Efficiency with Endogenous Population Growth Technical Appendix*

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Abstract

In this Appendix we provide additional details on some of the ideas developed in the paper, "Efficiency with Endogenous Population Growth," (2006). We first give the formal proof that $\mathcal{A}$-efficiency is generically non-empty. The second section extends the notions of efficient fertility developed in the paper for the discrete case to environments in which fertility is a continuous choice variable. It also provides an extension of the First Welfare Theorem, given in the paper for the discrete case, to the continuous case. The second section gives an explicit example of an economy with negative externalities. We show that a negative externality can lead to too many people in equilibrium in the $\mathcal{A}$-sense, and also show how to decentralize $\mathcal{A}$-efficient allocations through Pigouvian tax systems. The third section provides a proof that the limit of the equilibria of the finite horizon B&B games exists, and is an equilibrium of the infinite horizon B&B game.

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1 Generic Non-emptiness of $\mathbb{A}$

Here we prove Result 3b formally. Suppose that there are multiple solutions to the maximization problem:

$\max_{\{f,x\}} \sum_{i\in\mathcal{P}_0} \alpha_i u_i(f,x)$

s.t., $\{f,x\} \in A$

for some choice of α_i with $\alpha_i > 0$ for all $i \in \mathcal{P}_0$.

Define $U(f,x;\alpha) = \sum_{i\in\mathcal{P}_0} \alpha_i u_i(f,x)$ and choose, arbitrarily, a one solution (f^*, x^*).

For $i \in \mathcal{P}_0$, define:

$v_i(f^i, x^i) = u_i(f^i, x^i) - \varepsilon_i \frac{d((f^i,x^i),(f^{i*},x^{i*}))}{1+d((f^i,x^i),(f^{i*},x^{i*}))}$.

Then $|v_i(f^i, x^i) - u_i(f^i, x^i)| < \varepsilon_i$ for all i and all (f^i, x^i), and $v_i(f^{i*}, x^{i*}) = u_i(f^{i*}, x^{i*})$.

Defining

$V(f,x) = \sum_{i\in\mathcal{P}_0} \alpha_i v_i(f,x)$,

note that:

$|V(f,x) - U(f,x;\alpha)| \leq \sum_{i\in\mathcal{P}_0} \varepsilon_i$

Finally note that $V(f^*, x^*) = U(f^*, x^*; \alpha)$ and that $V(f,x) < U(f,x;\alpha)$ for all $(f,x) \neq (f^*, x^*)$. Thus, it follows that the problem:

$\max_{\{f,x\}} V(f,x)$

s.t., $\{f,x\} \in A$

has a unique solution, (f^*, x^*), and because of this, it follows that (f^*, x^*) is $\mathcal{A}$-efficient for the economy with period 0 utility functions given by v_i instead of u_i and nothing else changed.

2 Non-Integer Fertility

In this section of the Appendix we present a version of our environment that can be used to show that the Barro and Becker (BB) model is efficient. There are two reasons why the BB model doesn't quite fit into the framework of Sections 2 and 3 in our paper. First, in the BB model, there is no integer constraint, i.e. the number of children can be anything in $\mathbb{R}_+$, while our framework constrains children to be natural numbers. Second, the BB model imposes symmetry, i.e. only allocations where all siblings do the same thing are considered feasible, while we do not impose symmetry.

2.1 Notation and Definitions

Assume that there is a finite number of dynasties, each associated with one dynastic head $i \in \mathcal{P}_0 = \{1, \ldots, N\}$. The maximal number of children per person is $\bar{f}$. Let $\mathcal{F} = [0, \bar{f}]$. Then we can define the set of potential people recursively as $\mathcal{P}_1 = \mathcal{P}_0 \times \mathcal{F}$, and $\mathcal{P}_t = \mathcal{P}_{t-1} \times \mathcal{F}$. As before, let $\mathcal{P} = \cup_t \mathcal{P}_t$ be the set of all potential people in this economy. A person $i \in \mathcal{P}_t$ can be written as $i^t = (i^{t-1}, i_t)$ where i^{t-1} is i^t's parent and i_t specifies i^t's position in the sibling order. The measure of children actually born to person i is $f(i) \in \mathbb{R}$. Then $\bar{f} - f(i)$ is the measure of potential children of person i who are not born. To simplify what follows we will assume that the children that are born have indexes $[0, f(i)]$. Then, $i_t \leq f(i^{t-1})$ means that i^t is born and $i_t > f(i^{t-1})$ implies that i^t is not born.

We assume that there are k goods available in each period. There is one representative firm, which behaves competitively. The technology is characterized by a production set: $Y \subset \mathbb{R}^{k\infty}$. In other words, an element of the production set is an infinite sequence of k-tuples, that describes feasible input/output combinations. Note that goods are defined in a broad sense here, it can include labor, leisure, capital stock, etc. An element of the production set will be denoted by $y \in Y$. We can write $y = \{y_t\}_{t=0}^{\infty}$, where $y_t = (y_t^1, \ldots, y_t^k)$

is the projection of the production plan onto time t.

An allocation is a fertility, a consumption and a production plan, (f, x, y), where $(f, x) = \{f(i), x(i)\}_{i \in \mathcal{P}}$. Define a consumption set $Z \subset \mathcal{F} \times \mathbb{R}^k$. We require that $(f(i), x(i)) \in Z$ if i is born.

