranging from 25 to 45 years.

Table B.2: Changing productivity growth

Productivity Growth	$\sigma = 0.5$		$\sigma = 1.0$		$\sigma = 3.0$	
γ_{ann}	$\gamma_{N,ann}$	$CBR_{s,ann}$	$\gamma_{N,ann}$	$CBR_{s,ann}$	$\gamma_{N,ann}$	$CBR_{s,ann}$
1.00	0.98	5.25	1.00	15.38	1.013	21.37
1.01	0.99	10.49	1.00	15.38	1.007	18.45
1.02	1.00	15.38	1.00	15.38	1.00	15.38

Table B.3: Changing survival to adulthood (STA)

	π_{cy} (STA)	θ_s/w	$\gamma_{N,ann}$	CBR_{ann}	$CBR_{s,ann}$
	0.6	0.85	0.993	19.68	11.81
$\sigma = 0.5$	0.8	0.80	0.997	17.51	14.01
	1.0	0.77	1.00	15.38	15.38
	0.6	0.65	0.996	22.46	13.48
$\sigma = 1.0$	0.8	0.61	0.998	18.31	14.65
	1.0	0.59	1.00	15.38	15.38
	0.6	0.26	0.998	24.16	14.50
$\sigma = 3.0$	0.8	0.24	0.999	18.80	15.04
	1.0	0.23	1.00	15.38	15.38

Table B.4: Changing expected lifetime (EL)

	π_{ann}	$\frac{T}{1-\pi}$ (EL)	$\gamma_{N,ann}$	CBR_s
	0.923	25	0.979	15.60
$\sigma = 0.5$	0.959	35	0.993	15.26
	0.971	45	1.00	15.38
	0.923	25	0.991	19.56
$\sigma = 1.0$	0.959	35	0.997	17.00
	0.971	45	1.00	15.38
	0.923	25	0.998	21.84
$\sigma = 3.0$	0.959	35	0.999	18.01
	0.971	45	1.00	15.38

C.2.2 Changing base costs θ_b

In the main text, we have assumed that the base costs (relative to wage income) of raising a child to adulthood have been unchanged over the period. However, when one adopts a broad view of what determines these costs—e.g., subtracting out any direct input from the child on a farm—this is clearly a strong assumption. Mateos-Planas (2002) adjusts base costs residually in order to match the entire path of population growth rates in several European countries and finds large increases in these costs since 1900. A similar exercise in the present model with U.S. data would require base costs, θ_b/w, to have increased threefold over and above the assumed increase proportional to wages to capture the full change in CBR or twofold to capture the full change in population growth rates when $\sigma = 3$. Clearly the analysis would benefit greatly from a more direct accounting of the costs of children along these dimensions but is beyond the scope of this paper.

C.2.3 Increasing σ

Similarly, given the results in the main text, one could ask: how low does the IES have to be to fit fertility in 1800? Although when σ increases the implied levels for the CBR and the population growth rate are ever higher in the earlier periods, this change is not large. Even levels of σ close to 1,000 do not generate the entire change seen in the data.

C.2.4 Deviating from $\eta = 1 - \sigma$

In all the quantitative experiments, we assume that $\eta = 1 - \sigma$. Without this assumption, given our two admissible parameter configurations, AI and AII, η would have to satisfy $\eta < 1 - \sigma$ whenever $\sigma > 1$ or $\eta > 1 - \sigma$ whenever $\sigma < 1$. Hence, the assumption that $\eta = 1 - \sigma$ makes results more readily comparable. In this case, the two utility effects of increasing dynasty size cancel out—independently of σ.

We performed sensitivity with respect to η and found that, for $\eta < 1 - \sigma < 0$, results are not very sensitive to the value of η, holding $\sigma = 3$ (see Figure B.11). For the case, $\eta = 0.8 > 1 - \sigma = 0.5$, the effect of mortality on CBR starts to be negative, so that the model predicts an upward hump until 1880 and a slight decrease thereafter (see Figure B.12).

