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Abstract

Consider an economy in which a fixed supply of unskilled labor can be combined with knowledge capital to produce consumption. The technology for accumulating knowledge capital is linear in knowledge capital. This leads to long-term growth if the production function for consumption goods is approximately Cobb-Douglas for large values of the stock of knowledge capital. The quality-ladder economy of Boldrin and Levine [2010] generates a menu of Leontief technologies with this feature. If the initial capital stock is low, there can be a long period of stagnation before unskilled wages start to grow, as in Lewis [1954]. A small open economy with a sufficiently low initial capital stock will run a trade surplus during its initial stages of development.

1. Introduction

This paper describes a model of endogenous growth with a fixed factor. It can be viewed as a smooth version of the quality-ladder model developed by Boldrin and Levine [2010]. These authors describe an economy with a sequence of Leontief technologies that can be used to combine knowledge capital and unskilled labor to produce final consumption. Unskilled labor is a fixed factor and there is a linear technology for accumulating knowledge capital. The equilibrium allocation exhibits cycles as well as long-run growth.

Technically, the smooth economy presented here is a special case of Lucas [1988] in which the physical capital stock is fixed (here, unskilled labor) and only human capital

*Work in progress. Comments welcome.

(here, knowledge capital) can be accumulated. For most production functions, such an economy would not generate long-run growth with a stable factor shares. This paper shows that the Leontief technologies described in Boldrin and Levine [2010] generate an approximate Cobb-Douglas technology for consumption.[1] Because of this, the economy does exhibit long-run growth with stable long-run factor shares. An attractive feature of this economy is that one does not have to account for long-run growth by having the effective per-capita supply of unskilled labor grow exponentially.

Augmented with a linear (unskilled) labor-only technology, the economy exhibits a potentially interesting "Malthus-to-Solow" transition. Starting from low levels of knowledge capital, consumption is initially constant as knowledge capital is accumulated at its maximal rate. As the price of knowledge capital declines, this is followed by a transitory phase of rapid consumption growth, and then by slower but long-term growth in consumption. In contrast to the Malthus-to-Solow model of Hansen and Prescott [2002], this does not rely on unexplained technological progress. Permanent stagnation is also a possible outcome, if knowledge capital cannot be accumulated sufficiently fast. An improvement in this accumulation technology can trigger a growth transition in consumption that will occur only after a possibly extended delay.

This growth transition is also reminiscent of the Lewis [1954] account of development with unlimited supplies of labor. Here the supply is not unlimited, but there is a sector of the economy, initially very large, from which unskilled labor can be re-allocated to the sector in which knowledge capital is used to produce consumption. Wages of unskilled labor are determined by the linear unskilled labor-only technology of this sector as long as the sector is active. This will be the case as long as the economy has not reached its balanced growth path. If capital is interpreted as knowledge capital embodied in skilled workers, then the wages of skilled workers will pull ahead relative to the wages of unskilled workers during the initial stages of development.

A key feature of this economy is the fact that the price of capital in units of consumption is declining over time. As more and more capital is used in combination with the fixed labor factor, the rental price of capital has to decline. The rapidly growing stock of capital combined with a declining relative price of capital is of course familiar from Greenwood, Hercowitz and Krusell [1997]. But there growth was the result of exogenous technical progress, and here it depends on how the economy allocates capital

[1]This approximation result differs from the way in which Houthakker [1955-1956] and Jones [2005] generate Cobb-Douglas technologies. It does not rely on unexplained Pareto distributions. See Section 3.5 below for a discussion.

between capital accumulation and the production of consumption goods. More importantly, this type of balanced growth crucially relies on having a Cobb-Douglas technology for consumption. This paper shows how such a Cobb-Douglas technology can arise.

A correctly calculated Solow residual in this economy will be zero. In both the consumption and capital sectors of the economy, the residual is zero. A Divisia quantity index then implies a zero Solow residual in the aggregate. But because the price of capital is declining over time in units of consumption, the implicit price deflator for GDP in units of consumption is also declining over time. In turn this implies that the Divisia index for aggregate output grows at a higher rate than output measured in units of consumption, than consumption itself, and than the value of aggregate investment. As a result, the consumption-output ratio converges to zero when output is measured using a Divisia quantity index. As in Greenwood, Hercowitz and Krusell [1997], the fact that this is not what we observe in the data has to be attributed to a systematic mismeasurement of the quantities of new capital produced, one that understates the extent to which economies accumulate capital.

The distinction between exogenous technical progress and knowledge capital accumulation matters for what one expects to happen to small open economies that start out on a path of development. In a Solow world, a country that gains the ability to costlessly adopt the world technological frontier experiences a large gain in wealth, even if only consumption goods can be traded and it still needs to expand its capital stock to take full advantage of these new technological possibilities. Such a country will run large trade deficits in the early stages of development. But for the technologies described in this paper, the main impact of opening up to trade on a country in its early stages of development is an increase in interest rates that causes it to substitute away from early consumption. In the extreme case of a country with no capital, this is all that happens, implying an initial trade surplus. The trade balance is a continuous function of the initial capital stock at zero capital, and so small open economies with very little capital will also run a trade surplus. This can explain why capital does not always flow from rich to poor countries (Lucas [1990]).

2. A Smooth Economy

There is a representative consumer whose preferences are determined by

$$\int_0^\infty e^{-\rho t} u(c_t) \mathrm{d}t.$$

Consumption is produced using capital x_t and a fixed factor, according to

$$c_t = f(x_t) \tag{1}$$

where f is strictly increasing and exhibits decreasing returns to scale. The stock of capital k_t can be accumulated according to

$$\mathrm{D}k_t = A \cdot (k_t - x_t), \quad x_t \in [0, k_t], \tag{2}$$

where A is a positive coefficient. The initial capital stock is given by some $k_0 > 0$.

Throughout, the production function for consumption goods is taken to be time-invariant. There is no technological change. Exogenous technological progress at an exponential rate would change the production function to $e^{\gamma t} f(x_t)$. This is the same as replacing $u(c_t)$ by $u(e^{\gamma t} c_t)$ instead. If utility is homothetic, this just changes the subjective discount rate ρ. Of course, the units in which consumption is measured matter for interpreting data.

One can take k_t to represent knowledge capital embodied in skilled workers and interpret the fixed factor implicit in (1) as "unskilled" labor. In this interpretation, skilled workers divide their time between accumulating more knowledge and using their knowledge in a team with unskilled workers to produce consumption goods. Knowledge is a purely rival good in this economy: a skilled worker who knows something cannot help all unskilled workers in the economy produce more, only those with whom this skilled worker is teamed up.[2] As in the span-of-control model of Lucas [1978], there are decreasing returns to adding unskilled workers to a team. The knowledge capital accumulation technology (2) is the same as the human capital accumulation technology used in Lucas [1988]. But there output can also be used to add to the other factor of production. Here that other factor is unskilled labor, and assumed to be fixed.

To begin studying the properties of this economy, it will initially be assumed that f is sufficiently smooth. In the application below, $f(0)$ is positive, implying that consumption can be positive even if the capital stock grows at its maximal rate A. As will be made explicit, an aggregate production function with this property arises when there is an unskilled labor-only backstop technology.

2.1 Equilibrium Conditions

Assume the initial capital stock is positive. Capital does not depreciate, and so capital will be held in equilibrium. Throughout, let consumption be the numeraire. The price of

[2] A natural place to re-introduce the nonrival aspects of knowledge is the knowledge accumulation equation (2). For some recent examples, see Luttmer [2007, 2011b].

capital is q_t, and its rental price is v_t. Since capital can be used to produce new capital or consumption, it must be that

$$Aq_t \leq v_t, \text{ w.e. if } x_t < k_t \tag{3}$$

and

$$\mathrm{D}f(x_t) \leq v_t, \text{ w.e. if } x_t > 0, \tag{4}$$

where x_t is capital rented out to produce consumption goods and k_t is the capital stock at time t. Since the capital stock is positive, these conditions imply $v_t = \max\{Aq_t, \mathrm{D}f(x_t)\}$.

Write r_t for the real interest rate. Again, because it is positive, the capital stock must earn a return equal to the real interest rate,

$$r_t q_t = v_t + \mathrm{D}q_t. \tag{5}$$

The representative consumer earns wages

$$w_t = f(x_t) - \mathrm{D}f(x_t)x_t. \tag{6}$$

Thus unskilled wages increase if and only if the output of consumption goods increases in this economy. If capital is interpreted as knowledge capital embodied in skilled labor, then $v_t k_t$ can be interpreted as the labor income of skilled labor.