As in the paper, for each $i \in \mathcal{P}_0$, we define D_i to be the set of potential descendants of i, including i. Further, we define $I(f)$ to be the set of people that are alive in an allocation with the fertility plan f. Then we define $I(f_i) = I(f) \cap D_i$ to be the set of people of dynasty i alive under the dynastic fertility plan f_i. Finally, let $I_t(f_i) = I(f_i) \cap \mathcal{P}_t$ denote the set of descendants of i that are alive at date t under allocation (f, x, y).

Definition 1 *An allocation is feasible if*

1. $(f(i), x(i)) \in Z$ *for almost every* $i \in I(f)$.[1]
2. $\sum_{i \in \mathcal{P}_0} \left(\int_{I_t(f_i)} x_j dj + \int_{I_t(f_i)} c(f_j) dj \right) \leq \sum_{i \in \mathcal{P}_0} \int_{I_t(f_i)} e_j dj + y_t \quad \forall t \geq 1$
3. $y \in Y$

Note this formulation allows for different children of the same parent to be treated differently. This is more general than the Barro-Becker formulation where all children of the same parent receive the same allocation by assumption.

The profits earned by dynasty i, Π_i, are defined exactly as in the paper.

Definition 2 *An allocation* $(f, x, y) = (\{(f_j, x_j)\}_{j \in \mathcal{P}}, y)$ *is* $\mathcal{P}$*-efficient if it is feasible and there is no other feasible allocation* $(\hat{f}, \hat{x}, \hat{y})$ *s.t.*

1. $u_j(\hat{f}, \hat{x}) \geq u_j(f, x)$ *for almost every* $j \in P$

[1]Note that since fertility is a continuous variable, it is possible to change assignments of endowments, consumption, fertility, etc., for measure zero sets of children (for periods beyone period 0) without affecting aggregate resources and/or utilities. Thus, all statements are 'almost everywhere.' Formally, we use the measure which is the counting measure on period 0 individuals and Lebesgue measure for all other periods.

2. *There exists* $S \subset \mathcal{P}$ *with positive measure s.t.* $u_j(\hat{f}, \hat{x}) > u_j(f, x)$ *for all* $j \in S$.

Definition 3 *Given* (p, y), *a dynastic allocation for dynasty* i $(f_i, x_i) = \{f(j), x(j)\}_{j \in D_i}$ *is said to be Dynastically* $\mathcal{P}$*-maximizing if* $(f(j), x(j)) \in Z$ *for almost every* $j \in I(f_i)$ *and* $\sum_t p_t \sum_{j \in I_t(f_i)} [x(j) + c(f(j))] \leq \sum_t p_t \sum_{j \in I_t(f_i)} e(j) + \psi_i \sum_t p_t y_t$ *and if* $\nexists (\hat{f}_i, \hat{x}_i)$ *s.t.*

1. $(f(j), x(j)) \in Z$ *for almost every* $j \in I(\hat{f}_i)$.
2. $u_j(\hat{f}_i, \hat{x}_i) \geq u_j(f_i, x_i)$ *for almost every* $j \in D_i$.
3. *There exists* $S \subset D_i$ *s.t.* $u_j(\hat{f}_i, \hat{x}_i) > u_j(f_i, x_i)$ *for all* $j \in S$ *and* S *has a positive measure.*
4. $\sum_t p_t \int_{I_t(\hat{f}_i)} (\hat{x}(j) + c(\hat{f}(j)))dj \leq \Pi_i + \sum_t p_t \int_{I_t(\hat{f}_i)} e(j)dj$

Next we define the analogue of a competitive equilibrium among the dynasties in the partition (exactly the same as before).

Definition 4 (p^*, f^*, x^*, y^*) *is a dynastic* $\mathcal{P}$*-equilibrium if*

1. *For all dynasties* i, *given* (p^*, y^*), (f_i^*, x_i^*) *is dynastically* $\mathcal{P}$*-maximizing.*
2. (f^*, x^*, y^*) *is feasible.*
3. *Given* p^*, y^* *maximizes profits, i.e.* $p^* y \leq p^* y^* \quad \forall y \in Y$.

The definitions of $\mathcal{A}$-efficient allocations, Dynastically $\mathcal{A}$-maximizing decisions and dynastic $\mathcal{A}$-equilibria are modifications of these definitions along the lines in the paper and are not included here.

2.2 First Welfare Theorem and Proof

Assumption 1 *(No negative externalities)*
We assume that u_j is monotone increasing in $x_{j'}$, that is each agent is weakly better off when consumption is increased for a set of agents of positive measure. Thus, there are no negative external effects in consumption.

Assumption 2 *(Positive externalities only within a Dynasty)*
For all $i \in \mathcal{P}_0$, we assume that if (f, x, y) and $(\hat{f}, \hat{x}, \hat{y})$ are two allocations such that $(f(j), x(j)) = (\hat{f}(j), \hat{x}(j))$ for almost every $j \in D_i$, then, $u_j(f, x)) = u_j(\hat{f}, \hat{x})$ for almost every $j \in D_i$.