Figure B.11: Sensitivity with respect to $\eta < 1 - \sigma$, for $\sigma = 3$

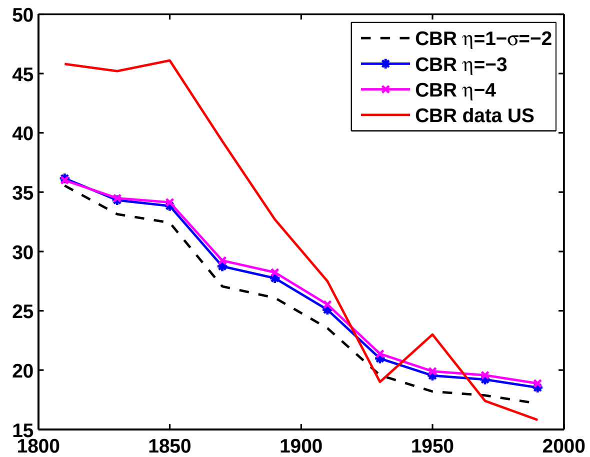
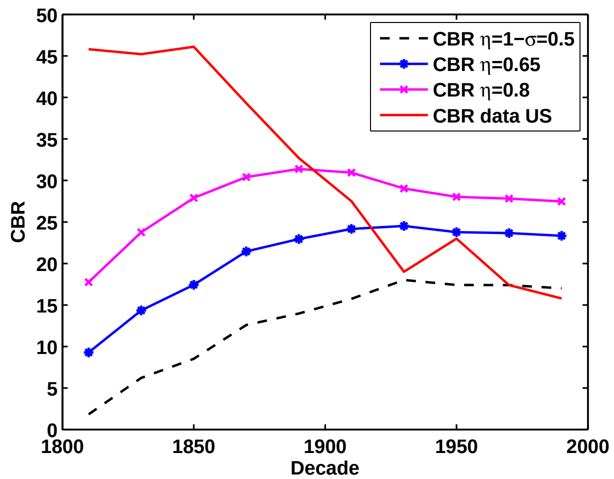


Figure B.12: Sensitivity with respect to $\eta > 1 - \sigma$, for $\sigma = 0.5$



C.2.5 Using other estimates of early productivity growth

In Section 5, for the period from 1800 to 1830, we follow Lucas (2002) in assuming $\gamma = 1.00$ and use data on real wages in Greenwood and Vandenbroucke (2005) for a consistent series for γ from 1830 to 1988.

Other estimates suggest slightly higher productivity growth for the early period. For example, Geib-Gundersen and Zahrt (1996) estimate the annual growth rate in U.S. agriculture between 0.10-0.14 from the 1800-1820 period and between 0.44-1.33 in the 1820-1840 period. However, according to Lucas (2002), early increases in productivity differ from those later on in that they were one-time improvements in technology, rather than sustained growth–the relevant object for our argument. Note that our results are slightly sensitive to the choice of the early growth rate. For example, assuming $\gamma_{ann} = 1.001$ in 1800 changes the results for $\sigma = 3$ in Table 2 from $\gamma_{ann,N} = 1.0142$ to $\gamma_{ann,N} = 1.0135$, from $CBR_{ann} = 36.3$ to $CBR_{ann} = 35.9$ and from $CTFR = 4.2$ to $CTFR = 4.14$.

C.2.6 Using estimates from the U.K. to infer youth mortality rates before 1850

Since data is not available for the U.S. mortality experience before 1850, we assume that π_i, π_{ic}, π_{cy} and π_{ann} were constant at their 1850 values before 1850.

We could use data for the England and Wales to approximate π_i, π_{ic} and π_{cy} for 1800 to 1840, namely 1800 to 1830 from Wrigley et al. (1997, Table 6.1, pp.215), and 1840 from Human Mortality Database. With this data, the results for $\sigma = 3$ for 1800 in Table **??** change from $\gamma_{ann,N} = 1.0142$ to $\gamma_{ann,N} = 1.0135$, from $CBR_{ann} = 36.3$ to $CBR_{ann} = 34.4$ and from $CTFR = 4.2$ to $CTFR = 4.0$. Since mortality rates are lower in England and Wales than in the U.S. in 1850 suggesting they may have been lower before as well, these results should be taken with a grain of salt. Interestingly, since mortality rates decreased from 1800 to 1820 but then sharply increased in 1830 and 1840, the predicted pattern of population growth when only youth mortality changes (i.e. Table 3, column 1, line 3) is a slight increase from 1800 to 1820, a sharp decrease from 1820 to 1830 followed by a sustained increase as in the baseline experiment.