Given these prices, the representative consumer chooses to consume c_t, hold a real bank account valued at b_t units of consumption, and a stock of capital k_t, subject to the flow budget constraint

$$\mathrm{D}(b_t + q_t k_t) = r_t(b_t + q_t k_t) + w_t - c_t$$

and the borrowing constraint

$$\liminf_{T\to\infty} \exp\left(-\int_0^T r_t \mathrm{d}t\right)(b_T + q_T k_T) \geq 0.$$

These constraints are equivalent to the present-value budget constraint,

$$\int_0^\infty \exp\left(-\int_0^t r_s \mathrm{d}s\right) c_t \mathrm{d}t \leq q_0 k_0 + \int_0^\infty \exp\left(-\int_0^t r_s \mathrm{d}s\right) w_t \mathrm{d}t.$$

Consumer wealth is simply the value of the capital stock plus the present value of labor income.

The usual first-order condition is

$$e^{-\rho t}\mathrm{D}u(c_t) = \lambda \exp\left(-\int_0^t r_s \mathrm{d}s\right), \tag{7}$$

for some Lagrange multiplier λ. The present-value budget constraint will bind and since b_t must be zero in equilibrium, this implies

$$\lim_{T\to\infty} \exp\left(-\int_0^T r_t \mathrm{d}t\right) q_T k_T = 0. \tag{8}$$

The equilibrium conditions are (1)-(8). The state of the economy is its capital stock, and there will be only one equilibrium for a given value of this state.

For future reference, note that the no-arbitrage condition (5), the capital stock dynamics (2) and the first-order condition (3) imply $r_t q_t k_t = v_t x_t + \mathrm{D}[q_t k_t]$, and then (8) gives

$$q_t k_t = \int_t^\infty \exp\left(-\int_t^s r_u \mathrm{d}u\right) v_s x_s \mathrm{d}s. \tag{9}$$

Alternatively, one can infer this from the present-value budget constraint and the market clearing condition $c_t = f(x_t)$, together with (4) and (6). Capital is only valued because it can be used to produce consumption goods.

To simplify some preliminary calculations, define

$$\sigma_t = -\frac{c_t \mathrm{D}^2 u(c_t)}{\mathrm{D}u(c_t)}$$

so that $1/\sigma_t$ is the intertemporal elasticity of substitution. The first-order condition (7) then implies

$$\frac{\mathrm{D}c_t}{c_t} = \frac{r_t - \rho}{\sigma_t}, \tag{10}$$

which is the usual Euler condition.

2.2 Stagnation or Growth

There is no technological change in this economy, only accumulation of knowledge capital. The economy can only grow if the technology for accumulating this knowledge is sufficiently productive.

2.2.1 Permanent Stagnation

If $x_0 = k_0$ then the state of the economy will not change. Thus $x_0 = k_0$ implies $k_t = k_0$ for all t. Conjecture that this is the equilibrium. Consumption will be constant and thus the interest rate must be $r_t = \rho$. The first-order condition (4) and the valuation condition for the capital stock (9) then imply $q_t = \mathrm{D}f(k_0)/\rho$. The first-order conditions (3)-(4) imply that $x_t = k_0 > 0$ can only hold if $Aq_t \leq \mathrm{D}f(k_0)$, and hence it must be that $A \leq \rho$.

That is, much as in an AK economy, permanent stagnation is the equilibrium here if and only if the technology for producing capital is insufficiently productive.

2.2.2 Growth

Now assume $A > \rho$, so that permanent stagnation is not the equilibrium. More precisely, $x_t < k_t$ at all times, and so the capital stock will grow forever.

To interpret the historical record, one can imagine that initially $A < \rho$, and then someone in the early 15$^{\text{th}}$ century figured out a way to more quickly reproduce knowledge capital, raising A above ρ. Or perhaps progress in agriculture led to an improvement in diet, resulting in longer life-expectancies, lowering the effective rate ρ at which agents were discounting the future.

The Constant Consumption Phase Consider first a situation in which $Aq_t > \mathrm{D}f(0)$. Then (3)-(4) implies $x_t = 0$, and so $c_t = f(0)$. While this situation lasts, consumption is constant and thus the first-order condition for consumption (7) implies $r_t = \rho$. The no-arbitrage restriction (5) simplifies to $\mathrm{D}q_t/q_t = -(A-\rho) < 0$. Hence q_t declines exponentially over time. This means that the condition $Aq_t > \mathrm{D}f(0)$ can only last for a finite amount of time. While it lasts, the market value of gross output $y_t = c_t + q_t\mathrm{D}k_t$ is given by

$$y_t = f(0) + Aq_tk_t.$$

Although there is no physical depreciation in this economy, the economic depreciation rate is $-\mathrm{D}q_t/q_t = A-\rho > 0$. Net output would account for this. The capital stock grows at the maximal rate A, but the value of the capital stock grows at the more modest rate $\mathrm{D}[q_tk_t]/(q_tk_t) = \rho$. This implies $\mathrm{D}y_t = A\rho q_tk_t = \rho q_t\mathrm{D}k_t$, and thus

$$\frac{\mathrm{D}y_t}{y_t} = \frac{q_t\mathrm{D}k_t}{c_t + q_t\mathrm{D}k_t} \times r_t = \frac{q_0k_0}{f(0)e^{-\rho t} + q_0k_0} \times \rho.$$

That is, the growth rate of output accelerates towards ρ as the value of the capital stock grows relative to consumption. Unskilled wages are constant because consumption is constant. Skilled labor income is $v_tk_t = Aq_tk_t$, and this grows at the rate ρ. Thus during this phase, skilled and unskilled labor incomes diverge.

The Growing Consumption Phase The exponential decline in q_t implies that the condition $Aq_t > \mathrm{D}f(0)$ will eventually be violated. The assumption $A > \rho$ implies that

$x_t < k_t$, and so (3)-(4) gives $Aq_t = \mathrm{D}f(x_t)$. This and the fact that $c_t = f(x_t)$ imply

$$\frac{\mathrm{D}c_t}{c_t} = \left(\frac{x_t \mathrm{D}f(x_t)}{f(x_t)}\right)\frac{\mathrm{D}x_t}{x_t}, \qquad \frac{\mathrm{D}q_t}{q_t} = -\left(-\frac{x_t \mathrm{D}^2 f(x_t)}{\mathrm{D}f(x_t)}\right)\frac{\mathrm{D}x_t}{x_t}, \tag{11}$$

where we have used the assumed smoothness of f. Consumption can grow only if more capital is used to produce consumption, and this can happen only if the price of capital declines. Since the rental price of capital must be equal to Aq_t, the no-arbitrage condition (5) now implies

$$\frac{\mathrm{D}q_t}{q_t} = -(A - r_t). \tag{12}$$

Combining (10), (11) and (12) gives an interest rate

$$r_t = \frac{\left(-\frac{x_t \mathrm{D}^2 f(x_t)}{\mathrm{D}f(x_t)}\right)\rho + \sigma_t \left(\frac{x_t \mathrm{D}f(x_t)}{f(x_t)}\right) A}{\left(-\frac{x_t \mathrm{D}^2 f(x_t)}{\mathrm{D}f(x_t)}\right) + \sigma_t \left(\frac{x_t \mathrm{D}f(x_t)}{f(x_t)}\right)} \tag{13}$$

and a consumption growth rate

$$\frac{\mathrm{D}c_t}{c_t} = \frac{\sigma_t \left(\frac{x_t \mathrm{D}f(x_t)}{f(x_t)}\right)}{\left(-\frac{x_t \mathrm{D}^2 f(x_t)}{\mathrm{D}f(x_t)}\right) + \sigma_t \left(\frac{x_t \mathrm{D}f(x_t)}{f(x_t)}\right)} \times \frac{A - \rho}{\sigma_t}. \tag{14}$$

That is, the interest rate must be a weighted average of ρ and A, and the resulting consumption growth rate is some fraction of the growth rate $(A-\rho)/\sigma_t$ that arises in an AK economy, where f is linear. Since $A > \rho$, consumption grows at a positive rate and the price of capital falls over time. The interest rate satisfies $r_t \in (\rho, A)$ and the economic depreciation rate is now $A - r_t \in (0, A - \rho)$.