Lemma 1 *Assume that u_i is strictly increasing in own consumption for all $i \in \mathcal{P}_0$. Let (f_i^*, x_i^*) be dynastically $\mathcal{P}$-maximizing for dynasty D_i, given prices p and production y. Then $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for all $j \in D_i$ implies that $\sum_t p_t \int_{I_t(f_i)}(x(j) + c(f(j)))dj \geq \Pi_i^* + \sum_t p_t \int_{I_t(f_i)} e_j dj$.*

Proof. This will be proved by contradiction. Suppose not. Then there exists a (f_i, x_i) such that $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for all $j \in D_i$ and $\sum_t p_t \int_{I_t(f_i)}(x(j)+ c(f(j)))dj < \Pi_i^* + \sum_t p_t \int_{I_t(f_i)} e_j dj$. Then construct a new dynastic allocation $(\tilde{f}_i, \tilde{x}_i)$ as follows: $(\tilde{f}_i, \tilde{x}_i) = (f_i, x_i + \epsilon)$ for $i \in \mathcal{P}_0 \cap D_i$ and $(\tilde{f}_i, \tilde{x}_i) = (f_i, x_i) \forall$ other $j \in D_i$. Then $\exists \epsilon > 0$ such that the dynastic allocation $(\tilde{f}_i, \tilde{x}_i)$ does not violate the dynastic budget constraint. Moreover, by Assumption 1 $(\tilde{f}_i, \tilde{x}_i)$ is weakly preferred over (f_i, x_i) by all j in the dynasty, and hence also over (f_i^*, x_i^*). Finally, by strict monotonicity, $u_i(\tilde{f}_i, \tilde{x}_i) > u_i(f_i^*, x_i^*)$, but this contradicts the assumption that (f_i^*, x_i^*) was dynastically $\mathcal{P}$-maximizing. □

Proposition 1 *Suppose $u_i(x_i, f_i)$ is strictly increasing in own consumption for all $i \in \mathcal{P}_0$. If (p^*, f^*, x^*, y^*) is an $\mathcal{P}$-dynastic Walrasian equilibrium, then $\sum_t [p_t y_t^* + \int_{\mathcal{P}_t \cap I(f)} p_t e(j) dj] < \infty$, and (f^*, x^*, y^*) is $\mathcal{P}$-efficient.*

Proof. First, note that since $u_i(x_i, f_i)$ is strictly monotone in own consumption, for all $i \in \mathcal{P}_0$, for the given allocation to be a dynastic $\mathcal{P}$-equilibrium,

(f_i, x_i) must be dynastically $\mathcal{P}$-maximizing, and hence, $\Pi_i + \sum_t p_t \int_{I_t(f_i)} e(j)dj < \infty$, for all i. Summing over all dynasties and substituting in the definition of profits Π_i, gives $\sum_t [p_t y_t^* + (\int_{\mathcal{P}_t \cap I(f)} e(j))dj] < \infty$, which proves the first part.

Suppose now that (f^*, x^*, y^*, p^*) is a dynastic $\mathcal{P}$-equilibrium and by way of contradiction, assume that it is not $\mathcal{P}$-efficient. Then there exists an alternative feasible allocation (f, x, y) that is $\mathcal{P}$-superior to (f^*, x^*, y^*). That is, $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for almost all $j \in P$ and there exists $S \subset P$ s.t. $\forall j \in S$, $u_j(f_i, x_i) > u_j(f_i^*, x_i^*)$. For some i, it must be that $S \cap D_i$ has strictly positive measure. Then, for this dynasty i, since (f_i^*, x_i^*) is dynastically $\mathcal{P}$-maximizing, and since there are no external effects across dynasties (Assumption 2), it must be that (f_i, x_i) was not affordable, i.e.

$$\sum_t p_t^* \int_{I_t(f_i)} (x(j) + c(f(j)))dj > \Pi_\tau^* + \sum_t p_t^* \int_{I_t(f_i)} e(j)dj$$

Moreover, by Lemma 2, we know that for all other dynasties the following must hold.

$$\sum_t p_t^* \int_{I_t(f_i)} (x(j) + c(f(j)))dj \geq \Pi_\tau^* + \sum_t p_t^* \int_{I_t(f_i)} e(j)dj$$

Summing up over all dynasties, we get

$$\sum_t p_t^* \int_{I_t(f)} (x(j) + c(f(j)))dj > \sum_t p_t^* [y_t^* + \int_{I_t(f)} e(j)di] \tag{1}$$

Since the right hand side is finite, the strict inequality is preserved. Profit maximization implies that $p^* y^* \geq p^* y$ for all other production plans $y \in Y$. Using this, we can rewrite equation (1) as

$$\sum_t p_t^* \int_{I_t(f)} (x(j) + c(f(j)))dj > \sum_t p_t^* [y_t + \int_{I_t(f)} e(j)dj] \tag{2}$$

Finally, feasibility of (fmx, y) implies that

$$\int_{I_t(f)} (x(j) + c(f(j)))dj \leq y_t + \int_{I_t(f)} e(j)dj \text{ for all } t$$

Multiplying the above by p_t^* and summing up over all t gives

$$\sum_t p_t^* \int_{I_t(f)} (x(j) + c(f(j)))dj \leq \sum_t p_t^*[y_t + \int_{I_t(f)} e(j)dj]$$

But this contradicts equation (2). This completes the proof. □

The proof that $\mathcal{A}$-dynastic Walrasian equilibrium allocations are $\mathcal{A}$-efficient is similar and is omitted.

3 Pollution

In this section, we outline the details of the example discussed in section 6.2 of the main paper. The example features a negative external effect across agents. We characterize the set of symmetric $\mathcal{A}$-efficient and symmetric $\mathcal{P}$-efficient allocations and show how to implement them using Pigouvian taxes. We find that typically both a pollution tax and a 'child' tax are necessary to implement efficient allocations.

