Properties of the Growth Model

Above, we have seen that the standard growth model has a much richer interpretation than it first appears. In certain cases, it is equivalent to a complex environment with heterogeneity with many consumers, sectors and firms, each of which is taking prices as exogenous to its own decision problem. This does require assumptions however, and they are often quite strong.

What are the benefits of this? All of the properties of the standard model that come from its formulation:

1 The characterization of the problem as a Dynamic Programming problem if preferences and production functions satisfy certain key assumptions.

2 The characterization of the solution as a first order difference equation.
in the optimal choice of all of the variables as functions of the current, and only the current value of the state variable $k_t$.

2. Uniqueness of the steady state of the solution.

2. Global convergence of the system to the steady state under stronger assumptions.

2. The explicit analytic solution of the problem under even stronger assumptions.

2. The host of numerical techniques available for the solution of DP's that have been developed over the years.

I probably should add more detail to this discussion at some point.

One question that has come up in past discussions is: How fast does the solution to these problems converges to their steady state values? There are two ways to approach this precisely. For global issues, we can look at either numerical simulations, or those special cases where analytic solutions exist. It is also possible to get some idea of the answer to this question by linearizing the system around the steady state to get some idea of what the policy function looks like in a neighborhood of its steady state value.
Typically this convergence is quite rapid. The following discussion is meant to give you some feeling for why.

\[
\frac{U^0(\alpha)}{U^0(\alpha+1)} = -(1_i \pm f_0^0(k_{t+1}))
\]

\[
\frac{U^0(\alpha)}{U^0(\alpha+1)} = \frac{\alpha+1}{\alpha} = -(1_i \pm f_0^0(k_{t+1})) \text{ this is under log preferences}
\]

If \( f^0(k) \) is very high then interest rate is also high. People save more and consume less. This accounts for the fact that the transition is really fast.

\[
1 + R = 1_i \pm f^0
\]

\[
\frac{U^0(\alpha)}{U^0(\alpha+1)} = 1 + R
\]

Can you use this to get some idea about cross country comparisons?

What happens if you \( \text{t} \) in same coefficient for USA into other three countries?
According to the model, we will be able to tell when Japan catches US.

But the implied interest rate differentials are quite extreme? This would imply very high growth rates in consumption where countries are at a lower level of development.

Thus, it would say Japan had higher interest rate in the beginning. Actually it is true that poor countries have higher interest rates than rich ones as a rule, but these differences are not large typically. Also, it is difficult to know to what extent this is due to the fact that $k$ is lower, and to what extent it is related to the fact that investments in poor countries seem to be riskier.
An alternative hypothesis might be that the production functions are different in different countries. It is hard to know what this means. Literally, it says that something are possible in countries with high A's, that are NOT possible in countries with low A's. Thus, it would say that poor countries are poor because it is not POSSIBLE for them to be rich. For example, suppose:

\[ y_t(\text{US}) = A_t(\text{US}) k_t^\circ \]
\[ y_t(\text{JAP}) = A_t(\text{JAP}) k_t^\circ \]

What part of the differences in \( y_t \) should be traced to differences in \( k_t \) and what part to differences in \( A_t \)?

Note the Main point however: because we are explicit, we can solve the model for different assumptions and generate the time-series of the solution.
Figure 3:

![De-trended plot](image)

...to compare them with actual data. Thus, at least we can have a sensible discussion about it!

**<Policy in the Growth Model>**

**Remark:** Why do we need a model or a theory at all? Why don't we just look at data to ask the questions that we are interested in? One problem is the difficulty with doing controlled experiments. But even beyond this, (i.e., in fields where they can do controlled experiments), models/theories provide useful devices for organizing our thinking. For this, the theory needs to be sufficiently 'concrete' so that we can solve it explicitly to:

2. To see if the theory is right.
To check 'What if policy questions. That is to answer the question: What would happen if we did X? When we have no data on situations where X has been done.

To ask what policy 'should be' to characterize optimal policy.

**Examples:** What if? (Policy changes)

1) We changed the current US tax system to one in which there was a flat rate tax on income from the current progressive system?

2) We changed the way we fund social security payments from the current system to one in which social security accounts are run like individual pension accounts?

3) We changed from an income tax based system to a consumption tax based system?

What effects would these changes have on \(y_t; x_t; c_t, \ldots\)?

What effects would these changes have on \(U_i; \mu; \bar{c}; \bar{l}\)? Would they improve welfare? Would they lessen it? Would they increase utility for some people and lower it for others? If so, who would bene\^t, who would be hurt? Can we find other changes that might improve everyone's welfare?
A Price Taking Model of Equilibrium with Taxes and Spending

To adress these issues we'll develop a version of the model CE, price taking model we described above and introduce taxes and government spending to the mix:

We'll want to add taxes:

1) on $c_t - \xi_c$
2) on $x_t - \xi_x$
3) on labor income, $w_t n_t - \xi_{nt}$
4) on capital income, $r_t k_t - \xi_{kt}$

and

5) lump sum transfers - $T_t$
6) spending - $g_t$

<T D C E>

A Tax Distorted Competitive Equilibrium CE with taxes and spending is given by the sequences $\xi_c; \xi_x; \xi_{nt}; \xi_{kt}; T_t; g_t$ is:

(i) Plans for households $(c_t^i; x_t^i; n_t^i; l_t^i; k_t^i)_{t=0}^1$

(ii) Plans for firms (assuming there is only one) $(c_t^f; x_t^f; n_t^f; g_t^f; k_t^f)_{t=0}^1$
(iii) Prices \((p_t; r_t; w_t)\)

such that,

a) Firms and houses are maximizing given prices, taxes, transfers and spending and

b) the usual accounting identities for quantities hold.

\[\begin{align*}
\text{Maximization} & \\
\text{(HH)} \text{ M ax } & U_i \quad c_l^t \\
\text{s.t.} & \\
i) & P_t^1 (1 + \xi_{ct}) c_l^t + P_t^1 (1 + \xi_{xt}) x_l^t \cdot P_t^1 (1 + \xi_{nt}) w_t n_l^t + (1 + \xi_{kt}) r_t^t k_l^t + T_l^t \\
ii) & k_{l+1}^t \cdot (1 + \beta) k_l^t + x_l^t \\
iii) & n_l^t + l_l^t \cdot h_l^t = 1 \\
\end{align*}\]

and \(k_0^t\) is fixed

Remark: Note that we have assumed that the tax system is linear

no progressivity/regressivity.

\[\begin{align*}
(\text{FIRM}) \quad & (c_t^f; x_t^f; g_t^f; k_t^f; n_t^f)_{t=0}^1 \text{ solves} \\
& \text{M ax } p_t^f c_t^f + x_t^f + g_t^f \cdot i \cdot r_t^f k_t^f \cdot w_t^f n_t^f \\
\text{s.t.} & c_t^f + x_t^f + g_t^f \cdot F_t(k_t^f; n_t^f) \\
\end{align*}\]

Remark: We have directly jumped to the assumption that \(p_c = p_{x_t} = \)
\( p_3 = p_t \). Given our assumption that \( c, x, \) and \( g \) are perfect substitutes in the output of the firm, this would follow automatically in any equilibrium and in any period in which all three are positive.