Since $Aq_t = \mathrm{D}f(x_t)$, gross output $y_t = c_t + q_t \mathrm{D}k_t$ equals

$$y_t = f(x_t) + \mathrm{D}f(x_t)(k_t - x_t).$$

The concavity of f implies that $y_t \geq f(k_t)$. As expected, gross output is no less than the amount of consumption that could be produced if all capital were used for that purpose. Taking a time derivative gives

$$\mathrm{D}y_t = \mathrm{D}f(x_t)A(k_t - x_t) + \mathrm{D}^2 f(x_t)\mathrm{D}x_t(k_t - x_t)$$

or

$$\frac{\mathrm{D}y_t}{y_t} = \frac{\mathrm{D}f(x_t)(k_t - x_t)}{f(x_t) + \mathrm{D}f(x_t)(k_t - x_t)} \times \left\{A - \left(-\frac{x_t \mathrm{D}^2 f(x_t)}{\mathrm{D}f(x_t)}\right)\frac{\mathrm{D}x_t}{x_t}\right\}.$$

Combining this with (11) and (12) shows that the second factor is just the interest rate, and therefore

$$\frac{\mathrm{D}y_t}{y_t} = \frac{q_t \mathrm{D}k_t}{c_t + q_t \mathrm{D}k_t} \times r_t. \tag{15}$$

As in the constant-consumption phase, the growth rate of output equals the share of investment in output times the interest rate.

A calculation using $w_t = f(x_t) - \mathrm{D}f(x_t)x_t > c_t$ and (11) gives

$$\frac{\mathrm{D}w_t}{w_t} = \frac{-\frac{\mathrm{D}^2 f(x_t)x_t}{\mathrm{D}f(x_t)}}{1 - \frac{\mathrm{D}f(x_t)x_t}{f(x_t)}} \times \frac{\mathrm{D}c_t}{c_t}.$$

This is positive because consumption grows at a positive rate. But the precise relation to consumption growth depends on the factor share of unskilled labor and the curvature of the production function f at x_t. For skilled labor, $v_t k_t = Aq_t k_t$ together with (12) gives

$$\frac{\mathrm{D}[v_t k_t]}{v_t k_t} = r_t - \frac{Ax_t}{k_t}.$$

Even the sign of this cannot be determined without knowledge of the equilibrium ratio x_t/k_t. Of course, $y_t = w_t + v_t k_t$, and so the factor-share weighted growth rates of unskilled and skilled labor income have to add up to (15).

At the point in time when q_t enters the range $Aq_t \leq \mathrm{D}f(0)$, the consumption growth rate jumps up from a zero growth rate. As a result, the interest rate must jump up as well. But consumption itself, and thus x_t cannot jump. And neither can the price of capital. Thus the share of investment in output cannot jump at this point in time. It follows from (15) that the growth rate of output jumps up as consumption begins to grow. Thus the initial acceleration towards ρ is followed by an upward jump in the growth rate of gross output.

3. Leontief Approximations

It is immediate from (14) that this economy will grow at a constant rate when consumer preferences are CES over time and the production function is Cobb-Douglas. More generally, this economy will grow at an approximately constant rate as long as preferences are CES and $x_t \mathrm{D}f(x_t)/f(x_t)$ is approximately constant. For example, if preferences are CES and $x\mathrm{D}f(x)/f(x)$ converges to a positive constant as x becomes large, then the economy will settle down to a constant growth rate, even if growth rates vary initially.

The economy would also grow at an asymptotically constant rate for a CES production function with an elasticity of substitution greater than 1, as in Jones and Manuelli [1990]. But the income share of unskilled labor would converge to zero. The data suggest that this is not the case, although historically there seem to have been stages of economic development during which the income share of unskilled labor has lagged behind.

Despite its extremely pervasive use, the Cobb-Douglas technology is very special, and one should question any account of growth that critically relies on it. The following describes circumstances under which a collection of Leontief technologies gives rise to an approximate Cobb-Douglas technology. Leontief technologies are easy to interpret and recognize in micro data, and thus it is of interest to understand what kind of collections of Leontief technologies will behave like a Cobb-Douglas technology.

3.1 A Sequence of Leontief Technologies

Consider an economy with a countable number of Leontief technologies indexed by $n \in \mathbb{N}$. Technology n can be used to combine capital k and labor l to produce output

$$y = \min\left\{\frac{k}{A_n}, \frac{l}{B_n}\right\}$$

for some positive A_n and B_n. Thus A_n and B_n are the input requirements for capital and labor, respectively. Every technology produces the same type of output. Suppose the aggregate supplies of capital and labor are K and L. Then aggregate output will be

$$F(K,L) = \max_{\{k_n,l_n\}_{n=1}^{\infty} \in \mathbb{R}_+^{\infty} \times \mathbb{R}_+^{\infty}} \left\{\sum_{n=1}^{\infty} \min\left\{\frac{k_n}{A_n}, \frac{l_n}{B_n}\right\} : \sum_{n=1}^{\infty} k_n \leq K, \sum_{n=1}^{\infty} l_n \leq L\right\}. \tag{16}$$

This is a constant returns to scale production function that is entirely standard except for the fact that it is not smooth.

The isoquants $1 = \min\{k/A_n, l/B_n\}$ for three different technologies are shown in Figure 1. In the example shown, A_n is decreasing and B_n is increasing in n. It is always possible to rank the technologies by labor productivity to ensure that B_n is decreasing. If the resulting sequence A_n is not increasing, then there are technologies that will never be used in any equilibrium. These technologies can be omitted from the description of the economy to ensure that

$$\begin{aligned} A_1 &< A_2 < \ldots \\ B_1 &> B_2 > \ldots \end{aligned} \tag{17}$$

Note that this implies $A_1/B_1 > A_2/B_2 > \ldots$. The capital-labor ratios increase with n. High-n technologies are capital-intensive and have a high labor-productivity.

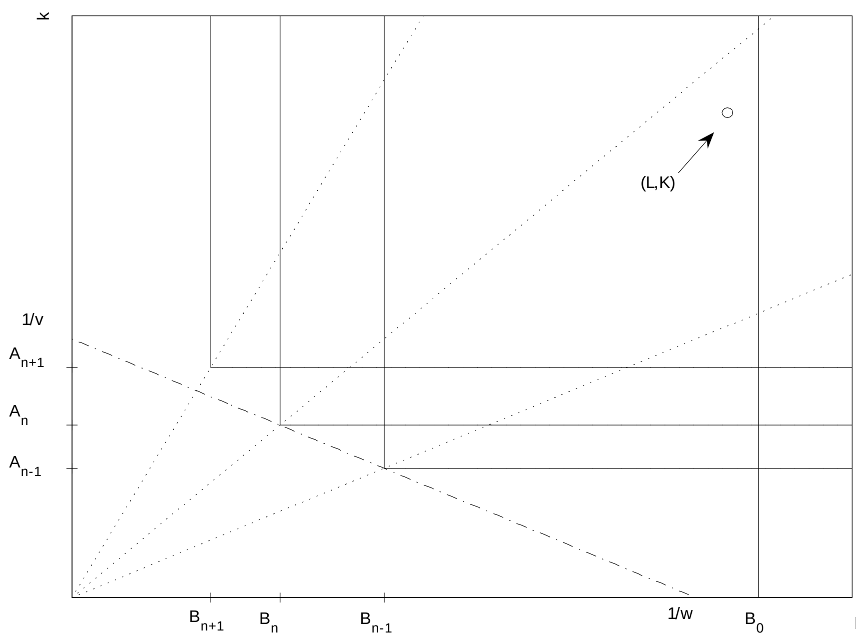

FIGURE 1 The Isoquants $\min\{k/A_n, l/B_n\} = 1$

Let v and w be the factor prices of capital and labor, respectively. It is immediate from Figure 1 that only two technologies will be used. Suppose the $n-1$ and n technologies are used. Since both technologies must imply zero profits, this gives

$$\begin{bmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which yields

$$\begin{bmatrix} v \\ w \end{bmatrix} = \frac{1}{\frac{B_{n-1}}{B_n} - \frac{A_{n-1}}{A_n}} \begin{bmatrix} \frac{1}{A_n}\left(\frac{B_{n-1}}{B_n} - 1\right) \\ \frac{1}{B_n}\left(1 - \frac{A_{n-1}}{A_n}\right) \end{bmatrix}. \tag{18}$$

The conditions (17) ensure that both factor prices are strictly positive. The factor price of capital will shrink to zero if A_n grows without bound, and wages grow without bound if $1/B_n$ does.