Assume that there are a continuum of agents in period 1, indexed by $i \in [0, 1]$. Each period 1 agent has a unit endowment of the consumption good, $e_1(i) = 1$ for all i. This endowment is divided between own consumption, $c_1(i)$, and child rearing. The cost of rearing n children is θn. The $j - th$ child of the $i - th$ period one agent is denoted by (i, j). Agents in period 1 are altruistic toward their own children as in Barro and Becker (1989) and have utilities given by $u(c(i)) + \beta n_i^\alpha \int_0^{n_i} V(i, j)dj$, where $V(i, j)$ is the utility received by child (i, j). We assume that u satisfies the Inada condition, $u'(0) = \infty$.

Each agent in period 2 can sell labor that can be transformed into a consumption good according to the linear technology $c = l$. The utility of the agent is $V(i, j) = v(c_2(i, j), C) - l(i, j)$, where $c_2(i, j)$ is consumption of individual (i, j), $C = \int_0^1 \int_0^{n_i} c_2(i, j)didj$ is aggregate production in the economy, and $l(i, j)$ is the amount of time that (i, j) works. We abstract from constraints on leisure and only assume that there is disutility from work for simplicity.

Finally, feasibility requires:

$\int_0^1 \int_0^{n_i} c_2(i,j)djdi = \int_0^1 \int_0^{n_i} l(i,j)djdi$, and

$\int_0^1 \theta n_i di + \int_0^1 c_1(i)di = \int_0^1 e_1(i)di = 1.$

We assume that the economy lasts for only two periods. Note that this is a simplified version of a Barro-Becker economy, where if $v_2 = 0$ the equilibrium is efficient as in Theorem 2 in the paper. The effect of pollution is captured by assuming that $v_2 < 0$. For simplicity, we assume that there is no pollution in the first period.

Define $\bar{n} = \int_0^1 n_i di$ as the number of people born in period 1 and alive in period 2.

3.1 Characterization of Symmetric, Efficient Allocations

Claim 1 *If $v(c, C)$ is strictly concave in c for each C, then, in every $\mathcal{P}$-efficient and in every $\mathcal{A}$-efficient allocation, the second period consumptions are equal across agents, $c(i,j) = c$ for all (i,j) where c must satisfy:*

$$v_1(c, \bar{n}c) + \bar{n}v_2(c, \bar{n}c) = 1. \tag{3}$$

Proof of the Claim: First, from Result 1 in the paper, all $\mathcal{P}$-efficient and $\mathcal{A}$-efficient allocations are Pareto optimal given the set of people. As is standard with quasi-linear preferences, without limits on labor supply, a necessary condition for Pareto optimality is that the allocation solve a planner's problem with equal weights (or the agent with the lowest weight will work an infinite amount of hours). The allocation of labor effort across the agents is ambiguous, each giving the planner the same utility (but corresponding to different optima among the agents). Because of this, it follows that, given n_i, second period consumption must solve:

$$\max \int_0^1 \int_0^{n_i} \left[v\left(c_2(i,j), \int_0^1 \int_0^{n_i} c_2(i,j) \right) - l(i,j) \right] djdi$$

$$s.t.(\mu) \quad : \quad \int_0^1 \int_0^{n_i} c_2(i,j)djdi \leq \int_0^1 \int_0^{n_i} l(i,j)djdi$$

Since $v(c, C)$ is strictly concave in c for each given value of C, it follows that the optimal choice of $c(i,j)$ is a constant, $c(i,j) = c$ for some c. (If not, the planner's utility can be increased by giving each agent the average c since this does not change C or the aggregate labor supply required.) Thus, the problem can be rewritten as:

$$\max_c \bar{n} v(c, \bar{n}c) - \bar{n}c$$

The result follows directly from this. □

For symmetric, efficient allocations, it follows that $l(i,j) = c_2$ as well, and so, in this case, this equation can also be written as

$$v_1(l, nl) + n v_2(l, nl) = 1. \tag{4}$$

Assumption 3 *Assume that there is a unique solution to (4) for every n.*

Under Assumption 3, equation (4) implicitly defines a relationship between l and n, call this $l^*(n)$, *i.e.* $l^*(n)$ is the efficient labor supply, given population size n, in any efficient, symmetric allocation. Then, the utility of a person alive in period 2 at the symmetric efficient labor choice, given n, is given by:

$$V(n) \equiv v(l^*(n), nl^*(n)) - l^*(n) \tag{5}$$

Lemma 2 $\frac{\partial V}{\partial n} < 0$

Proof of the lemma. $\frac{\partial V}{\partial n} = v_1 l'(n) + v_2[l^*(n) + nl'(n)] - l'(n) = l'(n)[v_1 + nv_2 - 1] + l^*(n)v_2 = l^*(n)v_2$, where the third equality uses equation (4). But this last expression is negative because $v_2 < 0$ by assumption and $l(n) > 0$ from feasibility. □

Next, we want to discern what restrictions efficiency places on $n = \bar{n}$ in symmetric, efficient allocations. The conclusions from the following discussion are summarized in Proposition 2. Define $U(n)$ to be the utility of the typical parent in a symmetric efficient allocation in which (3) is satisfied in the second period. That is,

$$U(n) = u(e_1 - \theta n) + n^{\alpha+1} V(n) = u(e_1 - \theta n) + \beta n^{\alpha} \int_0^n (v(c, nc) - c) di.$$

Assumption 4 *$U(n)$ is strictly concave in n and $U'(0) > u'(e_1)$.*

Note that the feasible choices for n in symmetric allocations are bounded above by $n = e_1/\theta$ and here, $U'(e_1/n) < 0$. Define $n_{\mathcal{A}}$ to be the unique value of n that maximizes $U(n)$. Then, $n_{\mathcal{A}}$ satisfies:

$$-\theta u'(e_1 - \theta n_{\mathcal{A}}) + \beta(\alpha + 1)n_{\mathcal{A}}^{\alpha} V(n_{\mathcal{A}}) + \beta n_{\mathcal{A}}^{\alpha+1} V'(n_{\mathcal{A}}) = 0. \tag{6}$$