**Remark:** We have assumed that households are the ones that are responsible for paying the taxes.

**Markets Clear**

i) \( 8t \quad \sum_{i=1}^{P} n_i^t = n_t^f \)

ii) \( 8t \quad \sum_{i=1}^{P} k_i^t = k_t^f \)

iii) \( 8t \quad \sum_{i=1}^{P} (c_i^t + x_i^t + g_i^t) = F_t(k_t^f; n_t^f) \)

**The Budget of the Government is Balanced in Present Value**

\[
P \frac{1}{h} \sum_{t=0}^{T} \left( \sum_{i=1}^{P} p_i \right) c_i^t + \sum_{t=0}^{T} \left( \sum_{i=1}^{P} p_i \right) x_i^t + \sum_{t=0}^{T} \left( \sum_{i=1}^{P} p_i \right) n_i^t + \sum_{t=0}^{T} \left( \sum_{i=1}^{P} p_i \right) k_i^t = p \frac{1}{h} \sum_{t=0}^{T} \left( T_t^i + p \sum_{i=1}^{P} g_i^t \right)
\]

(Remove revenue from consumption tax + revenue from investment tax + revenue from income tax

Expenditure side = lump sum transfers + government expenditure)

**Remark:** Note that I have set this up with an infinite horizon BC for both the HH and government and hence free, perfect lending markets are being assumed.
Problem: Show that: If a price system and allocation satisfy everything except government budget balance, it must also be satisfied.

Problem: Set up the problem with sequential BC's for both HH's and the government and show that these two ways are equivalent.

Problem: Set up the problem with firms paying taxes and show equivalence.

Problem: How should capital formation be included in this version of the model? Does it matter if the firm or HH does the investment for the properties of equilibrium?

Remark: What would it mean for \( \delta_{xt} \) to be negative? Or any of the other taxes?

Remark: In this formulation, it is assumed that consumers take prices, tax rates, and transfers as given. That is, unaffected by how they make their consumption, savings, labor supply and investment decisions.

Remark: If \( T^i_t < 0 \), then it is interpreted as a lump sum tax, if \( T^i_t > 0 \), then it is a lump sum transfer.

Theorem) Ricardian Equivalence

The timing of the \( T^i_t \) is irrelevant. (i.e. same equilibrium prices and
allocations)

pf) Obvious since only the present value of transfers appears in the BC.

Remark: That is, you can move $T^i_t$ back and forth in time without changing the equilibrium allocations and prices; the only thing that matters is $(P^1_{t=0} T^1_t, P^1_{t=0} T^2_t, P^1_{t=0} T^3_t, \ldots)$.

Remark. (Stanley wrote Dirk Krueger next to this remark. I think that what that probably means is that he took the following formalization of the above from Dirks class notes, but I'm not sure.)

Take as given a sequence of government spending $(g_t)_{t=0}^1$ and initial debt $B_0$. Suppose that allocations $c^i_t$, prices $p^e_t$ and taxes $T^i_t$, etc. form an Arrow-Debreu equilibrium. Let $\hat{T}^i_{t}$ be an arbitrary alternative tax system satisfying $P^1_{t=0} T^1_t = P^1_{t=0} \hat{T}^i_{t}$ \textit{8 i}.

Then $c^i_t$, $p^e_t$ and $\hat{T}^i_{t}$, etc., form an Arrow-Debreu equilibrium as well.

Remark: A more subtle version of this same result is due to Barro. This is that it does not matter whether you tax father or son. The idea is that if any redistributive taxation you do across generations will be undone through bequests among the different individuals in the family.

Remark: What if there were more than one firm in a sector but all firms within a sector had identical CRS production functions in every period.
Would our earlier aggregation results still hold? What if there were more than one sector, but with identical production functions? Would our earlier aggregation results still hold in this formulation with taxes?

**Remark:** What if all agents have the same homothetic utility function. Would our aggregation results still hold? What about non-linear tax systems (i.e., progressive or regressive systems).

**Remark:** In some ways, this approach to policy is a bit odd. It is what is known as the 'throw it in the ocean' model of government spending. That is, \( g \) does not enter \( U \) (e.g., parks or schools), and it does not enter \( F \) (e.g., roads or bridges). Of course, it would be better to explicitly include those kinds of considerations in the model. It would also be more difficult! So, this formulation is used as a simple 'starter' version. Unfortunately.. often, no one goes beyond this version! It has kind of funny implications for policy:

**Optimal Policy**

What is optimal policy under the assumptions made so far?

i) \( g = 0 \) for all \( t \)

ii) \( c_t = x_t = n_t = k_t = 0 \) for all \( t \)

iii) any desired redistribution can be done through \( T_t \), this follows from
the 2nd welfare theorem, i.e. PO! CE under appropriate transfers. Although given the structure, it's not clear why redistribution would be desirable.

Remark: For your own sanity in thinking about this, it's probably best to either just think of \( g \) as being given outside the model for some reason the government HAS to have \( g \) in each period. Or, you could think about ways to put \( g \) directly into either \( U \) or \( F \) where we implicitly assume that consumer views himself as having no influence on \( g \) and takes it given. If for example, \( g \) enters the utility function of the consumer in and additively separable way, you can check that you will get exactly the same equilibrium relationships as in the model we have outlined above.

11/09/2000

Recall:

Given a fiscal policy \( (\xi_t; \xi_xt; \xi_nt; \xi_kt; T^i_t; g)_{t=0}^1 \)

a TDCE is

i) \( (c^i_t; x^i_t; k^i_t; n^i_t; l^i_t)_{t=0}^1 \) \( i = 1; 2; \ldots; I \)

ii) \( (c^f_t; x^f_t; g^f_t; k^f_t; n^f_t)_{t=0}^1 \)

iii) \( (p_t; r_t; w_t)_{t=0}^1 \)

such that,
1) solves consumers' problem (CP) person $i$ given $(p_t; r_t; w_t)_{t=0}^1$ and 
$$(\ell_{ct}; \ell_{xt}; \ell_{nt}; \ell_{kt}; T_{t}^1; g)_{t=0}^1$$
2) solves firms' problems (FP) given $(p_t; r_t; w_t)_{t=0}^1$ and $(\ell_{ct}; \ell_{xt}; \ell_{nt}; \ell_{kt}; T_{t}^1; g)_{t=0}^1$
3) $S=D$ for all markets
4) Government's budget balance (in PV sense)

This definition allows the model to be solved for 'any' specification of fiscal policy. However, it implicitly assumes that there is an equilibrium. This can't be true in general! For example, suppose spending is positive in every period, but taxes are zero in every period! In that case, there can be no prices... such that all are maximizing and quantities add up. Thus, the assumption that an equilibrium exists implicitly puts some restriction on the combinations of taxes, transfers and spending that the government is doing. There is no simple way of summarizing what this set of restrictions entails. A more general approach allows spending and transfers by the government to be contingent on (i.e., be functions of) the revenue raised. This in turn depends on what prices are in addition to quantities chosen and tax rates. If this function satisfies Budget Balance by the government at all revenue possibilities, then typically an equilibrium will exist. (This requires some additional assumptions.) An easy way to guarantee this is to have transfers
be dependent on tax revenue and spending, so that they always make up the difference between direct tax revenue and spending. Under some further assumptions on \( g \) this is sufficient to guarantee that an equilibrium will exist.