By itself, condition (17) does not imply that every technology will be used for some aggregate capital-labor ratio K/L. It could be that a convex combination of the $n-1$ and $n+1$ technologies dominates technology n. This possibility is ruled out if and only if

$$\frac{A_{n+1} - A_n}{B_n - B_{n+1}} > \frac{A_n - A_{n-1}}{B_{n-1} - B_n}. \tag{19}$$

This condition is equivalent to saying that the equilibrium w/v if n and $n+1$ are used is higher than the equilibrium w/v if $n-1$ and n are used. With this assumption, the isoquants of all available technologies touch the envelope of all isoquants. If (19) is not satisfied, one can omit technologies from the description of the economy until it is. In other words, given a countable collection of Leontief technologies with various input requirements, the efficient frontier of this collection can be characterized by a sequence $\{A_n, B_n\}_{n=1}^{\infty}$ that satisfies (17) and (19).

With (17) and (19) Figure 1 shows that $F(A_n, B_n) = 1$ and that the marginal product of labor at $K/L = A_n/B_n$ is in between the equilibrium wage when technologies $n-1$ and n are used and the equilibrium wage when n and $n+1$ are used. The resulting factor share of labor therefore satisfies

$$\frac{B_n \partial F(A_n, B_n)}{F(A_n, B_n)} \subset \left[\frac{1 - \frac{A_{n-1}}{A_n}}{\frac{B_{n-1}}{B_n} - \frac{A_{n-1}}{A_n}}, \frac{B_n}{B_{n+1}} \frac{1 - \frac{A_n}{A_{n+1}}}{\frac{B_n}{B_{n+1}} - \frac{A_n}{A_{n+1}}} \right]. \tag{20}$$

Observe that the right-hand side only depends on the ratios A_n/A_{n-1} and B_n/B_{n-1}, and not on the levels of A_n and B_n. Thus, the factor share of labor will move in a range that is determined by how fast A_n increases and B_n decreases with n.

3.1.1 Adding a Labor-Only Backstop Technology

If the capital-labor ratio is sufficiently low, $K/L < A_1/B_1$, then there is more labor than can be employed with any technology. In that case, $F(K, L) = K/A_1$ and wages will be zero.

One can add a linear labor-only technology to give the suppliers of labor an outside option. This will then determine the wage if $K/L < A_1/B_1$. This has the natural implication that wages will be constant as long as there not much capital. Specifically, suppose $A_0 = 0$ and $B_0 > B_1$, large enough so that (19) holds for $n = 1$. Then one unit of labor can produce $1/B_0$ by itself, and all the other technologies are still on the isoquant for the aggregate technology. Wages will never be below $1/B_0$.

With this labor-only technology included, we have

$$F(K, L) = K + \frac{L - K}{B_0}, \quad \frac{K}{L} \in \left[0, \frac{A_1}{B_1}\right] \tag{21}$$

If n and $n+1$ are used, then output is determined by

$$\begin{bmatrix} A_n & A_{n+1} \\ B_n & B_{n+1} \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix}.$$

Solving for $[y_n, y_{n+1}]$ and using $F(K, L) = y_n + y_{n+1}$ gives

$$F(K, L) = \frac{(A_{n+1} - A_n)L + (B_n - B_{n+1})K}{A_{n+1}B_n - A_n B_{n+1}}, \quad \frac{K}{L} \in \left[\frac{A_n}{B_n}, \frac{A_{n+1}}{B_{n+1}}\right]. \tag{22}$$

This is positive because of (17) and satisfies $F(A_n, B_n) = F(A_{n+1}, B_{n+1}) = 1$ by construction.

3.2 The Geometric Case

Suppose now that

$$A_n = \alpha^n, \quad B_n = \beta^n$$

for some $\alpha > 1 > \beta$. Clearly, (17) holds and one can verify that (19) holds as well. Thus these technologies are all efficient for some factor prices. Adding a labor-only backstop technology and imposing (19) simply requires $B_0 > \beta(\alpha - \beta)/(\alpha - 1)$. This backstop technology will become efficient to use when the capital-labor ratio rises above α/β.

With these input requirements, $F(A_n, B_n) = 1$ becomes $F((\alpha/\beta)^n, 1) = 1/\beta^n$, and hence

$$F(k, 1) = \frac{1}{\beta^{\ln(k)/\ln(\alpha/\beta)}} = k^{1 - \frac{\ln(\alpha)}{\ln(\alpha) + \ln(1/\beta)}}, \quad k \in \left\{ \left(\frac{\alpha}{\beta}\right)^n \right\}_{n=1}^{\infty}$$

Essentially, a countable collection of Leontief technologies approximates a Cobb-Douglas technology if the logarithmic capital and labor input requirements are on a uniform grid.

By making the grid sufficiently dense, one can make the approximation error arbitrarily small. A precise argument is given in Appendix B. Here, suppose

$$F(K, L) = \max_{k(\cdot), l(\cdot) \geq 0} \left\{ \int_0^\infty \min\left\{ \frac{k(x)}{\alpha^x}, \frac{l(x)}{\beta^x} \right\} \mathrm{d}x : \int_0^\infty k(x) \mathrm{d}x \leq K, \int_0^\infty l(x) \mathrm{d}x \leq L \right\}$$

Then the optimal allocation of capital and labor is given by

$$k(x) = K, \; l(x) = L, \; \frac{K}{L} = \left(\frac{\alpha}{\beta}\right)^x,$$

provided $K > L$ so that $x > 0$, and hence

$$F(K, L) = K^{1 - \frac{\ln(\alpha)}{\ln(\alpha) + \ln(1/\beta)}} L^{\frac{\ln(\alpha)}{\ln(\alpha) + \ln(1/\beta)}}, \quad \text{if } K > L.$$

If there is not much capital then

$$F(K, L) = K, \quad \text{if } K < L.$$

Thus the technology is Cobb-Douglas provided the capital stock is large enough. The labor share of this technology depends on how "close" alternative technologies are in terms of input requirements. If $\alpha > 1$ is large compared to $1/\beta > 1$, then nearby technologies differ more in terms of labor input requirements than in terms of capital input requirements. The similarity of nearby technologies in terms of capital input requirements gives labor a larger share of the pie.

3.3 Cobb-Douglas Utility

The above argument can also be used to construct a Cobb-Douglas utility function. Suppose consumers can buy the services of various types of capital and combine them with leisure time to produce utility. Total utility is the sum of all utility produced from leisure and the various types of capital. The resulting utility function will be $u(c,l) = c^{1-\sigma}l^{\sigma}$, where $\sigma \in (0,1)$ and $l \in [0,1]$. Over time, one obtains CES utility with an elasticity $1/\sigma > 1$, or risk-aversion $\sigma \in (0,1)$.

Note that maximizing $c^{1-\sigma}l^{\sigma}$ subject to $c + zl \leq z$ gives $c = (1-\sigma)z$ and $l = \sigma$, and hence

$$\max_{c,l} \left\{ c^{1-\sigma}l^{\sigma} : c + zl \leq z \right\} = (1-\sigma)^{1-\sigma}\sigma^{\sigma} \times z^{1-\sigma}.$$

Thus, if there is heterogeneity in effective labor endowments, then utility will scale with $z^{1-\sigma}$ in the cross-section. More importantly, high-ability individuals will combine high-quality consumption goods with their leisure time.