It follows from Lemma 2 that $n_{\mathcal{A}}^{\alpha+1} V'(n_{\mathcal{A}}) < 0$. Finally, it must also be true that $V(n_{\mathcal{A}}) > 0$ since, if this does not hold, U is maximized at $n_{\mathcal{A}} = 0$. To see that the allocation characterized by (4) and (6) is $\mathcal{A}$-efficient, note that any potentially superior allocation must involve the same $n_{\mathcal{A}}$, c_1 and c_2, because otherwise the period 1 agents would be strictly worse off. That leaves only rearrangements of labor among period 2 agents, which would immediately make some period 2 agents strictly worse off.

We now show that any allocation with $n < n_{\mathcal{A}}$ can be dominated in the $\mathcal{A}-$ and the $\mathcal{P}$-sense. Suppose $n < n_{\mathcal{A}}$ and consider the following change in plan:

1. Increase n to $n + \Delta n$;

2. Have $c(i, j) = \hat{c}$ for all i and all $0 \leq j \leq n + \Delta n$, where $\hat{c}$ is the solution to $v_1(\hat{c}, (n + \Delta n)\hat{c}) + (n + \Delta n)v_2(\hat{c}, (n + \Delta n)\hat{c}) = 1$, i.e., $\hat{c}$ is the new optimal consumption level for all agents in the second period given that n has been increased to $n + \Delta n$;

3. Let $l(i, j) = \hat{l}$ for all i and for all $0 \leq j \leq n$, where $\hat{l}$ is defined by: $v(\hat{c}, (n + \Delta n)\hat{c}) - \hat{l} = v(c, nc) - c$. That is, $\hat{l}$ is just enough extra leisure so that all children in period 2 are indifferent to this change.

4. Let $l(i, j) = \tilde{l}$ for all i and for all $n \leq j \leq n + \Delta n$, where $\tilde{l}$ is defined by feasibility: $n\hat{l} + \Delta n\tilde{l} = (n + \Delta n)\hat{c}$.

Note, this has children (i, j) with $n \leq j \leq n + \Delta n$, working more than those children (i, j) with $0 \leq j \leq n$ (to compensate them for the loss in utility they experience from the increased population).

Finally, it is left to show that the new allocation is strictly better for the typical parent. The utility received by the typical parent from the new allocation is given by

$$W(\Delta n) = u(e_1 - \theta n - \theta\Delta n) + \beta(n + \Delta n)^\alpha \int_0^{n+\Delta n} [v(\hat{c}, (n + \Delta n)\hat{c}) - l(i, j)]\, dj$$

$$\begin{aligned}
&= u(e_1 - \theta n - \theta\Delta n) + \beta(n + \Delta n)^\alpha \left[(n + \Delta n)v(\hat{c}, (n + \Delta n)\hat{c}) - \int_0^{n+\Delta n} [l(i, j)]\, dj\right] \\
&= u(e_1 - \theta n - \theta\Delta n) + \beta(n + \Delta n)^\alpha \left[(n + \Delta n)v(\hat{c}, (n + \Delta n)\hat{c}) - (n + \Delta n)\hat{c}\right] \\
&= u(e_1 - \theta n - \theta\Delta n) + \beta(n + \Delta n)^{\alpha+1} \left[v(\hat{c}, (n + \Delta n)\hat{c}) - \hat{c}\right] \\
&= u(e_1 - \theta n - \theta\Delta n) + \beta(n + \Delta n)^{\alpha+1} \left[V(n + \Delta n)\right]
\end{aligned}$$

Differentiating this with respect to Δn we obtain:

$$W'(\Delta n) = -\theta u'(e_1 - (n+\Delta n)\theta) + \beta(\alpha+1)(n+\Delta n)^\alpha \left[V(n + \Delta n)\right] + \beta(n+\Delta n)^{\alpha+1} \left[V'(n + \Delta n)\right].$$

Evaluating this at $\Delta n = 0$, we obtain:

$$W'(0) = -\theta u'(e_1 - n\theta) + \beta(\alpha + 1)n^\alpha V(n) + \beta n^{\alpha+1}V'(n).$$

This is positive for all $n < n_{\mathcal{A}}$. Thus, by construction, this change improves the welfare of the parent and leaves indifferent all children, (i, j) with $0 \leq j \leq n$.

The change in utility of the children (i, j) with $n \leq j \leq n + \Delta n$ is given by $v(\hat{c}, n\hat{c}) - \tilde{l}$. By continuity, this is approximately equal to $V(n)$ when Δn is sufficiently small. Thus, this gives a Pareto Improvement anytime $n < n_{\mathcal{A}}$ and $V(n) > 0$.

Moreover, it follows that if $n > n_{\mathcal{A}}$, the symmetric allocation satisfying (3) is not $\mathcal{A}$-efficient. To see this, simply note that by reducing n to $n_{\mathcal{A}}$, and using the corresponding allocation satisfying 3 improves the welfare of the surviving period 2 agents (since $V'(n) < 0$) and improves welfare of the period one agents as well (since W is maximized at $n = n_{\mathcal{A}}$).

The following Proposition summarizes this discussion.