**Examples of Fiscal Policies**

(FP1) What would the behavior of the economy be if \( c_3 = 0.2 \) (i.e., a 20% tax on consumption at period 3), \( c_t = 0 \) \( t \neq 3 \), \( x_t = k_t = n_t = 0 \) \( 8 t \), \( g_t = 0 \), \( T_3^t = c_3 \phi_3^t \)? That is, what would happen if we taxed consumption in period 3, and used the revenue to finance lump sum transfers back to the consumer in the same period? (Note, as above, it doesn't matter if it's \( T_0^t \) due to Ricardian Equivalence.)

(FP2) Is the TDCE for this economy the same as \( c_t = x_t = k_t = n_t = 0 \) \( 8 t \), \( g_t = 0 \), \( T_t^t = 0 \)?

That is, is FP1 the same as a fiscal policy where you do nothing?

Answer) No.

(FOC) \( \frac{U_{i3}^i}{(1 + c_3)w_t} = \frac{U_{i3}^i}{(1 + n_3)w_t} \)

In FP1, \( \frac{U_{i3}^i}{U_{i3}^i} = \frac{(1 + c_3)w_t}{w_t} = \frac{(1 + c_3)w_t}{w_t} = 1.2 \frac{F_{n3}}{F_{n3}} \)  

In FP2, \( \frac{U_{i3}^i}{U_{i3}^i} = \frac{w_t}{w_t} = \frac{1}{F_{n3}} \)
Thus, these cannot be the same since in this case, $F_{n3} = \hat{F}_{n3}$ would hold, and hence $MRS_1 = MRS_2$ would have to hold too. Contradiction =) ( =

That is:

* If you take any stu® from $i$ in one way & give it back in another way, he $i$ doesn't take into account that he gets back the tax revenue, since he is taking tax rates and transfers as fixed in his maximization problem.

* Thus, if the consumer thought perfectly that $T_t^i = \hat{c}_t$, then $\hat{c}_t$ is irrelevant.

For example, you might want to consider what would happen if instead of giving back the revenue as a lump-sum transfer, what happens if you subsidize leisure in a way that is balanced budget in equilibrium? Or:

$$(FP3) \hat{c}_3 = 0.2, \hat{c}_t = \hat{c}_nt = \hat{c}_kt = 0 \; \hat{c}_t \; 0 \; \hat{c}_t \; 0 \; \hat{c}_t \; 6 \; 3; 4, \hat{c}_ct$ is chosen to balance the budget { i.e., tax in period 3 and subsidize in period 4}

Is the equilibrium allocation same as FP2 in this case? NO.

$$\frac{U_{13}}{(1+\hat{c}_3)p_3} = \frac{U_{14}}{(1+\hat{c}_4)p_4}$$

$$\frac{U_{13}}{U_{14}} = \frac{(1+\hat{c}_3)p_1}{(1+\hat{c}_4)p_4}$$

but $(1 + \hat{c}_3) > 1$ and $(1 + \hat{c}_4) < 1.$
Therefore, $\text{LHS} > \frac{p_3}{p_4}$ and hence the allocations must be different.

**Question:** Are TDCE Pareto Optimal? In general, NO. (Look at the FOC, $\text{MRS} = \text{MRT}$ needed for PO)

Of course, we need a definition of PO here, but it is the obvious one:

**Definition:** An allocation $(c_t^i; x_t^i; n_t^i; l_t^i; k_t^i)_{t=0}^1; (c_t^f; x_t^f; n_t^f; g_t^f; k_t^f)_{t=0}^1$ is PO given $(g_{t=0})$ if there does not exist another feasible allocation $(c_t^i; x_t^i; n_t^i; l_t^i; k_t^i)_{t=0}^1; (c_t^f; x_t^f; n_t^f; g_t^f; k_t^f)_{t=0}^1$ given $(g_{t=0})$ such that $U_i (c_t^i; l_t^i) > U_i (c_t^f; l_t^f)$ with strict inequality for at least one $i$ and $g_{t=0} > g_{t=0}$.

As is probably not surprising, there is a tight relationship between PO given $g$ and the use of lump-sum taxes:

**Theorem** If $\ell_{ct} = \ell_{xt} = \ell_{nt} = \ell_{kt} = 0$, (there are no distortionary taxes), then the TDCE allocation is PO. given $g$.

**Proof:** Obvious

**Remark:** This is the analogue of the 1st welfare theorem

**Corollary:** Optimal financing of government expenditures is to use lump sum transfers ($T$) only.

**Remark:** (Alice Schoonbroodt) There are also other ways of getting
allocations that are like lump sum tax allocations given our set up. For example: suppose $\xi_{ct} = \xi_{xt} \land 1 + \xi_{ct} = 1 \land \xi_{nt} = 1 \land \xi_{kt} = 0$. Then, this is the same as having lump sum transfers equal to $\frac{T}{\pi}$.

<Second Welfare Theorem>

Let $(c^i_t; x^i_t; l^i_t; k^i_t; n^i_t)_t \in \mathbb{R}; (c^f_t; x^f_t; g^f_t; k^f_t; n^f_t)_t \in \mathbb{R}$ be PO given $(g^t)_t \in \mathbb{R}$

Then $\exists T^i_t$ s.t.

$(c^i_t; x^i_t; l^i_t; k^i_t)_t \in \mathbb{R}$ is TDCE

given the FP ($\pi$-iscal policy) $(g^t)_t \in \mathbb{R}; (T^i_t)_t \in \mathbb{R}$, $\xi_{ct} = \xi_{xt} = \xi_{nt} = \xi_{kt} = 0$.

Pf) Obvious.

<Solving the Model- Representative Consumer Case>

1. System of equations approach. ! representative consumer (either all identical or homothetic)

2. Planner's problem equivalence? This won't work in general since TDCE is not PO in general! You can if $\xi = 0$, i.e., you only use transfers to raise revenues; with distortionary taxes, this will typically not work.

However, sometimes this works!
Proposition:

(A) Assume that there is a representative consumer and that \( c_t = x_t = 0 \) \( 8 t \) (Only using lump sum tax to finance spending), OR

(B) there is a representative consumer and \( c_t = x_t = T_t = 0 \) \( 8 t \), but, \( n_t = k_t = 0 \) \( 8 t \), and \( p_t = w_t n_t + r_t k_t \) \( 8 t \), i.e., period by period budget balance.