3.4 Boldrin and Levine [2010]

The quality-ladder economy studied in Boldrin and Levine [2010] is equivalent to $u(c) = \ln(c)$ and $f(x) = F(x,1)$ were F is determined by $A_n = \alpha^n$ and $B = \beta^n$. Note that one can write

$$\begin{aligned} F(K,L) &= \max\left\{ \sum_{n=1}^{\infty} \min\left\{ \frac{k_n}{\alpha^n}, \frac{l_n}{\beta^n} \right\} : \sum_{n=1}^{\infty} k_n \leq K,\ \sum_{n=1}^{\infty} l_n \leq L \right\} \\ &= \max\left\{ \sum_{n=1}^{\infty} \frac{1}{\beta^n} \min\left\{ \frac{k_n}{(\alpha/\beta)^n}, l_n \right\} : \sum_{n=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^n \frac{k_n}{(\alpha/\beta)^n} \leq K,\ \sum_{n=1}^{\infty} l_n \leq L \right\} \\ &= \max\left\{ \sum_{n=1}^{\infty} \zeta^n \min\left\{ h_n, l_n \right\} : \sum_{n=1}^{\infty} \gamma^n h_n \leq K,\ \sum_{n=1}^{\infty} l_n \leq L \right\} \end{aligned}$$

where $\zeta = 1/\beta > 1$ and $\gamma = \alpha/\beta > \zeta$. One unit of type-$n$ capital must be combined with one unit of labor to use the type-n technology, and the constraint on capital says that one unit of type-n capital can be transformed into $1/\gamma < 1$ units of type-$n+1$ capital. In Boldrin and Levine [2010], it is not possible to convert type-$n+1$ capital back into type-n capital, but this constraint never binds if all capital is initially of the lowest type.

There are many other ways to choose sequences of Leontief input requirements $\{A_n, B_n\}_{n=1}^{\infty}$ to approximate the Cobb-Douglas production function. For example, $A_n = n^{\alpha}$ and $B_n = n^{\beta}$ for some $\alpha > 0 > \beta$ would work as well. What is special about the geometric coefficients used here is that the amount of type-n capital needed to produce a unit of type-$n+1$ capital is the same for all n.

3.5 Other Interpretations of Cobb-Douglas Technologies

Probably the most well-known interpretation of the Cobb-Douglas production function is the one given by Houthakker [1955, 1956]. There, there is a measure of heterogeneous fixed factors that can be combined with unskilled labor. Labor can be allocated freely across the fixed factors. Output is $z \min\{1, l\}$, where l is unskilled labor and z is the productivity of the fixed factor. If the distribution of z is Pareto, and the aggregate labor supply exceeds the measure of heterogeneous fixed factors, then the aggregate technology will be a Cobb-Douglas function of the total measure of heterogeneous fixed factors and the aggregate supply of unskilled labor. One obvious interpretation is that there is a measure of skilled workers who can form teams with unskilled workers, and whose abilities follow a Pareto distribution. In contrast to Lucas [1978], every skilled worker can be matched with only one unskilled worker. The Pareto distribution of ability shows up only in the distribution of factor payments to skilled workers, not in the sizes of the teams they manage.

Closer to the present paper is Jones [2005], who considers the production function $\max\{F(aK, bL) : (a, b) \in \Gamma\}$, where Γ is some collection of parameters that index different technologies, and F is a constant-returns to scale production function. Jones [2005] specializes to $F(aK, bL) = \min\{aK, bL\}$ and then takes Γ to be a collection of N techniques drawn randomly from a joint distribution for (a, b) that is the product of two independent Pareto distributions. Output is Y_N, and the result is that $Y_N/(K^\alpha L^{1-\alpha})$ converges in distribution to a Fréchet distribution as N becomes large. Thus, if there is a large population of ex ante identical firms that get to choose (K, L) and then draw techniques independently, then there will be an ex post distribution of output among these producers that is approximately Fréchet, and aggregate output will scale with $K^\alpha L^{1-\alpha}$.

In this paper, there is no randomness and individual producers get to pick techniques (a, b) and input choices (k, l) jointly. The collection of available techniques is non-random, but evenly spaced in terms of logarithmic input requirements. The Boldrin and Levine [2010] quality ladder economy shows how this spacing arises if a proportional reduction in labor input requirements can be obtained with a proportional increase in the effective capital input requirements. When capital can be accumulated geometrically, this leads to unbounded growth with stable factor shares.

4. Interpreting Sir Arthur Lewis

The combination of a labor-only technology and the collection of Leontief technologies for capital and labor described in the previous section can be used to give a natural account of how an economy can transition from stagnation to long-term balanced growth. The labor-only technology describes a "primitive sector" from which labor can be re-allocated to capital-intensive sectors, much like Lewis [1954].

A countable collection of Leontief technologies is somewhat inconvenient to work with. Letting the spacing between the Leontief technologies become small results in a combination of two technologies: a linear technology that requires only labor and no capital, and a constant-returns Cobb-Douglas technology for capital and labor. As calculated in Appendix A, if the limiting labor share is α, then the implied production possibilities are given by

$$f(x) = \begin{cases} \alpha\xi^{1-\alpha} + (1-\alpha)\xi^{-\alpha}x, & x \le \xi \\ x^{1-\alpha}, & x \ge \xi \end{cases} \tag{23}$$

where $\alpha \in (0,1)$ and $\xi > 0$. Note that $f(0)$ is positive, and that the marginal product of capital at 0 is finite, and equal to what it is at ξ. The marginal product of capital is continuous everywhere. But it is not differentiable at $x = \xi$. Not all of the formulas in Section 2 apply at $x_t = \xi$.

Observe that $f(x) - \mathrm{D}f(x)x = \alpha f(x)$ for $x \ge \xi$, and so the wage will be the usual Cobb-Douglas share of production if enough capital is used to produce consumption goods. On the other hand, $f(x) - \mathrm{D}f(x)x = \alpha\xi^{1-\alpha} = f(0) = \alpha f(\xi)$ for $x \le \xi$, and so $f(0)$, the implied productivity of the linear labor-only technology, will be the unskilled wage when fewer than ξ units of capital are used to produce consumption. The continuity of marginal products ensures that wages will not jump as x passes through ξ. The more productive the labor-only technology, the larger ξ, and thus the larger the range of capital inputs over which the marginal product of capital will not run into decreasing returns.

Assume preferences are determined by $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ for some positive σ. Thus $\sigma_t = \sigma$. The parameters are assumed to satisfy

$$A > \rho > (1-\sigma)(1-\alpha)A. \tag{24}$$

The first inequality implies growth, and the second inequality ensures finite utility when k_t grows at anywhere near its maximal rate.

4.1 Balanced Growth

To construct the equilibrium for this economy, begin by considering situations in which k_t is large enough so that the equilibrium satisfies $Aq_t = \mathrm{D}f(x_t) = (1-\alpha)x_t^{-\alpha} < (1-\alpha)\xi^{-\alpha}$. Then (13)-(14) gives

$$r_t = \frac{\alpha\rho + (1-\alpha)\sigma A}{\alpha + (1-\alpha)\sigma} = r \tag{25}$$

and

$$\frac{\mathrm{D}c_t}{c_t} = \frac{(1-\alpha)\sigma}{\alpha + (1-\alpha)\sigma} \times \frac{A-\rho}{\sigma} = g, \tag{26}$$

which satisfies $r > g$ because of (24). Consumption grows at a constant rate if and only if x_t grows at a constant rate. The growth rate of capital must then be at least that of x_t, and the transversality condition will be violated if k_t grows faster than x_t. Thus $\mathrm{D}k_t/k_t = \mathrm{D}x_t/x_t = (\mathrm{D}c_t/c_t)/(1-\alpha) = g/(1-\alpha)$. The resource constraint $\mathrm{D}k_t = A \cdot (k_t - x_t)$ then implies

$$\frac{x_t}{k_t} = 1 - \frac{1}{A}\frac{g}{1-\alpha} = \frac{1}{A}\frac{\rho - (1-\sigma)(1-\alpha)A}{\alpha + (1-\alpha)\sigma} = \frac{r-g}{A} \tag{27}$$

which is in $(0,1)$ because of (24). Alternatively, the valuation equation for the capital stock (9) together with $v_t = Aq_t$ and $v_t x_t = (1-\alpha)c_t$ immediately gives the right-hand side of (27).