Proposition 2 *In every symmetric $\mathcal{P}$-efficient allocation we have:*

- $n \geq n_{\mathcal{A}}$,
- $c_2(i,j) = c_2$ *satisfies:* $\quad v_1(c_2, nc_2) + nv_2(c_2, nc_2) = 1,$
- $l(i,j) = l = c_2,$
- $c_1(i) = c_1 = 1 - \theta n,$
- $V(n) \geq 0.$

Also, there is a unique symmetric $\mathcal{A}$-efficient allocation with $n = n_{\mathcal{A}}$, as defined by equation (6).

3.2 Implementation of Efficient Allocations

Next, we characterize equilibrium choices of fertility and consumption by private agents acting in a decentralized way in markets. Since there is no way to physically transfer goods across the periods, and all dynasties are identical, we can, without loss of generality, assume that bequests are not allowed.

In the market equilibrium allocation with a tax of τ_c on the production of c_2 we have that each period 2 child takes as given C, τ_c and T_2, the transfer received from the government. Thus, a period 2 agents solves:

$$\max_{\{c_2,l\}} v(c_2, C) - l$$
$$\text{s.t. } (1+\tau_c)c_2 = l + T_2$$

Let $\tilde{c}_2(\tau_c, C, T_2)$ denote the individual's solution to this problem. Note that, it does not depend on the period 1 choice of n due to the assumed linearity of the period 2 production function. In equilibrium, $\tilde{C} = \tilde{n}\tilde{c}_2$, and so the FOC is

$$v_1(\tilde{c}_2, \tilde{n}\tilde{c}_2) = (1 + \tau_c). \tag{7}$$

If $\tau_c = 0$, we have the standard, fixed population, result that each agent sets his own MRS equal to the price ratio, ignoring the external effect on the other agents. This induces too much output in the equilibrium and hence, it follows that the undistorted market equilibrium quantities (i.e., with $\tau_c = 0$) are neither $\mathcal{P}$ nor $\mathcal{A}$ efficient.

In period 1, the parent maximizes utility knowing what will happen in period 2. However, each period one parent views that their own choice of n will have no effect on $\tilde{C}$. Thus, a parent solves:

$$\max_{\{c_1,n\}} u(c_1) + \beta n^{\alpha+1} \left[v(\tilde{c}_2, C) - \tilde{c}_2\right]$$

s.t.

$$c_1 + (\theta + \tau_n)n = 1 + T_1.$$

The FOC imply that

$$(\theta + \tau_n)u'(\tilde{c}_1)/\beta = (\alpha + 1)\tilde{n}^{\alpha} \left[v(\tilde{c}_2, \tilde{n}\tilde{c}_2) - \tilde{c}_2\right]. \tag{8}$$

Comparing these conditions with those derived above for symmetric, efficient allocations, we see that $\tau_c = -\tilde{n}v_2(\tilde{c}_2, \tilde{n}\tilde{c}_2)$ is required (to see this, compare equation (7) with (4)). This is standard, each agent must be induced to set his consumption so that his marginal rate of substitution equals the marginal social cost, not the marginal private cost.

It also follows that privately chosen fertility, n, will generally not be $\mathcal{A}$-efficient. For any given n, $n \geq n_{\mathcal{A}}$, equation (8) defines the unique level of τ_n necessary to implement that n as an equilibrium choice (in conjunction with $\tau_c = -nv_2(c_2, nc_2)$). For example, to implement $n_{\mathcal{A}}$ as an equilibrium outcome, it is necessary that $\tau_c = -n_{\mathcal{A}}v_2(l^*(n_{\mathcal{A}}), n_{\mathcal{A}}l^*(n_{\mathcal{A}}))$ and $\tau_n = -n_{\mathcal{A}}^{\alpha+1}\tilde{c}_2 v_2(\tilde{c}_2, \hat{n}\tilde{c}_2)/u'(\tilde{c}_1)$ per child (this follows from (6) and (8)). Note that

$\tau_n > 0$. This (τ_c, τ_n) decentralizes the unique symmetric $\mathcal{A}$-efficient allocation, and hence, it follows that the resulting allocation is both $\mathcal{P}$- and $\mathcal{A}$-efficient. Whether or not a given symmetric, $\mathcal{P}$-efficient allocation will require non-trivial taxation also follows from these results. Examination of (8) shows that implementation with $\tau_n = 0$ is equivalent to:

$$\theta u'(e_1 - \theta n)/\beta = (\alpha + 1)n^{\alpha} \left[v(l^*(n), nl^*(n)) - l^*(n) \right].$$

Typically, there will be only one such n. Note however, that for large n, one would typically expect $\tau_n < 0$ to be required for implementation. Summarizing this discussion:

Proposition 3 *The unique symmetric, $\mathcal{A}$-efficient allocation can be implemented with positive taxes on both the number of children and second period consumption. Most symmetric, $\mathcal{P}$-efficient allocations require non-zero taxes on children (but they could be negative) and all require a positive tax on second period consumption.*

As argued above, with only $\tau_n = 0$, the resulting equilibrium will not be $\mathcal{A}$-efficient. In other words, standard Pigouvian taxes are not sufficient to implement $\mathcal{A}$-efficient allocations. A fertility tax is needed in addition. But are equilibrium allocations with $\tau_n = 0$ (and τ_c set as described above) $\mathcal{P}$-efficient? The answer is typically yes. The lack of a fertility tax will lead to overproduction of kids from the perspective of period 1 agents, however, there is no allocation that is superior in the $\mathcal{P}$-sense, as it would necessarily require less children to be born, which would make those children strictly worse off. Note that this logic would be different if the externality was positive! Then, an equilibrium with $\tau_n = 0$ would lead to too low fertility, and a $\mathcal{P}$-superior allocation could easily be constructed.