Then the TDCE allocation solves:

\[
\text{Max } U(c; l)
\]
\[
s.t. \quad c_t + x_t \cdot (1 - n_t) F_t(k_t; n_t) \quad 8 t
\]
\[
\quad k_{t+1} \cdot (1 - n_t) k_t + x_t
\]
\[
\quad l_t + n_t \cdot 1 \quad 8 t
\]
\[
\quad k_0 \text{ fixed}
\]

pf) Obvious

Corollary: Under these conditions, the equilibrium allocation is also unique.

pf) Concave maximization problems have unique solution.

Question) If \( g_t = 0 \) instead and \( T_t = c_t(w_t n_t + r_t k_t) \) (lump sum rebate), is the THM still true?
No! Why not? g takes out real goods and services but transfers don't. It follows that feasibility is messed up.

That is, feasibility in the TDCE is
\[ c_t + x_t + g - c_t + x_t = F_t(k_t; n_t) \theta (1 - \xi_t) F_t(k_t; n_t) \]
In the planner's problem, feasibility is:
\[ c_t + x_t = (1 - \xi_t) F(k_t; n_t; t) \]
and thus these are not the same.

If the revenue is used to purchase goods and services however, the feasibility constraint in the planner's problem is
\[ c_t + x_t \cdot (1 - \xi_t) F(k_t; n_t; t) \]
Since \( g \cdot \xi_t F(k_t; n_t; t) \), this is the same as that in the equilibrium:
\[ c_t + x_t + g \cdot F(k_t; n_t; t) \]
\[ c_t + x_t \cdot (1 - \xi_t) F(k_t; n_t; t) \]
Are there other results like this? A few....

Proposition: Suppose there is a representative consumer, and that:

i) \( \xi_{xt} = \xi_{nt} = \xi_{kt} = T^t_i = 0 \) 8 t; i {no transfers,

ii) \( p_t g_t = c_t p_t c_t \)

{consumption tax revenue is spent on \( g_t \) in every period,
then TDCE solves

\[ \text{Max } U(\tilde{c}_t; \tilde{l}_t) \]

s.t. \( x_t \cdot F_t(k_t^x; n_t^x) \leq t \) \( x_t \cdot F_t(k_t^x; n_t^x) \leq t \) \( x_t \cdot F_t(k_t^x; n_t^x) \leq t \) \( x_t \cdot F_t(k_t^x; n_t^x) \leq t \)

\( (1 + \tilde{c}_t) c_t \cdot F_t(k_t^c; n_t^c) \leq t \) \( (1 + \tilde{c}_t) c_t \cdot F_t(k_t^c; n_t^c) \leq t \) \( (1 + \tilde{c}_t) c_t \cdot F_t(k_t^c; n_t^c) \leq t \) \( (1 + \tilde{c}_t) c_t \cdot F_t(k_t^c; n_t^c) \leq t \)

\[ k_{t+1} \cdot (1_i \not= k_t + x_t) \]

\[ n_t^x + n_t^c + l_t \leq 1 \]

\[ k_t^x + k_t^c \leq k_t \]

\[ k_0 \text{ given.} \]

pf) sorta obvious

**Corollary**) Equilibrium is unique.

\( (\Pi) \& (\Pi) \not= (1 + \tilde{c}_t) c_t + x_t \cdot F_t(k_t; n_t) \) (CRS)

**First Order Conditions in General Tax Problems**

\( (CP) \text{ Max } P^{-t} U(c_t; 1_i, n_t) \)

\( \text{s.t. } P_t [(1 + \tilde{c}_t) c_t + (1 + \tilde{x}_t) x_t] \cdot P_t [(1_i \not= n_t) w_t n_t + (1_i \not= k_t) r_t k_t + T_t] \)

99K multiplier \( , \)

\[ k_{t+1} \cdot (1_i \not= k_t + x_t) 99K \text{ multiplier } t_{1_t} \]

(FOC)
² w.r.t. \( c_t \)

\[-^tU_c(t) + \mathcal{P}_t(1 + \xi_c) = 0\]

\[-^tU_c(t) = \mathcal{P}_t(1 + \xi_c)\]

\[\frac{^tU_c(t)}{^0U_c(0)} = \frac{\mathcal{P}_t(1 + \xi_c)}{\mathcal{P}_0(1 + \xi_c)} \quad (\text{wlog let } \mathcal{P}_0 = 1)\]

Note: if \( \xi_c > \xi_{c_0} \) then people artificially consumes less in period \( t \). * tax distorts people's choice. Without tax, \( \frac{\mathcal{M}_U}{\mathcal{M}_U} = \mathcal{P}_t \).

² w.r.t. \( n_t \)

\[i \cdot ^tU_l(t) + \mathcal{P}_t(1 + \xi_n) w_t = 0\]

\[i \cdot ^tU_l(t) = \mathcal{P}_t(1 + \xi_n) w_t\]

\[\frac{^tU_l(t)}{^0U_l(0)} = \frac{w_t(1 + \xi_n)}{\mathcal{P}_0(1 + \xi_n)}\]

² w.r.t. \( x_t \)

\[i \cdot \mathcal{P}_t(1 + \xi_x) + ^t_1 = 0\]

\[^t_1 = \mathcal{P}_t(1 + \xi_x)\]

² w.r.t. \( k_{t+1} \)

\[i \cdot ^t_1 + (1_i \pm)^{^t+1_1}_{t+1} + \mathcal{P}_t(1 + \xi_{kt+1}) r_{t+1} = 0\]

or \( \mathcal{P}_t(1 + \xi_x) = (1_i + \xi_{x_{t+1}})^{^t_1} \mathcal{P}_{t+1} + (1_i + \xi_{kt+1}) r_{t+1}\)

(?)

This last condition is called an Arbitrage condition it constrains how prices and taxes must move together in any TDCE.
If there were no tax, this would be, $p_t = (1 \ i \ \pm b) p_{t+1} + r_{t+1}$

or $p_t = p_{t+1} \ i \ \pm r_{t+1} = p_{t+1} (1 \ i \ \pm F_k (t+1))$

and then $-U_t(t) / U(t)(0) = (1 \ i \ \pm F_k (t+1))$

thus (?) is tax-distorted version of Euler equation

Recall that investment is CRS, and hence, (?) is the condition that implies that the after-tax profits from investment is zero (no quantities involved).

**Euler equation for Tax Model**

$$\frac{U_k(t)(1+\xi t)}{(1+\xi ct)} = \frac{-U_k(t+1)}{(1+\xi ct+1)} \ h (1 \ i \ \pm (1 + \xi xt + 1) + (1 \ i \ \xi kt + 1) \ r_{t+1})$$

Remark: This is a necessary condition for an interior solution. Are they sufficient? Need transversality condition. BC holding with equality.