C.2.7 Using expected working life data, instead of life expectancy data

In this section we compare results when expected working lifetime instead of expected lifetime conditional on reaching age 20 is used in the experiments. The range for survival probabilities conditional on reaching age 20, π, in the main text are deduced from measures of expectation of life (EL) at age 20 (see Table A.1., column g, in the main text). EL increased from 38.5 to 53 years. One issue related to this is that expected time in retirement has increased dramatically over the past 150 years (see Lee (2001)). We performed the same experiment using expected working life at age 20 (EWL) (i.e. the difference between expected lifetime and expected years of retirement) which implied an increase from 36.7 to 40.3 years (taken from Lee (2001), Column C). We obtained very similar results since the effect of longevity on births and population growth rates are small in the experiments above. In particular, we find a decrease in $CTFR$ from 4.21 to 2.3, a decrease in CBR from 36.95 to 19.6 and a decrease in population growth from 1.44 percent per year to 0.65 percent per year. The trade-off between using either one of these measures is that on the one hand, EL overstates the benefits in terms of income from having children, while EWL understates the benefits from dynasty size (since retirement is analogous to death in this case).

C.2.8 Perfect foresight transition path versus balanced growth path

In this section, we discuss an alternative to the calculation given in Section 5. There, the simulated data were calculated assuming that the agents believed that their current circumstances, in terms of child costs, productivity growth rates and survival probabilities, would prevail indefinitely into the future when making their decisions—i.e., the calculations are balanced growth path to balanced growth path. The weakness of this is that it assumes that agents act as if circumstances will not change in the future, even though they actually will. At the other extreme, one could assume that agents in a give period t, anticipate exactly all future changes that will occur—i.e., there is perfect foresight with respect to future values of γ, π and θ_s. Here we give the calculations for the model under this alternative assumption. We find that this makes very little difference in the end.

Solving the model with perfect foresight
From the Planner's problem in Section 3, the first-order condition for N_{t+1} is:

$$\theta_{s,t}N_t^{\eta+\sigma-1}C_t^{-\sigma} = \beta\left[\tfrac{(\eta+\sigma-1)}{(1-\sigma)}N_{t+1}^{\eta+\sigma-2}C_{t+1}^{1-\sigma} + \left[w_{t+1}+\theta_{s,t+1}\pi_{t+1}\right]N_{t+1}^{\eta+\sigma-1}C_{t+1}^{-\sigma}\right]$$

The other equation determining the solution is:

$$C_t = \left[w_t + \theta_{s,t}\pi_t\right]N_t - \theta_{s,t}N_{t+1}.$$

After some algebra, these two equations can be rewritten as:

$$\gamma_{Nt}^{1-\eta}\left[\frac{\left[\frac{w_{t+1}}{\theta_{s,t+1}}+\pi_{t+1}\right]-\gamma_{Nt+1}}{\left[\frac{w_t}{\theta_{s,t}}+\pi_t\right]-\gamma_{Nt}}\right]^{\sigma}$$

$$= \beta\left[\frac{\theta_{s,t+1}}{\theta_{s,t}}\right]^{1-\sigma}\left[\frac{(\eta+\sigma-1)}{(1-\sigma)}\left[\left[\frac{w_{t+1}}{\theta_{s,t+1}}+\pi_{t+1}\right]-\gamma_{Nt+1}\right]+\left[\frac{w_{t+1}}{\theta_{s,t+1}}+\pi_{t+1}\right]\right]$$

$$\frac{C_t}{N_t}\frac{1}{\theta_{s,t}} = \left[\frac{w_t}{\theta_{s,t}}+\pi_t\right]-\gamma_{Nt}.$$

The first of these is a first order difference equation in γ_N. It has time varying coefficients however.