4 The Limit of the SPE of the T Period B&B Game

In this section we fill out the details of the claim (footnote 19, page 25 in the paper) that the limit of the SPE of the finite horizon games of the B&B model as presented in the paper, both exists and is an SPE of the infinite horizon version of the game. This is done in two steps. We start with a brief discussion of each step, and then provide details.

Step 1: *The limit of the equilibrium strategies of the T horizon game exists.*

To do this, use the characterization of the equilibrium strategies in period t of the T period game as the first period of the solution to a planning problem that runs from period t to period $T+1$. The planning problems for period t in the T period game can all be embedded in a common 'space,' one with an infinite horizon, but in which after period $T+1$ all variables are required to be 0. The solutions to this sequence of planning problems converge to the solution to an infinite horizon planning problem state by state and period by period. Thus, the equilibrium strategies of the period t players of the T period game have a limit, given by the first period actions of the solution to the infinite horizon planning problem.

Step 2. *These limiting strategies form an SPE of the infinite horizon game. Moreover, the characterization of the strategies as the solution of the appropriate infinite horizon planner's problems also holds for this limit. I.e., the first period actions of the solution to the infinite horizon continuation planner's problems are an SPE.*

To do this step, suppose it is false. Then there is some node and some player who can do better by using a different strategy. We argue that this better strategy has to be symmetric across siblings by the strict concavity of the continuation utility. Use this better strategy to construct a better strategy for the finite horizon games by having it be the better strategy for the infinite horizon game. Argue for large enough T, this is better for the finite horizon game since the utilities of the two relevant possibilities converge to the limiting utilities, and there is strict improvement in the limiting utilities by assumption. This contradicts the assumption that the original strategies were an SPE for the finite horizon game.

4.1 Step 1

From the characterization result in the current appendix to the paper, we have that there is a unique SPE of the T-period horizon game, and that for each $t \leq T$ and for every node, h^t the only thing that matter for each alive individual (i.e., those $i^t = (i^{t-1}, i_t)$ such that $i_t \leq f(i^{t-1})$ for all $j \leq t$) is the wealth passed forward to him by his parent, $a(h^t)$. This SPE can be calculated through the solution to a planner's problem which is given by:

For every (a, t, T), and every history up to t, the outcome of the continuation subgame is unique, symmetric, depends only on the bequest given to each agent, a, and solves:

$$\begin{aligned}
\max_{X_t, F_t, X_{t+1}, F_{t+1} \ldots, X_{T+1}, F_{T+1}} U_t &= \sum_{s=0}^{T+1-t} \beta^s \left[F_{t+s} g(F_{t+s}) u(X_{t+s}/F_{t+s}) \right] \\
& \quad s.t. \sum_{s=0}^{T+1-t} p_s \left[X_s + c(F_s) \right] \leq \sum_{s=0}^{T+1-t} p_s F_s e_s + a \\
F_t &= 1
\end{aligned}$$

We can embed this problem in an infinite horizon one to have them all defined on the same space:

For every (a, t, T), and every history up to t, the outcome of the continuation subgame is unique, symmetric, depends only on the bequest given to each agent, a, and solves:

$P(a, t, T)$:

$$\max_{X_t, F_t, X_{t+1}, F_{t+1} \ldots} U_t = \sum_{s=0}^{\infty} \beta^s \left[F_{t+s} g(F_{t+s}) u(X_{t+s}/F_{t+s})\right]$$

$$s.t. \quad \sum_{s=0}^{\infty} p_s \left[X_s + c(F_s)\right] \leq \sum_{s=0}^{\infty} p_s F_s e_s + a$$

$$F_t = 1$$

$$F_s = X_s = 0, s > T + 1$$

Denote the solution to this problem by $((X_s(a, t, T), F_s(a, t, T))_{s=0}^{\infty}) \in \Re^{\infty} \times \Re^{\infty}$.

These belong in a bounded set from our assumption that the set of feasible allocations is bounded (Assumption 6.5).

To find the equilibrium strategies for the period t agent in the T horizon game at the node a define:

$X_t^T(a) = X_t$: this is the consumption of the agent.

$F_t^T(a) = F_{t+1}$: this is the number of children the agent has, and,

$B_t^T(a) = B_t = q_t e_t - a - (X_t + c(F_{t+1}))$: this is the bequest he leaves in total to his children, or, per person:

$b_t^T(a) = b_t = B_t/F_{t+1}$.

These are the equilibrium strategies for the period t agent in the T horizon game at the node summarized by a.

Similarly, use $((X_s(a, t, \infty), F_s(a, t, \infty))_{s=0}^{\infty}) \in \Re^{\infty} \times \Re^{\infty}$ and $(X_t^{\infty}(a), F_{t+1}^{\infty}(a), B_t^{\infty}(a))$ to denote the solutions to the infinite horizon problem:

$P(a, t, \infty)$:

$$\max_{X_t,F_t,X_{t+1},F_{t+1}\ldots} U_t = \sum_{s=0}^{\infty}\beta^s\left[F_{t+s}g(F_{t+s})u(X_{t+s}/F_{t+s})\right]$$
$$s.t.\ \sum_{s=0}^{\infty}p_s\left[X_s+c(F_s)\right]\leq\sum_{s=0}^{\infty}p_sF_se_s+a$$
$$F_t = 1$$

In what follows, we denote first period components of the solution as: $(X_t^T(a), F_{t+1}^T(a), B_t^T(a))$, $(X_t^\infty(a), F_{t+1}^\infty(a), B_t^\infty(a))$, etc.