Unskilled wages $w_t = \alpha c_t$ grow at the same rate as aggregate consumption. The price of capital follows from $Aq_t = \mathrm{D}f(x_t)$. It declines at the rate $g \times \alpha/(1-\alpha)$. It follows that the value of the aggregate capital stock $q_t k_t$ and capital income $v_t k_t = Aq_t k_t$ grow at the same rate g as aggregate consumption. More specifically,

$$\frac{q_t k_t}{c_t} = \frac{\mathrm{D}f(x_t)k_t}{f(x_t)} = (1-\alpha) \times \frac{k_t}{x_t} = \frac{(1-\alpha)A}{r-g}.$$

and the value of aggregate investment is given by

$$\frac{Aq_t(k_t - x_t)}{c_t} = \frac{q_t k_t}{c_t}\frac{1}{A}\frac{g}{1-\alpha} = \frac{g}{r-g}.$$

Hence aggregate output grows at the same rate as consumption, and $c_t/y_t = 1 - g/r$. The value of the aggregate capital stock relative to output is constant along the balanced growth path. But the capital-output ratio k_t/y_t is not: the physical capital stock grows at the rate $g/(1-\alpha) > g$ while output only grows at the rate g.

4.1.1 The Balanced Growth Threshold

The above balanced growth calculations apply as long $x_t > \xi$, and the corresponding lower bound $k_t > \kappa$ is implied by $x_t/k_t = \xi/\kappa$ and (27). This gives

$$\frac{\xi}{\kappa} = \frac{1}{A}\frac{\rho - (1-\sigma)(1-\alpha)A}{\alpha + (1-\alpha)\sigma}. \tag{28}$$

Observe from (25) and (26) that one can write (28) as

$$\kappa = \frac{A\xi}{r-g}.$$

The threshold level of capital is the threshold level of investment discounted at the rate $r - g$. Since $r \in (\rho, A)$, this threshold satisfies $\kappa = A\xi/(r-g) > A\xi/r > \xi$, as expected.

4.2 Take-Off and Transition

Suppose k_0 is small enough that $x_0 = 0$. Consumption is produced using (unskilled) labor only and all capital is used to produce more capital. Let $\tau > 0$ be the time when x_t becomes positive and let $T > \tau$ the time when x_t rises above ξ. Thus

$$c_\tau = \alpha\xi^{1-\alpha}, \quad c_T = \xi^{1-\alpha} \tag{29}$$

and $k_T = \kappa$, where κ is defined in (28).

For $t \in (\tau, T)$, $x_t \in (0, \xi)$ and hence $\mathrm{D}f(x_t) = \mathrm{D}f(0) = \mathrm{D}f(\xi)$. Capital is used both to produce consumption and to produce more capital, and thus $Aq_t = \mathrm{D}f(x_t)$. It follows that the price of capital is constant and determined by $Aq_t = \mathrm{D}f(0) = \mathrm{D}f(\xi)$. Since $\mathrm{D}^2f(x_t) = 0$ over this episode, (13)-(14) gives

$$r_t = A, \quad \frac{\mathrm{D}c_t}{c_t} = \frac{A-\rho}{\sigma}.$$

That is, the economy behaves like an AK economy during this episode. Because of (29), $c_T/c_\tau = 1/\alpha$, and thus the length of this episode must be given by

$$T - \tau = \frac{\ln(1/\alpha)}{(A-\rho)/\sigma}. \tag{30}$$

Given the length of this episode, and given that $k_T = \kappa$ at the end of this episode, one can work backwards to infer the size of the capital stock at $t = \tau$. The capital accumulation technology (2) together with $k_T = \kappa$ implies the present-value restriction

$$k_\tau = \int_0^{T-\tau} e^{-As}Ax_{\tau+s}\mathrm{d}s + e^{-A(T-\tau)}\kappa.$$

Since $c_\tau = \alpha\xi^{1-\alpha}$ and consumption grows at the rate $(A-\rho)/\sigma$ during the interval $[\tau, T]$, it must be that x_t solves

$$\alpha\xi^{1-\alpha}e^{\left(\frac{A-\rho}{\sigma}\right)(t-\tau)} = \alpha\xi^{1-\alpha} + (1-\alpha)\xi^{-\alpha}x_t$$

for all $t \in [\tau, T]$. Inserting this into the expression for k_τ gives

$$\begin{aligned} k_\tau &= \frac{\alpha A\xi}{1-\alpha}\int_0^{T-\tau} e^{-As}\left(e^{\left(\frac{A-\rho}{\sigma}\right)s} - 1\right)\mathrm{d}s + e^{-A(T-\tau)}\kappa \\ &= \frac{\alpha A\xi}{1-\alpha}\left(\frac{1-e^{-\left(A-\frac{A-\rho}{\sigma}\right)(T-\tau)}}{A-\frac{A-\rho}{\sigma}} - \left(\frac{1-e^{-A(T-\tau)}}{A}\right)\right) + e^{-A(T-\tau)}\kappa \qquad (31) \end{aligned}$$

Since $T-\tau$ is given in (30), this determines k_τ. Note that $x_t \in [0, \xi]$ during this phase, and the growth rate of the capital stock declines from A to $A(1-\xi/\kappa) = A - (r-g) = g/(1-\alpha)$, which is its growth rate along the balanced growth path.

During the interval $[0, \tau]$, all capital is used to produce more capital, and thus consumption is constant at $c_t = f(0)$. It follows that $r_t = \rho$, and hence $\mathrm{D}q_t/q_t = -(A-\rho)$. Thus $q_0 = q_\tau e^{(A-\rho)\tau}$, where $q_\tau = \mathrm{D}f(0)/A$ because capital begins to be used to produce consumption at time $t = \tau$. Capital grows according to $\mathrm{D}k_t/k_t = A$ during the interval $[0, \tau]$ and hence

$$\tau = \frac{1}{A}\ln\left(\frac{k_\tau}{k_0}\right). \qquad (32)$$

The variables $(\tau, k_\tau, T, \kappa)$ that define the different regimes are determined by the threshold κ defined in (28), together with the equilibrium conditions (30)-(32).

An example is shown in Figure 2. To summarize, initially consumption is constant and the capital stock grows at the rate A. The price of capital falls at the rate $A-\rho$. When it reaches $\mathrm{D}f(0)/A$, it stops falling and the interest rate jumps up from ρ to A. Consumption grows at the rate $(A-\rho)/\sigma$, as in an AK economy. Once consumption reaches $f(\xi)$, the interest rate declines to somewhere in between ρ and A, and consumption continues to grow at a positive rate, but not as fast as $(A-\rho)/\sigma$. The price of capital continues it decline. In this regime the aggregate value of the capital stock and aggregate consumption grow at the same rate. But since the price of capital is declining, this means that the capital stock grows at a faster rate than consumption.

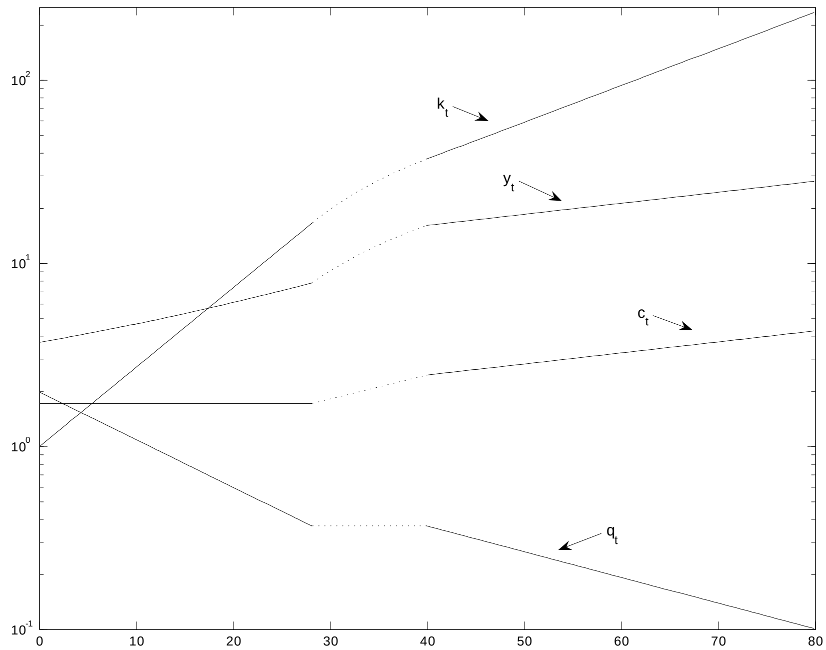

FIGURE 2 A Growh Transition

4.2.1 Factor Payments

Clearly, unskilled wages are constant during the period $[0, \tau]$, when no capital is used to produce consumption. The price of capital declines at the rate $A - \rho$ and the capital stock grows at the rate A. The value of the capital stock therefore grows at the rate ρ, and since $v_t = Aq_t$, so does the factor income $v_t k_t$. If capital is interpreted as human capital embodied in skilled labor, this implies growing factor payments to skilled labor and stagnant factor payments to unskilled labor during the take-off phase.

