Problem Set #5

Econ 8105-8106
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1. Consider the following optimal growth problem:

\[
\max_{\{c_t, k_{t+1}, x_t\}_{t=0,1,...}} \sum_{t=0}^{\infty} \beta^t \log c_t
\]

subject to:

\[
\begin{align*}
  c_t + x_t & \leq A k_t^\alpha \\
  k_{t+1} & \leq k_t (1 - \delta) + x_t \\
  c_t, k_{t+1} & \geq 0 \\
  k_0 & \text{ given.}
\end{align*}
\]

where \(0 < \alpha < 1, 0 < \beta < 1, \) and \(0 < \delta < 1.\)

(a) Use the first-order conditions of this problem to write an equation that a solution to the problem must satisfy, relating the loss and benefit from giving up a small amount of consumption in period \(t\) for consumption in period \(t+1\). (The Euler equation).

(b) Write a condition stating that, as time goes to infinity, the "value" of the capital stock, in terms of the discounted utility of current consumption, goes to zero. (The transversality condition).

(c) Define a steady-state for this problem. Use (a) to calculate the steady-state values of all variables.

Now suppose \(\delta = 1.\)

(d) Write the functional equation (Bellman’s equation) for this problem. Guess that the value function has the form \(v(k) = B_0 + B_1 \log k,\) and calculate \(B_0, B_1,\) and the optimal policy function \(g(k).\) Show that the sequence defined recursively by \(k_{t+1} = g(k_t)\) satisfies the Euler equation and transversality condition, and that, given any positive value for \(k_0,\) this sequence converges to the steady-state you found in part (c).

(e) Log-linearize the Euler equation and the feasibility constraint of this problem around the steady state. Guess that the optimal decisions for \(c_t\) and \(k_{t+1}\) take the form

\[
\begin{align*}
  \log c_t & = \gamma_c + \psi_c \log k_t \\
  \log k_{t+1} & = \gamma_k + \psi_k \log k_t
\end{align*}
\]

and solve for the coefficients \(\gamma_c, \psi_c, \gamma_k,\) and \(\psi_k.\)
2. Consider the problem in 1. with the following parameters:
\( \alpha = 0.3, \beta = 0.6, \delta = 1, A = 20. \)

(a) Take the discrete grid \( X = \{2, 4, 6, 8, 10\} \) for the values of the capital stock. Consider the procedure of directly iterating on the functional equation,
\[
v_{n+1}(k) = \max_{k' \in X} \left\{ \log(20k^{0.3} - k') + (0.6)v_n(k') \right\}
\]
with the initial guess \( v_0(k) = 0 \) for all \( k \). Calculate \( v_1 \) and \( v_2 \) by hand.

(b) Now consider the (much finer) grid \( X = \{0.05, 0.10, 0.15, \ldots, 9.90, 9.95, 10\} \). Use a computer to perform value function iteration. Perform the iteration until \( \max_{k \in X} |v_{n+1}(k) - v_n(k)| < 10^{-5} \), and report the resulting value function and policy function. Compare these functions to your answer in 1.(d) above. Use the policy function to calculate \( k_t \) and \( c_t \) for the first 25 periods, for the given initial condition \( k_0 = 1.00 \). Do the same for \( k_0 = 9.00 \) (Note that with the approximate policy function as calculated, \( k_t \) must be in the grid \( X \), but not \( c_t \).)

(c) Now, let \( \delta = 0.5 \). Repeat the computation in part (b) (Note: there is now no analytical solution for comparison)

3. Consider the following optimal growth problem:
\[
\max \sum_{t=0}^{\infty} \beta^t \log \left( \frac{C_t}{N_t} \right)
\]
subject to :
\[
C_t + K_{t+1} - (1 - \delta)K_t \leq (\gamma^{1-\alpha})^t AK_t^\alpha N_t^{1-\alpha}
\]
\( C_t, K_t \geq 0 \)
\( K_0 \) given, \( N_0 \) given
\( N_t = \eta t N_0 \)

Where \( C_t \) is aggregate consumption, \( N_t \) is population, which grows constantly at the rate \( \eta \), \( K_t \) is aggregate capital stock.

The parameters of this problem are \( \beta \), the discount rate, \( \delta \), the depreciation rate, \( \gamma \) and \( \eta \), the exogenously specified growth rates, and \( A \) and \( \alpha \), the parameters on the production function.

(a) Write the Euler equation for this problem.
(b) A \textit{balanced growth path} for this problem is defined as a solution in which the variables \( \frac{C}{N_t} \) (consumption per capita), \( \frac{K}{N_t} \) (capital stock per capita), and \( \frac{Y}{N_t} \) (output per capita), all grow at constant rates (though they may not be the same). Show that in a balanced growth path for this problem, the following are true:

1. \( \frac{C_{t+1}}{N_{t+1}}, \frac{K_{t+1}}{N_{t+1}}, \) and \( \frac{Y_{t+1}}{N_{t+1}} \), all grow at the constant (gross) rate \( \gamma \) - that is, for example, \( \frac{C_{t+1}}{C_t} = \gamma \) for all \( t \).
2. The real interest rate \( r_t = \delta \), the factor shares of output, \( \frac{w_t}{Y_t} \) and \( \frac{r_t}{K_t} \), and the capital-output ratio, \( \frac{K_t}{Y_t} \), are all constant. (\( r_t \) is the marginal product of capital and \( w_t \) is the marginal product of labor in the production function.)
3. The variable \( \frac{K_t}{Y_t} \) (capital-output ratio) is constant.

(c) Calibrate the parameters of this model by using data for a country other than the Netherlands, from the past 30 or 40 years, assuming that country was in a balanced growth path of this model during that time.