Lemma 3 *For each* a*,* $((X_s(a,t,T), F_s(a,t,T))_{s=0}^\infty) \to ((X_s(a,t,\infty), F_s(a,t,\infty))_{s=0}^\infty)$ *in the product topology.*

Proof. This is a standard result. The set containing the solutions of the finite horizon problems is bounded by feasibility in each period and hence, for each a, are all contained in a set that is compact in the product topology. Using the arguments in Jones and Manuelli (1990), it also follows that the utility function is continuous in this topology. Finally, for each fixed a, the constraint set is both uhc and lhc in T. That is, letting Z_T and Z_∞, if $z_T \in Z_T$ and if $z_T \to z_\infty$ in the product topology, then $z_\infty \in Z_\infty$ and if $z_\infty \in Z_\infty$ then letting z_T be the projection of z_∞ on the first T components, we have that $z_T \in Z_T$ and $z_T \to z_\infty$. From this, it follows that the solutions of the T problem converge to the solution of the ∞ problem in the product topology. (This is basically the Theorem of the Maximum.) ■

Lemma 4 *For each* a*,* $(X_t^T(a), F_{t+1}^T(a), B_t^T(a)) \to (X_t^\infty(a), F_{t+1}^\infty(a), B_t^\infty(a))$*.*

Proof. This follows immediately from the Lemma above since $(X_s(a,t,T), F_s(a,t,T))_{s=0}^\infty) \to ((X_s(a,t,\infty), F_s(a,t,\infty))_{s=0}^\infty)$ in the product topology. ■

4.2 Step 2

Proposition 4 *The strategies,* $(X_t^\infty(a), F_{t+1}^\infty(a), B_t^\infty(a))$*, form an SPE of the infinite horizon game.*

Proof. Suppose not. Then there is some player, and some node a, for whom the prescribed strategy is not optimal. Let $(\hat{X}_t^\infty(a), \hat{F}_{t+1}^\infty(a), \hat{B}_t^\infty(a))$ the supposed improving strategy choice at node a. Note that it is possible that $b_t(i^{t+1})$ might vary across $i^{t+1} = (i^t, i_{t+1})$. Finally, let

$((\hat{X}_s(a, t+1, \infty, i_{t+1}), \hat{F}_s(a, t+1, \infty, i_{t+1}))_{s=0}^\infty)$

denote the equilibrium outcome of the continuation game resulting from using the strategy $(\hat{X}_t^\infty(a), \hat{F}_{t+1}^\infty(a), \hat{B}_t^\infty(a))$ at the node a.

Then, by hypothesis,

$\hat{U} = U(\hat{X}_t^\infty(a)) + \beta g(\hat{F}_{t+1}^\infty(a)) \int_0^{\hat{F}_{t+1}^\infty(a)} \hat{U}(i^t, i_{t+1}) di_{t+1}$

$> U^* = U(X_t^\infty(a)) + \beta g(F_{t+1}^\infty(a)) \int_0^{F_{t+1}^\infty(a)} U(i^t, i_{t+1}) di_{t+1}$

where $\hat{U}(i^t, i_{t+1})$ and $U(i^t, i_{t+1})$ are, respectively, the utility of the children under the two strategies.

First use the characterization result from the old appendix to show that we can make b not depend on i_{t+1}. This follows from the convexity of the payoff function of the continuation payoffs from the succeeding node forward. (This is only shown for the T period games in the current appendix, but it follows by taking limits as $T \to \infty$ and continuity.)

Thus, without loss of generality, we can assume that:

$$\hat{U} = U(\hat{X}_t^\infty(a)) + \beta g(\hat{F}_{t+1}^\infty(a)) \hat{F}_{t+1}^\infty(a) u\left(\frac{\hat{X}_{t+1}^\infty(a)}{\hat{F}_{t+1}^\infty(a)}\right) + \beta^2 g(\hat{F}_{t+2}^\infty(a)) \hat{F}_{t+2}^\infty(a) u\left(\frac{\hat{X}_{t+2}^\infty(a)}{\hat{F}_{t+2}^\infty(a)}\right) + \dots$$

and,

$$U^* = U(X_t^\infty(a)) + \beta g(F_{t+1}^\infty(a)) F_{t+1}^\infty(a) u\left(\frac{X_{t+1}^\infty(a)}{F_{t+1}^\infty(a)}\right) + \beta^2 g(F_{t+2}^\infty(a)) F_{t+2}^\infty(a) u\left(\frac{X_{t+2}^\infty(a)}{F_{t+2}^\infty(a)}\right) + \dots$$

From this, we want to show that we can construct a strategy for this player, at the same node a, but in the finite horizon version of the game which also does better. This will give us a contradiction.

To see this consider the payoff that the agent will receive in the T period game if he adopts the strategy $(\hat{X}_t^\infty(a), \hat{F}_{t+1}^\infty(a), \hat{B}_t^\infty(a))$. Denote this by $\hat{U}^T$. By construction, it follows as in the argument given above that $\hat{U}^T \to \hat{U}$.

Similarly, letting U^{*T} denote the payoff from using the strategy $(X_t^\infty(a), F_{t+1}^\infty(a), B_t^\infty(a))$, we have that $U^{*T} \to U^*$.

But, then it follows that for large enough T, $\hat{U}^T > U^{*T}$ contradicting the fact that $(X_t^\infty(a), F_{t+1}^\infty(a), B_t^\infty(a))$ is an optimal strategy in the T period game. This completes the proof. ■