(Firm's problem)

Max $p_t F_t(k_t; n_t) + w_t n_t \ i \ r_t k_t$

$F_k(t) = \frac{L_t}{p_t}$

$F_n(t) = \frac{w_t}{p_t}$

So a TDCE is $(p_t; w_t; r_t); (c_t; x_t; n_t; k_t)$ such that

i) $8 t \ p_t (1 + \xi c_t) = \frac{-U_t(t)}{U(t)(0)} \ ; \ p_0 = 1$
ii) \( F_k(t) = \frac{r_k}{p_k} \cdot 8t \)

iii) \( F_n(t) = \frac{w_n}{p_n} \cdot 8t \)

iv) \( \frac{U_k(t)}{U(t)} = \frac{(1 + îc t)}{(1 + îc t)} F_n(t) \cdot 8t \)

v) \( \frac{U_n(t)(1 + îx t)}{(1 + îc t)} = \frac{-U_k(t+1)}{(1 + îc t+1)} (1 i î > (1 + îx t+1) + (1 i î k t + 1) \frac{r_k}{p_k+1} \cdot 8t \)

vi) \( c_t + x_t + g = F_t(k_t; n_t) \cdot 8t \)

vii) \( p \cdot [(1 + îc t) c_t + (1 + îx t) x_t] = p \cdot [(1 i î c t) w_t n_t + (1 i î k t) r_t k_t + T_t] \cdot 8t \)

viii) \( k_{t+1} \cdot (1 i î k_t + x_t) \cdot 8t \)

This is the system of equations that has to be solved.

What happened to Budget balance by government? As noted above, it automatically follows from the other conditions.

\[
\begin{align*}
\text{\( p_1 \cdot \prod_{t=0}^{3} \text{p}_t \cdot \prod_{i=1}^{3} \text{q}_t + \prod_{i=1}^{3} \text{x}_t + \quad \prod_{i=1}^{3} \text{w}_t + \quad \prod_{i=1}^{3} \text{r}_t + \quad \prod_{i=1}^{3} \text{k}_t \)} &= p \cdot \prod_{t=0}^{3} \text{h}_t \cdot \prod_{i=1}^{3} \text{t}_t + \prod_{i=1}^{3} \text{g}_t
\end{align*}
\]

< Steady State with and without taxes >

Suppose \( îc \neq îc \)

\( îc t \neq îc \)

\( îc t \neq îc \)

\( îc t \neq îc \)

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¿kt! $\xi_k^2$

$\alpha_k!g^2$

Can we characterize the steady state?

What should $c^2; n^2; k^2; x^2$ have to satisfy?

\[
i\mathbf{Q} \frac{U_i(c^2; n^2)}{U_k(c^2; n^2)} = \frac{1}{(1 + \xi^2)} \mathbf{F}_n(k^2; n^2)
\]

\[
i\mathbf{Q}^{-1} \frac{1 + \xi^2}{1 + \xi^2} = [1 i (1 + \xi^2) + (1 i \xi_k^2) \mathbf{F}_k(k^2; n^2)] ( ) (1 + \xi^2) =
\]

\[
[i 1 (1 + \xi^2) + (1 i \xi_k^2) \mathbf{F}_k(k^2; n^2)]
\]

\[
iii\mathbf{Q} c^2 + x^2 + g^2 = F^2 k^2; n^2 \xi^2
\]

\[
iv\mathbf{Q} k^2 \cdot (1 i (1 + k^2 + x^2) ( ) \quad x^2 = \pm k^2
\]

Four unknowns ($c^2; n^2; k^2; x^2$) and four equations $i\mathbf{Q}; ii\mathbf{Q}; iii\mathbf{Q}; iv\mathbf{Q}$

Note: $\xi^2_c$ doesn't appear in $ii\mathbf{Q}$.

A) At Steady State (SS) $\frac{k^2}{n^2}$ doesn't depend on $\xi^2_c$ or $\xi^2_n$ (only on $\xi^2_k; \xi^2_k; \pm$)

That is, different countries with different $\xi^2_c$ or $\xi^2_n$ will still have same $\frac{k^2}{n^2}$.

If $n^2 = 1$ (inelastic labor supply with $U_i = 0$)

then

1) $i\mathbf{Q}$ disappears

2) only equations $ii\mathbf{Q} \gg iv\mathbf{Q}$ will determine the SS
Figure 4:

3) since $F^i_k; n = F^i_k; 1$ depends on $\xi_k, \zeta_k, \pm$, $k'$, $k''$

! enables comparative statics

What is $\frac{\partial k}{\partial \xi}$?

$$1 = \left(1 - \frac{1}{(1+\xi_k^2)}\right) F_k(k; 1)$$

or $\frac{i}{\partial_i} (1 - \frac{1}{(1+\xi_k^2)} = F_k(k; 1) \frac{1}{i} = f^0(k)$.

Thus, $\frac{\partial k}{\partial \xi} < 0$. Similarly, $\frac{\partial k}{\partial \xi} < 0$.

Since $x^2 = \pm k^2$; it follows that we have the same signs for $\frac{\partial k}{\partial \xi}$ and $\frac{\partial k}{\partial \xi}$.

What about $\frac{\partial c}{\partial k}$? $\frac{\partial^2 c}{\partial k^2} = f^0(k) \frac{\partial k}{\partial k}$.

Remark: In a sense, this formulation of the problem has too many taxes. That is:
Show that given any TDCE with $T_t = 0$ but $\xi_{ct}; \xi_{nt}; \xi_{xt}; \xi_{kt} > 0$ (or any subset....) there is another tax system

$\xi_{ct}; \xi_{nt}; \xi_{xt}; \xi_{kt}$ with $\xi_{ct} = \xi_{xt} = 0$ but the same TDCE allocation. That is, you can support the same allocation through a tax system in which consumption and investment taxes are zero. In this sense, these taxes are redundant.

Note: This does not say that $\xi_{kt}; \xi_{nt} > 0; \xi_{xt} = 0; \xi_{ct} = 0$ supports the same allocation. That is, you may have to adjust $\xi_{kt}; \xi_{nt}$ (to $\xi_{nt}; \xi_{kt}$) to support the same allocation.

Remark: Can you think of other versions of this?

Remark: How would you include a provision for depreciation allowances in the tax code?

$Taxes_t = \xi_{ct} p_c c_t + \xi_{xt} p_x x_t + \xi_{nt} w_t n_t + \xi_{kt} (r_t k_t i \pm k_t)$

with the BC then being:

$P(p_c c_t + p_x x_t) \cdot P(r_t k_t + w_t n_t + T_t) \cdot Taxes_t$

Remark: How would you include progressive (or regressive) tax systems?

$Taxes_t = \xi_t (w_t; r_t; n_t; k_t)$
with the budget constraint being:

\[ P (p_t c_t + p_t x_t) \cdot P (r_t k_t + w_t n_t + T_t i T \text{axes}_t). \]

**What should taxes be?**

For a given streams of expenditures \( g_t \) would consumers be better off under

System A) Choose \( c_t \) so that \( p_t g_t = \langle c_t p_t c_t \rangle \)

OR

System B) Choose \( n_t \) so that \( p_t g_t = \langle n_t w_t n_t \rangle \).