Once capital starts to be employed to produce consumption goods, consumption begins to grow at the rate $(A - \rho)/\sigma$, but unskilled wages remain stagnant. Not enough capital is being used together with unskilled labor to make the labor-only technology obsolete. During this phase, the marginal product of capital is constant at $\mathrm{D}f(0) = \mathrm{D}f(\xi)$, and hence the rental price v_t is constant. The growth rate of the capital stock is declining from A to $g/(1 - \alpha)$. Factor payments to skilled labor are therefore still growing, but at a declining rate.

In sum, factor payments to unskilled labor remain stagnant during the take-off and transition phases of this economy, even as capital begins to be reallocated to the production of consumption goods. As long as there is "surplus labor" using the labor-only technology, unskilled wages cannot grow.

Factor shares are stable once the economy reaches its balanced growth path. This takes the Cobb-Douglas part of the technology literally rather than as an approximation. In the quality-ladder economy of Boldrin and Levine [2010], there will be fluctuations in the factor shares, with the factor share of unskilled labor falling behind when capital is being upgraded for the next Leontief technology.

4.3 An Aside on Growth Accounting

Output in the consumption sector is $f(x_t)$, and hence the Solow residual in this sector of the economy must be zero. Output in the investment sector of the economy is $A(k_t - x_t)$ in units of capital. Thus the Solow residual will be zero in this sector as well if sectoral output is measured in units of capital. But when measured in units of consumption, output in this sector is $q_t A(k_t - x_t)$. Along a balanced growth path, the "Solow residual" in this sector will grow at the negative rate $-g \times \alpha/(1-\alpha)$, the rate at which the price of capital in units of consumption declines, if sectoral output is measured in units of consumption.

To compute the Solow residual for the aggregate economy, let Y_t be the Divisia index for aggregate output,[3]

$$\frac{\mathrm{D}Y_t}{Y_t} = \frac{c_t}{y_t}\frac{\mathrm{D}c_t}{c_t} + \left(1 - \frac{c_t}{y_t}\right)\frac{A\mathrm{D}(k_t - x_t)}{A(k_t - x_t)}.$$

Equivalently, one can think of Y_t as the quantity index obtained by deflating the market value of aggregate output y_t using a continuously chained Laspeyres price index. If consumption is the numeraire, then this Laspeyres index will decrease over time because the price of newly produced capital is decreasing over time. Similarly, let X_t be the Divisia index for inputs,

$$\frac{\mathrm{D}X_t}{X_t} = \frac{w_t}{y_t} \times 0 + \left(1 - \frac{w_t}{y_t}\right)\frac{\mathrm{D}k_t}{k_t}.$$

Consider times when $x_t \in (0, k_t)$ so that capital is used both to produce consumption goods and to produce new capital. Note that

$$\mathrm{D}c_t = v_t \mathrm{D}x_t, \quad \mathrm{D}y_t = \mathrm{D}w_t + v_t \mathrm{D}k_t + k_t \mathrm{D}v_t.$$

Then, using $v_t = Aq_t$, $y_t - c_t = v_t(k_t - x_t)$, and $y_t - w_t = v_t k_t$,

$$\begin{aligned}
\frac{\mathrm{D}Y_t}{Y_t} - \frac{\mathrm{D}X_t}{X_t} &= \frac{c_t}{y_t}\frac{\mathrm{D}c_t}{c_t} + \left(1 - \frac{c_t}{y_t}\right)\frac{A\mathrm{D}(k_t - x_t)}{A(k_t - x_t)} - \left(1 - \frac{w_t}{y_t}\right)\frac{\mathrm{D}k_t}{k_t} \\
&= \frac{1}{y_t}\left\{v_t \mathrm{D}x_t + v_t \mathrm{D}(k_t - x_t) - v_t \mathrm{D}k_t\right\} \\
&= 0.
\end{aligned}$$

[3] See Richter [1966] for the crucial role of Divisia quantity indices in growth accounting.

As expected, the Solow residual for this economy is zero (one can do similar calculations for the case $x_t = 0$.) But along the balanced growth path,

$$\begin{aligned}\frac{\mathrm{D}Y_t}{Y_t} &= \frac{c_t}{y_t}\frac{\mathrm{D}c_t}{c_t} + \left(1 - \frac{c_t}{y_t}\right)\frac{A\mathrm{D}(k_t - x_t)}{A(k_t - x_t)} \\ &= \left(1 - \frac{g}{r}\right)g + \left(\frac{g}{r}\right) \times \frac{g}{1-\alpha} \\ &> g\end{aligned}$$

In fact, $\mathrm{D}Y_t/Y_t = Ag/r$. That is, the Divisia index of output is growing faster than the market value of output measured in units of consumption, and than consumption itself. So the consumption-output ratio must converge to zero if output is measured using the Divisia quantity index. Of course, this is perfectly consistent with the fact that expenditure shares are constant.

Note that along the balanced growth path

$$\frac{\mathrm{D}y_t}{y_t} - \left\{\frac{w_t}{y_t} \times 0 + \frac{v_t k_t}{y_t} \times \frac{\mathrm{D}[v_t k_t]}{v_t k_t}\right\} = \frac{w_t}{y_t}\frac{\mathrm{D}w_t}{w_t} = \left(1 - \frac{g}{r}\right)\alpha \times g.$$

Therefore, if one mistakenly uses the market value of output and the market value of capital services to calculate a Solow residual, instead of the Divisia quantity index for output and the quantity of capital inputs, then the Solow residual will be positive. Replacing only the market value of output with the Divisia index $\mathrm{D}Y_t/Y_t > g$, but not the market value of capital services, will give rise to an even larger Solow residual.

Needless to say, as in Greenwood, Hercowitz and Krusell [1997], everything hinges on obtaining reliable estimates of $\mathrm{D}v_t/v_t$ or $\mathrm{D}q_t/q_t$. This plays a critical role as well in calibrating the standard Solow economy (Young [1995], Hsieh [2002]). The difficulty of measuring these prices is daunting. See Luttmer [2011a] for a cautionary tale.

5. A Small Open Economy

In view of the difficulties in measuring capital inputs, it is useful to look for other features of the economy that distinguish it from the Solow economy. A country that can look forward to using technological progress in the rest of the world for free is richer than a country that must build up a stock of knowledge capital mostly by itself. If initial output is low, the former country tends to borrow a lot from the rest of the world, and the latter country may not.

Consider a small open economy that can buy consumption goods abroad but must locally produce and accumulate capital. Suppose the economy starts out with a small initial capital stock and no claims against the rest of the world. Meanwhile, the world economy is on a steady state growth path, for the same preferences and technologies. It turns out this small economy may initially run a trade surplus. Growth requires capital accumulation, and a country with a low initial capital stock is poor in present value, even if it will grow quickly. Since the economy is open, consumption will grow at the same rate as world consumption. But if the initial capital stock is sufficiently low, then initial consumption can be so low that the country actually exports consumption goods during the initial stages of development.

5.1 The Rest of the World

Preferences everywhere are determined by $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$ and the production function for consumption goods is the combination of labor-only and Cobb-Douglas technologies given in (23). The rest of the world has a high initial capital stock that exceeds κ and is therefore growing at the constant rate g given in (26). The associated world interest rate is $r = \rho + \sigma g$, which gives (25).

5.2 The Small Open Economy

Preferences and technology are the same as in the rest of the world, but the initial capital stock satisfies $k_0 \in (0, \kappa)$, where κ is the threshold defined in (28). In autarky, this economy would not yet be growing at the steady state rate g, and unskilled wages would not yet be growing at all. Let b_0 be the initial claims of this small open economy on the rest of the world. Consumers in this country maximize

$$\int_0^\infty e^{-\rho t} u(c_t) \mathrm{d}t$$

subject to

$$\mathrm{D}b_t = rb_t + f(x_t) - c_t$$

and

$$\mathrm{D}k_t = A \cdot (k_t - x_t), \qquad x_t \in [0, k_t]$$

and a borrowing constraint

$$\liminf_{t \to \infty} e^{-rt} q_t b_t \geq 0$$

that is imposed by the rest of the world. Consumption goods can be traded, but capital and the fixed factor implicit in $f(x_t)$ cannot. For example, the fixed factor could be unskilled labor and capital could be the human capital of skilled workers, and neither type of worker can move.