If $(\theta_{s,t}, w_t, \pi_t)$ converge in the sense that $\frac{w_t}{\theta_{s,t}} \to \frac{w}{\theta_s}$, $\frac{\theta_{s,t+1}}{\theta_{s,t}} \to \gamma$, $\pi_t \to \pi$, it can be shown that the solution to the model converges to the balanced growth path determined by $\frac{w}{\theta_s}$, γ, and π. Further, assuming that $\frac{w_t}{\theta_{s,t}} = \frac{w}{\theta_s}$, $\frac{\theta_{s,t+1}}{\theta_{s,t}} = \gamma$, and $\pi_t = \pi$ for all $t \geq t^*$ for some t^*, it can be shown that all of the relevant variables, measured in per capita terms, are constant after date t^*. Because of this, the model can be solved backwards from t^* in this case. Thus, suppose that the sequence of exogenous parameters is given by:

$$(\theta_{s,0}, w_0, \pi_0, ..., \theta_{s,t^*}, w_{t^*}, \pi_{t^*}, \gamma\theta_{s,t^*}, \gamma w_{t^*}, \pi_{t^*}, ...).$$

Then, the solution to the perfect foresight infinite horizon problem is of the form:

$$(C_0, N_0, ..., C_{t^*}, N_{t^*}, C_{t^*+1}, N_{t^*+1}, ...)$$

where:

1) for $t \geq t^* + 1$, $N_{t+1} = \gamma_N N_t$ with γ_N given by the solution to:

$$\gamma_N^{1-\eta} = \beta\gamma^{1-\sigma} \left[\frac{(\eta+\sigma-1)}{(1-\sigma)} \left[\left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N\right] + \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right]\right];$$

2) $$\frac{C_{t^*}}{N_{t^*}} \frac{1}{\theta_{s,t^*}} = \frac{C^*_{t^*}}{N_{t^*}} \frac{1}{\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N;$$

3) for $s \geq 1$,

$$\frac{C^*_{t^*}}{N_{t^*}} \frac{1}{\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N \Leftrightarrow \frac{C_{t^*+s}}{N_{t^*+s}} \frac{1}{\gamma^s\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N;$$

4) For $t < t^*$, γ_{Nt} evolves according to the difference equation:

$$\gamma_{Nt}^{1-\eta} \left[\frac{\left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1}\right] - \gamma_{Nt+1}}{\left[\frac{w_t}{\theta_{s,t}} + \pi_t\right] - \gamma_{Nt}}\right]^\sigma$$

$$= \beta \left[\frac{\theta_{s,t+1}}{\theta_{s,t}}\right]^{1-\sigma} \left[\frac{(\eta+\sigma-1)}{(1-\sigma)} \left[\left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1}\right] - \gamma_{Nt+1}\right] + \left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1}\right]\right];$$

5) For $t < t^*$, $\frac{C_t}{N_t}$ is given by:

$$\frac{C_t}{N_t} \frac{1}{\theta_{s,t}} = \left[\frac{w_t}{\theta_{s,t}} + \pi_t\right] - \gamma_{Nt}.$$

Numerical Implementation

We keep the length of a period at $T = 20$ years. Suppose from $t^* = 1990$ on the growth rate in productivity, γ, infant, child and youth mortality $(\pi_i, \pi_{ic}, \pi_{cy})$ (and hence, detrended costs of raising surviving children, θ_s) and adult mortality (longevity), π, are constant. Then, we can use 1) above to solve for the population growth rate, γ_N, on the balanced growth path using parameter values for 1990. We can then use 4) to solve backward for γ_{Nt}, $t = 1970$ using $\gamma_{Nt+1} = \gamma_{Nt^*}$ $t^* = 1990$ and so on. To do this, we have to make one additional assumption (similar to the balanced growth assumption), namely that base costs of raising children, $(\theta_i, \theta_c, \theta_y)$ grow at the same rate as wages every period but are otherwise constant while the cost of raising a surviving child, θ_s, may vary additionally because youth mortality varies.