Note that in both of these systems would require that the relevant prices (resp. wages) would depend also depend on the tax code.

**Remark:** At this point, it is not even obvious that we can find such a system? That is, the first question is: When can I find a system of \( c_t, n_t, \) etc., such that the there is an equilibrium supporting the given sequence of government expenditures, \( g_t \) (and hence, in particular, such that the government budget balances)?

More generally, what should \( c \) be?

**Ramsey Problems**
This is the name given to a class of optimal policy problems:

Maximize utility of consumers given revenue requirements and instrument availability."

That is, the Ramsey Problem is to choose tax rates to maximize the welfare of the representative agent subject to the constraints that the government budget be balanced in PV in the resulting CE.

This comes from a classic paper by Ramsey (1928). Ramsey took $g$ as given and asked what combination of excise taxes (taxes on consumption goods) should be used to finance a given level of expenditures, $g$. He phrased this as maximizing Consumer's Surplus and found this by integrating under
the demand curves:

Max $CS(\xi)$

s.t. $\sum_{i=1}^{n} \xi_i q_i = R$  i.e., the tax revenue from consumption good $i = 1; \ldots; n$ covers the required Revenue, $R$.

(Mechanism) Pick $\xi^0$ ! CE given $\xi$ ! Revenue raised $(R(\xi))$ $CS$ obtained $(CS(\xi))$

The more modern version of this problem is stated as:

Given any set of taxes, $\xi_i$, Consumers solve

Max $U(q_1; \ldots; q_n)$
s.t. \[ P \cdot (1 + \xi_i) \cdot p \cdot q_i \cdot W \]

**Remark:** In the TA session Mike (Golosov) showed: If in the case \[ P \cdot (1 + \xi_i) \cdot p \cdot q_i \cdot W \cdot T, \] i.e., lump sum taxation is allowed, then solution is \[ \xi_i = 0 \& T = R. \] If \( T = 0 \) is assumed, then solution is \[ \xi_i = \xi \] for \( i \) where \[ W_i \cdot \frac{w_i}{\xi_i} = R. \] Also BC becomes \[ P \cdot q_i \cdot \frac{W}{(1 + \xi_i)} \] and this is equivalent to lump sum taxes.

Ramsey solved

\[ \text{Max } U(q_1(\xi), q_2(\xi), \ldots, q_n(\xi)) \]

\[ \xi = (\xi_1, \xi_2, \ldots, \xi_n; T) \]

s.t. i) \[ \sum q(\xi) \xi_i + T = R \]

ii) \[ T = 0 \]

iii) \[ \xi_1 = 0 \]

Result) Tax goods according to their elasticity of demand. That is, low "! high \( \xi. \)

**Remark:** Ramsey did not really have a tight justification for not allowing lump sum taxation, or assuming that \( \xi_1 = 0. \) The only reason is that if you don't make these restrictions, the solution will be rather simple, use lump sum taxes, or the equivalent. He viewed this as unrealistic, but gave no
formal justification. The modern solution to this problem would be to assume that there is private information about earning abilities. This approach was pioneered by Mirlees. Including this at this point would complicate matters considerably and hence we won't do it here. It's also not true that this more complex approach is equivalent to disallowing lump sum taxes, and assuming that $\zeta_1 = 0$.

<Macro Version>

(Ramsey problem, Ramsey planner)

Choose $\zeta = (\zeta_{kt}; \zeta_{nt})_{t=0}^{1}$

to maximize $U(\zeta) = \frac{c(\zeta)}{l(\zeta)}$

s.t. $c(\zeta)$; $l(\zeta)$ is a TDCE allocation for the economy with tax system $\zeta$ and $g_{0}$; $\cdots$ given.

Remark: As in Ramsey, we will not allow lump sum taxes. Also, we will assume that $\zeta_{kt} \leq 1$ for $t$. The reason for this assumption is two fold. First, if we allow $\zeta_{k0} > 1$, this is equivalent to lump sum taxation. Second, if we do not assume this, it is questionable that individual household supply of capital is equal to the stock that they have on hand.

<Solving the Ramsey Problem>
Step 1) Characterize the set of $\hat{\eta}$'s that raise enough revenue in equilibrium, and the resulting utility.

\[
\mu \begin{array}{c}
\| \end{array}
\]

Step 2) Maximize $U_{c}(\hat{\eta})$.

If $(p_t; r_t; w_t) (c_t; x_t; k_t; n_t; l_t)$ is a TDCE with tax system $\hat{\eta}$ which runs $g$.

Step 1:

From the Firm's problem, we have:

i) $F_k(k_t; n_t) = \frac{r_t p_t}{p_t} \ t$

ii) $F_n(k_t; n_t) = \frac{w_t}{p_t} \ t$

iii) $c_t + x_t + g = F_t(k_t; n_t)$

From the Consumer's problem, we have:

i) $p_t = \frac{-\bar{U}_c(t)}{U_c(0)}$

ii) $\frac{U_c(t)}{U_c(0)} = (1 \ i \ \hat{\eta}_t n_t) \ \frac{w_t}{p_t}$

iii) $U_c(t) = \bar{U}_c(t + 1) [(1 \ i \ \hat{\eta}_t + 1 \ i \ \hat{\eta}_{k_t+1}) F_{k_t}(k_{t+1}; n_{t+1})] \ t$

iv) $k_{t+1} \cdot (1 \ i \ \hat{\eta}_t k_t + x_t$

v) $P [p_t^i (1 \ i \ \hat{\eta}_t n_t) w_t n_t] = P [(1 \ i \ \hat{\eta}_k) r_t k_t i \ p_t (k_{t+1} i \ (1 \ i \ \hat{\eta}_t))]$

(RHS) $= (1 \ i \ \hat{\eta}_k) r_k k_t + (1 \ i \ \hat{\eta}_k) \ k_0 + P \ \frac{1}{t=1} k_t [p_t (1 \ i \ \hat{\eta}_k) F_{k_t}(0)] i \ p_t i \ 1$

(LHS) $= P \ \frac{-1}{U_c(0)} [U_c(t) c_t i \ U_i(t) n_t]$

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Thus, v) becomes

Implementability Condition

\[
[k_0 (1_i \neq k_0) F_k (0) + (1_i \neq k_0) U_c (0) = \sum_{t \in T} (U_c (t) c_{i, t} U_i (t) n)]
\]

Conversely, if \((c_t; x_t; k_t; n_t; l_t)\) satisfy

FP 3 \(c_t + x_r + g \cdot F_t (k_t; n_t) \geq t\)

CP 4 \(k_{t+1} \cdot (1_i \neq k_t) + x_t \geq t\)

and CP 5 \{ The Implementability condition above written purely in terms of quantities

Then, \(9 (\xi_n t; \xi_k t) ; (p_t; r_t; w_t) \) s.t.