As in Section 5, we assume that base costs are constant fractions of calibrated costs when children survive with certainty. The results from this experiment are almost indistinguishable from the balanced growth path to balanced growth path experiment in Section 5. This is not surprising since changes in mortality and productivity growth are very smooth. That is, knowing that mortality and productivity change slightly in the next few periods induces very similar choices to the setting in which people believe today's parameters will prevail forever. Moreover, the length of a period being 20 years implies large discounts of future utility (children's utility) and hence changes expected in the future do not affect current decisions very much.

Figure B.13: Perfect Foresight versus Balanced Growth to Balanced Growth, CBR

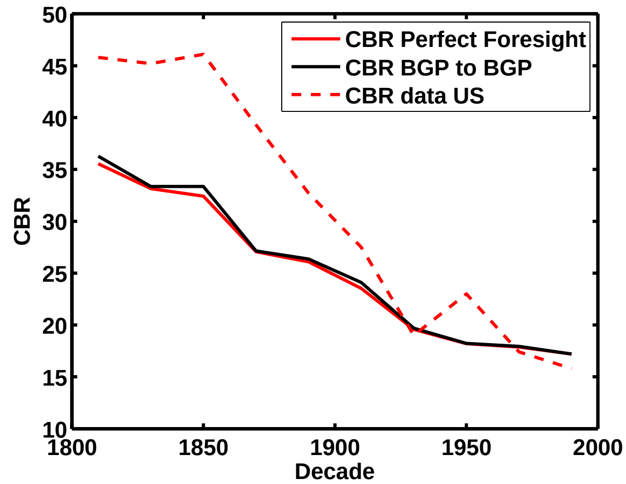
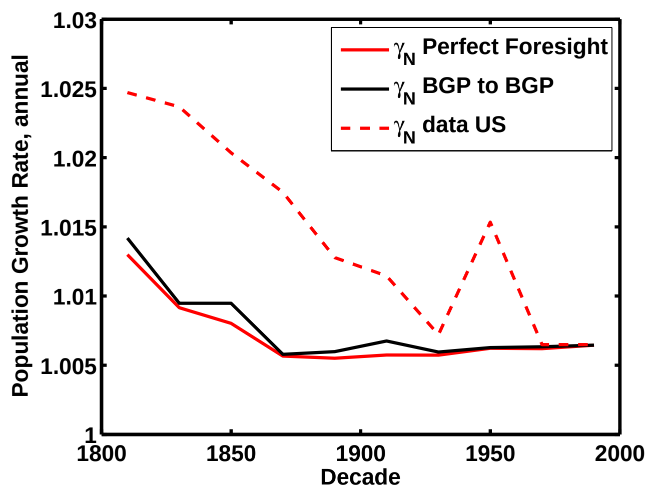


Figure B.14: Perfect Foresight versus Balanced Growth to Balanced Growth, γ_N



D The case of the U.K.

In this section, we perform the same experiment as in Section 5 but using data for the United Kingdom (England and Wales for the most part). The results are quite similar: we capture about two-thirds of the change in CBR and one half of the change in population growth.

The fertility experience in the U.K. over the past 200 years is similar to that of the U.S., except that levels in 1800 were lower already. Because of this the decrease, both in CBR and population growth, was smaller. Mortality was also lower in the U.K. than it was in the U.S. around that time. Finally, our estimates of productivity growth suggest the latter was higher in 1800 as well. Since, fertility, mortality and productivity growth are very similar in the two countries in 1990, all these observations are consistent with our theory and one would expect the model to capture the same fraction of changes in fertility and population growth.

Table B.5: U.K. Costs of children in 1990, Time Series Experiment

σ	$\frac{\theta_s}{w}$	$T \times \frac{\theta_s}{w}$	$Max\ CTFR$
0.5	0.76	15.19	2.63
1.0	0.56	11.27	3.55
3.0	0.20	4.04	9.90

Table B.6: U.K. Time Series Experiment: Data versus Model, for several values of σ

	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1801	1881	1986	1801	1881	1986	1801	1881	1986
Data	1.013	1.012	1.003	37.6	33.9	13.2	—	—	—
$\sigma = 0.5$	0.967	0.974	1.003	1.65	5.31	15.1	1.46	1.74	2.17
$\sigma = 1.0$	0.989	0.992	1.003	18.1	18.2	15.1	2.43	2.46	2.17
$\sigma = 3.0$	1.007	1.005	1.003	29.8	26.2	15.1	3.44	3.13	2.17

Table B.7: U.K. Decomposition: Productivity (γ) , Mortality (π_s) and Longevity (π)

$\sigma = 3$	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.013	1.012	1.003	37.6	33.9	13.2	—	—	—
Productivity (γ)	1.007	1.003	0.998	29.8	27.9	24.7	3.44	3.24	2.94
Mortality (π_s)	1.007	1.007	1.011	29.8	28.8	21.8	3.44	3.32	2.51
Longevity (π)	1.007	1.006	1.007	29.8	29.0	25.4	3.44	3.35	3.48

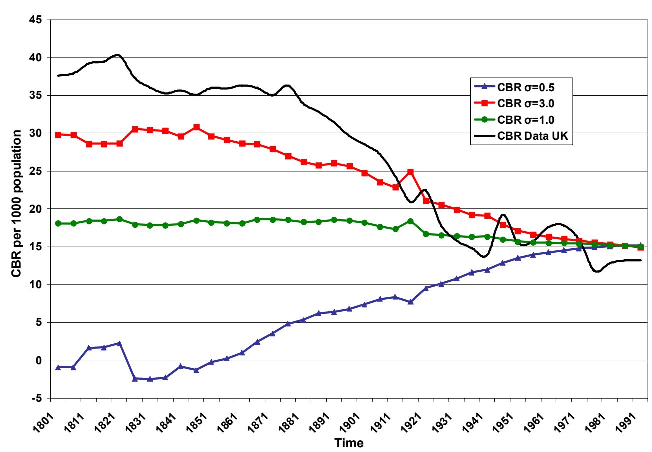
Table B.8: Annual Data Used for the Time Series Experiment in Section D, U.K.