\((p_t; r_t; w_t) \& (c_t; x_t; k_t; n_t; l_t)\) is a TDCE for the policy

\([\xi_n t; \xi_k t]) \cdot g^T_{t=0} :\)

That is, pick any quantities that satisfy the feasibility conditions and the implementability condition, CP5 and you can construct a system of taxes, such that the given allocation is a TDCE allocation given those taxes.

\(^2\) Thus, the RP is equivalent to:

\[
\text{Max } U \sum_{\xi} \mu(\xi) c(\xi); l(\xi)
\]
s.t. \( c_t + x_t + g \cdot F_t(k_t; n_t) \)

\[ k_{t+1} \cdot (1 \; \; \; \pm k_t + x_t \]

CP 5! \[ [k_0(1 \; \; \; \pm k_0) F_k (0) + (1 \; \; \; \pm k_0) U_c (0) = P^{-t}[U_c(t) \; \; \; \; 0 \; \; \; \; U_l(t) \; \; \; n_t] \]

That is,

Proposition:

\((\rho_t^2; r_t^2; w_t^2)\) and \((c_t^2; n_t^2; k_t^2; x_t^2)\) is a TDE with taxes \((\zeta_{nt}; \zeta_{kt})\) supporting \(g\):

\[ \]

\(i) \; F_{kt}(k_t^2; n_t^2) = \frac{r_t^2}{P_t} \]

\(ii) \; F_{nt}(k_t^2; n_t^2) = \frac{w_t^2}{P_t} \]

\(iii) \; p_t^2 = \frac{U_c(t)}{U_c(0)} - t \]

\(iv) \frac{U_c(t)}{U_c(0)} = (1 \; \; \; \pm \zeta_{nt}) \frac{w_t^2}{P_t} \]

\(v) \; U_c(t) = \bar{U_c}(t + 1) [(1 \; \; \; \pm \zeta_{kt+1}) F_k(t + 1)] \]

\(vi) \; k_{t+1} \cdot (1 \; \; \; \pm k_t^2 + x_t^2 \]

\(vii) \; c_t^2 + x_t^2 + g_t^2 \cdot F_t(k_t^2; n_t^2) \]

\(viii) \; U_c(0) k_0 [(1 \; \; \; \pm k_0) F_k(0; 0) + (1 \; \; \; \pm k_0) = P^{-t}[U_c(t) \; \; \; \; 0 \; \; \; \; U_l(t) \; \; \; n_t] \]

Remark: You can think of this as saying that \(i) \sim v)\) "determine" \((\rho_t^2; r_t^2; w_t^2)\)

\& \((\zeta_{nt}; \zeta_{kt})\) from an allocation determined by \(vi) \sim viii)\). Note that \(vi) \sim viii)\) de-
pend on quantities only.

From i)~ii), it follows that Firms are maximizing.

From iii)~v), and vii) Consumers are maximizing, assuming the solution is interior.

vi) and vii) are accounting identities. They merely make sure that physical feasibility is satisfied.

viii) is the Implementability constraint. This is what differentiates tax distorted equilibria from other feasible allocations. This is also what makes this problem different from standard growth model without distortions.

Ramsey planner's problem:

\[(RP\ I)\]
\[
\begin{align*}
\max_{\mu, \eta} \quad & U_c(\ell) ; l(\ell) \\
s.t. \quad & c(\ell) ; l(\ell) \text{ is the TDCE allocation given } \ell.
\end{align*}
\]

RP I is equivalent to RP II.

\[(RP\ II)\]
\[
\begin{align*}
\max_{c,n,k} \quad & U_c c ; l \\
s.t. \quad & (RPA) \quad c_t + x_t + g_t = F_t(k_t, n_t) \\
\quad & (RPB) \quad k_{t+1} \cdot (1_i - \delta) k_t + x_t
\end{align*}
\]
Feasible allocations

Figure 7:

\[(RPC)^P \sum_{t=0}^{\infty} [U_c(t) c_t \cdot U_l(t) n_t] = U_c(0) k_0 [1 + (1 - \ell_k) F_k(0) + (1 - \ell)]\]

**Remark:**. *RPC* is real version of *BC* after substitution. Implementability constraint. \(\ell_{k0}\) is the tax rate on initial capital. It is equivalent to a lump sum tax. Because of this, it is typically assumed that \(\ell_{k0} = 1\) (or 0):

Let \(\lambda\) denote the multiplier on *RPC* in this maximization problem.

Then, the Lagrangian is (letting \(\ell_{k0} = 1\))

\[P^{-t} U(c_t, \cdot, n_t) + \lambda U_c(0) k_0 (1 + \ell) + P^{-t} [U_c(t) c_t \cdot U_l(t) n_t] + \text{other terms}\]
\( U(c_0; 1_i, n_0) + \frac{1}{t} \sum_{i=1}^{\infty} [U(c; 1_i - n_i) + U(c; 1_i n_i) + Ui(0)] + \text{other terms.} \)

Let \( V(c; n; \lambda) = U(c; 1_i - n_i) + U(c; 1_i n_i) + Ui(0) \). Then, define

\[
W_0(c_0; n_0; k_0; \lambda) = U(c_0; 1_i - n_0) + U(c_0; 1_i n_0) + Ui(0) + \text{other terms.}
\]

Thus, \( \lambda \wedge V(c; 1_i n_i) = W_0(c_0; n_0; k_0; \lambda) + \frac{1}{t} \sum_{i=1}^{\infty} V(c; n_i + 1) \).

Thus, we can rewrite the Ramsey Problem as:

\[
\text{Max}_{c^*; x^*; k^*; n^*; \lambda^*} \quad V(c^*; \lambda^*)
\]

s.t. \( c_t + x_t + g_t = F_t(k_t; n_t; t) \)

\( k_{t+1} = (1_i - \Delta) k_t + x_t \)

\( \text{RP III} \) is a standard one-sector growth model where period utility is \( V(c; n; \lambda) \) not \( U(c; 1_i n_i) \). And of course, \( \lambda \) is endogenous.

It follows from the standard reasoning that the solution to \( \text{RP III} \) satisfies:

i) \( \frac{V(t)}{V(c(t))} = F_n(t) \quad t = 1, 2, \ldots \)

ii) \( V(c(t)) = V(c(t+1)) (1_i - \Delta) F_k(t+1) \)
iii) $k_{t+1} \cdot (1 \pm k_t + x_t$

iv) $c_t + x_t + g = F_t(k_t; n_t)$

In what follows, we will assume that the production function does not depend on time, $F_t = F$.

Let $\hat{c}^R_P; x^R_P; k^R_P; n^R_P$ solve RP III.

Assume $c^R_t^P \neq c^R_P$

$x^R_t \neq x^R_P$

$k^R_t \neq k^R_P$

$n^R_t \neq n^R_P$

so that the solution to this problem converges to a steady state.

Note: we also know that $g$ also converges to a constant in this case.