a $Year$	b γ	c π_i	d π_{ic}	e π_{cy}	f $\frac{T}{1-\pi}$ EL	g π_{ann} EL	h CBR data	i CBR HP	j PG data	k PG HP
1801	1.006	0.863	0.884	0.864	39.00	0.965	37.60	39.23	1.014	1.014
1806	1.006	0.863	0.884	0.864	39.00	0.965	37.90	38.98	1.013	1.014
1811	1.006	0.867	0.870	0.925	39.00	0.965	39.18	38.72	1.015	1.014
1816	1.006	0.867	0.870	0.925	39.00	0.965	39.48	38.46	1.015	1.014
1821	1.006	0.855	0.864	0.943	39.00	0.965	40.22	38.17	1.016	1.014
1826	1.006	0.855	0.864	0.857	39.00	0.965	37.30	37.87	1.014	1.014
1831	1.006	0.860	0.877	0.840	39.00	0.965	36.03	37.55	1.012	1.014
1836	1.007	0.860	0.877	0.840	39.00	0.965	35.27	37.21	1.012	1.014
1841	1.007	0.838	0.859	0.894	40.57	0.967	35.61	36.86	1.011	1.013
1846	1.007	0.826	0.847	0.886	38.99	0.965	35.06	36.47	1.012	1.013
1851	1.008	0.829	0.858	0.893	40.19	0.966	35.98	36.05	1.016	1.013
1856	1.008	0.830	0.856	0.898	40.63	0.967	35.89	35.57	1.012	1.013
1861	1.009	0.836	0.854	0.902	40.48	0.967	36.30	35.02	1.012	1.013
1866	1.009	0.826	0.864	0.909	39.93	0.966	35.95	34.39	1.012	1.012
1871	1.010	0.830	0.874	0.913	40.01	0.966	35.00	33.67	1.015	1.012
1876	1.010	0.839	0.882	0.926	40.29	0.966	36.30	32.84	1.015	1.012
1881	1.010	0.845	0.889	0.930	41.33	0.967	33.90	31.91	1.012	1.011
1886	1.010	0.844	0.898	0.941	41.79	0.968	32.80	30.88	1.010	1.011
1891	1.010	0.837	0.900	0.942	41.37	0.968	31.40	29.76	1.016	1.010
1896	1.010	0.825	0.907	0.950	42.85	0.969	29.60	28.57	1.013	1.009
1901	1.010	0.843	0.919	0.953	43.43	0.970	28.50	27.32	1.010	1.009
1906	1.010	0.868	0.932	0.957	44.59	0.971	27.20	26.04	1.010	1.008
1911	1.011	0.883	0.939	0.958	45.37	0.971	24.30	24.74	1.006	1.008
1916	1.011	0.900	0.936	0.919	38.3	0.964	20.90	23.47	0.984	1.007
1921	1.011	0.918	0.958	0.964	47.54	0.973	22.40	22.23	1.010	1.007
1926	1.011	0.927	0.963	0.968	47.85	0.973	17.80	21.06	1.005	1.007
1931	1.012	0.935	0.972	0.970	48.54	0.974	15.80	19.98	1.004	1.006
1936	1.013	0.942	0.981	0.975	49.32	0.974	14.80	18.99	1.005	1.006
1941	1.013	0.946	0.984	0.968	48.33	0.974	13.90	18.09	0.979	1.006
1946	1.014	0.960	0.992	0.982	50.76	0.975	19.20	17.29	1.050	1.006
1951	1.015	0.972	0.995	0.992	52.39	0.976	15.50	16.55	1.003	1.006
1956	1.015	0.976	0.996	0.993	53.17	0.977	15.70	15.87	1.005	1.006
1961	1.016	0.978	0.997	0.993	53.48	0.977	17.60	15.22	1.010	1.005
1966	1.017	0.981	0.997	0.993	53.97	0.977	17.80	14.60	1.006	1.005
1971	1.017	0.983	0.997	0.994	54.26	0.977	15.90	13.98	1.005	1.005
1976	1.017	0.986	0.998	0.994	54.79	0.978	11.80	13.36	1.000	1.004
1981	1.018	0.989	0.998	0.995	55.69	0.978	12.80	12.75	1.000	1.004
1986	1.018	0.991	0.998	0.996	56.46	0.978	13.20	12.15	1.003	1.003
1991	1.018	0.993	0.999	0.996	57.39	0.979	13.20	12.15	1.004	1.003

Data Sources for Table B.8:[2]

- Column b, γ (annual productivity growth rate) from annual growth rate in GDP per capita, (log GDP HP filtered $\lambda = 400$):
  1800 to 1865 from Clark (2001),
  1850 to 1990 from Maddison (1995), p. 194, rescaled to match Clark in 1850;
- Column c,d,e, $(\pi_i, \pi_{ic}, \pi_{cy})$ (survival rates from age specific mortality rates)
  1800 to 1837 from Wrigley et al. (1997), Table 6.1, p.215,
  1841 to 1990 from Human Mortality Database;
- Column f, $\frac{T}{1-\pi}$ (EL) (expectation of life at age 20):
  1841 to 1990 from Human Mortality Database,
  1800 to 1836 set constant at 39 years;
- Column g, π_{ann} (EL) (annual adult survival rate): derived from Column f;
- Column h, CBR (crude birth rate, annual):
  1800 to 1871 from Wrigley et al. (1997),
  1871 to 1986 from Mitchell (1998)
- Column i, CBR HP filtered (crude birth rate, annual): Column h HP filtered, $\lambda = 400$;
- Column j, PG (population growth rate, annual):
  1800 to 1837 from Wrigley et al. (1997), Table 6.1, p.215;
  1841 to 1990 from Human Mortality Database;
- Column k, PG HP filtered (population growth rate, annual): Column j HP filtered, $\lambda = 400$.

Figure B.15: The U.K. experience from 1800 to 1990, CBR



[2] We thank Michael Bar and Oksana Leukhina for help with data sources.

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