(Steady State)

i') $\frac{V^R_P}{V^R_x} = F_1 i^R_k; n^R_P$

ii') $1 = -i_1 \pm + F_k i^R_k; n^R_P$ (rmk. Describes after tax savings)

iii') $x^R = \pm^R_k$

iv') $c^R + x^R + g^* = F_1 i^R_k; n^R_P$

Recall that if $(c^n_t, x^n_t; k^n_t, n^n_t)$ is a T DCE allocation supporting $g$, it satisfies

(Euler equation) $U_c(c_t^P; 1_i; n_t^P) = -U_c i^R_t 1_i 1 i \pm \cdot (1 + \pm(x_{k+1}) F_k i^R_k; n^R_t^P)$
Hence, \( U_t \left( c_t^{RP}; 1 \right) n_t^{RP} \epsilon = -U_t \left( c_t^{RP}; 1 \right) n_t^{RP} \epsilon_f \left( 1 \right) \pm t \alpha \left( 1 \right) 1 + \epsilon_k^{RP} \epsilon F_k i k_t^{RP}; n_t^{RP} \epsilon \alpha \)

If \( t \neq 1 \), then \( 1 = -\epsilon_f \left( 1 \right) \pm t \alpha \left( 1 \right) 1 + \epsilon_k^{RP} \epsilon F_k i k_t^{RP}; n_t^{RP} \epsilon \alpha \).

This, along with ii)' implies \( \epsilon_{RP} \epsilon \left( \right) = 0 \).

That is, in the limit, the tax rate on capital income is zero.

What about limiting tax on labor?

\[
\frac{U_t(c_t^{RP}; 1; n_t^{RP})}{U_t(c_t^{RP}; 1; n_t^{RP})} = i \alpha \left( 1 \right) F_t i k_t^{RP}; n_t^{RP} \epsilon
\]

Comparing with i')}, we see that \( \epsilon_{RP} \epsilon \left( \right) = 0 \) as long as \( \alpha > 0 \).

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\[
\text{Max } \mathcal{V} c_t^{RP} = W_0(c_0; n_0; k_0; \alpha) + P \sum_{t=1}^{\infty} \mathcal{V} (c_t; 1; n_t; \alpha)
\]

s.t. \( c_t + x_t = F_t(k_t; n_t) \); \( \alpha \)

\( k_{t+1} \cdot (1 \pm) k_t + x_t \)

\( \mathcal{V} (c_t; 1; n_t; \alpha) = U(c_t; 1; n_t; \alpha) [U_t(c_t; 1; n_t) c_t U_t(c_t; 1; n_t)n] \)

< Labor income taxes >

From \( \frac{V_t(c_t^{RP}; 1; n_t^{RP}; \alpha)}{V_t(c_t^{RP}; 1; n_t^{RP}; \alpha)} = F_t i k_t^{RP}; n_t^{RP} \epsilon \), in T DCE, we see that
Combining the two,

\[
\frac{V_1(c^{RP}; n_{RP}^{RP}; \cdot)}{V_c(c^{RP}; n_{RP}^{RP}; \cdot)} = \frac{1}{1 - \xi_{nt}^{RP}} \frac{U_1(c^{RP}; l_{RP}; n_{RP}; \cdot)}{U_c(c^{RP}; l_{RP}; n_{RP}; \cdot)}
\]

From (5), we see that

\[
\frac{U_1(c^{RP}; l_{RP}; n_{RP}; \cdot)}{U_c(c^{RP}; l_{RP}; n_{RP}; \cdot)} = \frac{1}{1 - \xi_{nt}^{RP}} \frac{U_1(c^{RP}; l_{RP}; n_{RP}; \cdot)}{U_c(c^{RP}; l_{RP}; n_{RP}; \cdot)}
\]

It follows that if \( \xi = 0 \), then \( \xi_{nt}^{RP} = 0 \):

Remark: What is the interpretation of \( \xi \)? It is the multiplier on the BC in Ramsey problem.

\( \xi = 0 \), BC in RP is \"slack\", i.e. using distortionary taxes to "nance g has no welfare e®ects. This means that can drop this constraint and the solution would be unchanged. But, if you drop that constraint, the resulting problem has only feasibility. This means that in this case, the solution is the same as if you had lump sum taxes at your disposal.

Referring back to the original RP,

\[
\underset{c_t, x_t, k_t}{\text{Max}} U(c_t, l_t, n_t) \quad s.t. \quad c_t + x_t = F_t(k_t, n_t) \quad k_{t+1} = (1 + \gamma) k_t + x_t
\]
\[ P^{-t} [U_c(t) \xi_i U_i(t) n_t] = U_c(0) k_0 [(1 \xi_i \xi_{k_0}) F_k(0) + (1 \xi_i \xi_{k_0}) \text{multiplier } \xi_i] \]

Here, \( \xi_i \) is welfare cost of distortionary tax instead of lump sum tax which is more optimal.

\[ \xi_{kt}^{RP} = 0 \quad \text{and} \quad \xi_{nt}^{RP} > 0 \]

Remark: For early t's the tax rate on capital income is 100%, but it decreases over the transition period, and in the limit goes to zero.

\[
\begin{align*}
\text{Revt} &= \xi_{kt}^{RP} r_{kt}^{RP} k_{kt}^{RP} + \xi_{nt}^{RP} w_{kt}^{RP} n_{kt}^{RP} \\
\text{Revnt} &= \xi_{nt}^{RP} w_{nt}^{RP} n_{nt}^{RP}
\end{align*}
\]

(Special Case)

If \( U_i = 0 \) (inelastic labor supply), \( n_t = 1 \) for all \( t \) (as long as \( \xi_{nt} \cdot 1 \)), there is no distortion, but still generates revenue.

C.f. Alaska or Arab oil countries - Negative tax, and use revenues from oil instead to finance g

<Some Related Problems>

A) Show that the RP with \( \xi_{kt} \) and \( \xi_{nt} \) is equivalent to the one with \( \xi_{ct} \) and \( \xi_{nt} \).
Figure 8:
Figure 9:

Note that $\varepsilon_{nt}$ will not necessarily be the same in the two formulations.

B) Show that the TDCE CE with $\varepsilon_{nt}$ and $\varepsilon_{kt}$ is implemented with a sequence with $\varepsilon_{kt} \neq 0$, the TDCE is implemented with $\varepsilon_{nt}$ and $\varepsilon_{ct}$ such that $\varepsilon_{ct} \neq \varepsilon_{c1} < 1$.

C) Show that if we had formulated the Ramsey problem in terms of $\varepsilon_{nt}$ and $\varepsilon_{ct}$, we would have concluded that:

$\varepsilon_{nt} \neq \varepsilon_{n1} < 1$ and $\varepsilon_{ct} \neq \varepsilon_{c1} < 1$.

D) Show that the TDCE CE with $\varepsilon_{nt}$ and $\varepsilon_{kt}$ is implemented with a sequence with $\varepsilon_{kt} \neq \varepsilon_{k1} > 0$
the TDCE is implemented with \( \zeta_{nt} \) and \( \zeta_{ca} \) such that \( \zeta_{ca} \neq 1 \).