Consumers in the small open economy therefore solve

$$\max_{c_t \geq 0} \left\{ \int_0^\infty e^{-\rho t} u(c_t) \mathrm{d}t : \int_0^\infty e^{-rt} c_t \mathrm{d}t \leq a_0 + b_0 \right\}$$

given some initial net foreign assets b_0, and

$$a_0 = \max_{x_t \in [0, k_t]} \left\{ \int_0^\infty e^{-rt} f(x_t) \mathrm{d}t : \mathrm{D}k_t = A \cdot (k_t - x_t) \right\},$$

given the initial capital stock $k_0 \in (0, \kappa)$. It must be that $a_0 + b_0$ is positive. Initial net foreign liabilities, if any, cannot be to high. With that assumption, the consumer problem is well defined and standard. It implies $r = \rho + \sigma \mathrm{D}c_t / c_t$ and thus $\mathrm{D}c_t / c_t = g$. Consumption grows at the same rate as it does in the rest of the world. A present-value calculation gives $c_0 = (r - g)(a_0 + b_0)$. Other than the level of the consumption path, the small open economy may differ from the rest of the world in its allocation of capital and the resulting factor prices.

Let v_t and w_t be the factor prices of capital and labor, and write q_t for the price of capital. Then

$$a_0 = q_0 k_0 + \int_0^\infty e^{-rt} w_t \mathrm{d}t.$$

The optimal allocation of capital requires

$$\begin{aligned} \mathrm{D}f(x_t) &\leq v_t, \quad \text{w.e. if } x_t > 0, \\ Aq_t &\leq v_t, \quad \text{w.e. if } k_t > x_t. \end{aligned}$$

Capital does not depreciate and cannot be consumed. Thus a positive supply of capital is held in equilibrium. This implies $v_t = \max\{Aq_t, \mathrm{D}f(x_t)\}$ and therefore

$$rq_t = v_t + \mathrm{D}q_t.$$

Together with the differential equation for capital, this determines the dynamics of $q_t k_t$. The borrowing constraint together with the fact that consumers want to exhaust their present-value borrowing constraint implies that

$$\lim_{t \to \infty} e^{-rt} q_t k_t = 0$$

in any equilibrium. Gross output will again be $y_t = f(x_t) + q_t A \cdot (k_t - x_t)$. Since only consumption goods are traded, the balance of trade is simply $f(x_t) - c_t$.

5.2.1 The Equilibrium

Suppose $x_t \in (0, \xi) \cap (0, k_t)$. Then $Aq_t = v_t = \mathrm{D}f(x_t) = (1-\alpha)\xi^{-\alpha}$. But then also $\mathrm{D}q_t = -(A-r)q_t$. The world interest rate satisfies $r \in (\rho, A)$ and so the price of capital must be declining at a positive rate, and hence so does the rental price of capital. But this rental price is also equal to the marginal product of capital in the consumption goods producing sector of the economy, which is constant as long as $x_t \in (0, \xi)$. This means that x_t cannot be in this regime for more than an instant. As in the closed economy, one can also use $A > r$ to rule out the possibility of permanent stagnation.

Conjecture that initially $x_t = 0 < k_t$. Then

$$\mathrm{D}q_t = -(A-r)q_t, \quad \mathrm{D}f(0) \leq Aq_t, \quad \mathrm{D}k_t = Ak_t.$$

Thus capital is accumulated at its maximal rate. The price of capital declines at the rate $A - r$, and the value of the capital stock grows at the interest rate r. The declining price of capital implies that Aq_t will reach $\mathrm{D}f(0)$ in finite time, and so this regime can only last for a finite amount of time.

To summarize, either $x_t = 0$ or $x_t \geq \xi$, and the $x_t = 0$ regime can only last for a finite amount of time. The amount of capital assigned to producing consumption is either zero, or it is in the Cobb-Douglas range of the production function. In this open economy, there will be no extended transition period during which some part of the labor force is using the labor-only technology and another part is combined with capital. The transition will be instantaneous. In a closed economy, such an instantaneous transition would imply a suboptimal jump in consumption, but here trade in consumption goods can be used to smooth consumption. Of course, this instantaneous reallocation of capital and labor is extreme. But one expects the tendency of an open economy to specialize and transition more quickly than a closed economy to survive even when reallocation is costly.

High Initial Capital Suppose $x_t \in (\xi, k_t)$. Then it must be that

$$\mathrm{D}q_t = -(A-r)q_t, \quad \mathrm{D}f(x_t) = Aq_t, \quad \mathrm{D}k_t = A \cdot (k_t - x_t).$$

These are the same equilibrium conditions as faced by the rest of the world. The first two imply that $\mathrm{D}x_t/x_t$ is the same as in the rest of the world. The ratio x_t/k_t must then also be the same as in the rest of the world. If it were higher, $k_t - x_t$ would hit zero in finite time x_t cannot continue to grow. If it were lower, the discounted value of the capital stock would not converge to zero. Thus $x_t/k_t = (r-g)/A$, or (27), as in the rest of the world.

Given that $x_t/k_t = (r-g)/A$, the condition $x_t > \xi$ can only hold if $k_t \geq \kappa$, where κ is the same threshold (28) that applies in the rest of the world. For all $k_t \in (\kappa, \infty)$, capital is accumulated at the rate $(A-r)/\alpha$ and the price of capital will be $q_t = \mathrm{D}f(x_t)/A$. Given some $k_0 > \kappa$, the present value of all consumption produced is determine by

$$a_0 = \frac{f(x_0)}{r-g}, \quad \frac{x_0}{k_0} = \frac{r-g}{A}.$$

The price of capital is

$$q_0 = \frac{1}{A}\mathrm{D}f(x_0) = \frac{1-\alpha}{A}\left(\frac{(r-g)k_0}{A}\right)^{-\alpha}$$

and so $x_0/k_0 = (r-g)/A$ gives

$$q_0 k_0 = (1-\alpha)a_0.$$

As expected, the value of the capital stock is the capital share (taking into account the fact that $x_t > \xi$) of the present value of all consumption produced in the small open economy. This describes the equilibrium for any $k_0 > \kappa$.

Low Initial Capital Starting from a low initial capital stock $k_0 < \kappa$, it is not possible that $x_t > \xi$. As argued, the only alternative is then $x_t = 0$. Capital is used initially only to accumulate more capital, at the maximal rate A. The time it takes for the capital stock to reach the threshold κ is therefore given by

$$T = \frac{1}{A}\ln\left(\frac{\kappa}{k_0}\right). \tag{33}$$

Clearly, T can take only any value in $(0, \infty)$, depending on how far k_0 is from κ. Note from (32) that $T > \tau$, where $[0, \tau]$ is the period during which $x_t = 0$ a the closed economy with the same initial capital stock. In the closed economy, capital will be slowly reallocated from capital accumulation to the production of consumption, beginning at some time τ when the capital stock has not yet reached κ. In the small open economy, $x_t = 0$ as long as k_t is below κ, and when k_t reaches κ, x_t jumps from 0 to ξ.

Since unskilled wages are stuck at $f(0) = \alpha f(\xi)$ as long as k_t is below κ, one implication of this is that unskilled wages will be at this low level for a longer period of time in a small open economy than they are in a closed economy.[4] Since international trade can be used to smooth consumption, there is no need to begin producing more consumption before k_t has reached κ. Once time T arrives, x_t jumps from 0 to ξ, where the marginal product of unskilled labor is still $f(0) = \alpha f(\xi)$. From then on, the high-capital equilibrium prevails, and the output of consumption and unskilled wages will grow at the balanced growth rate g. The output of consumption jumps upwards at T but the path of wages is smooth. The price of capital cannot ever be expected to jump, and so at T it must be that $q_T = \mathrm{D}f(\xi)/A$. While $x_t = 0$, the price of capital declines at the rate $A - r$, and hence the initial price of capital must be $q_0 = e^{(A-r)T} q_T$.

Initial Wealth and Consumption So far, world interest rates have been used to determine production and factor prices in the small open economy. The budget constraint of the representative consumer implies $c_0 = (r - g)(a_0 + b_0)$, and so it only remains to calculate the domestic component of wealth.

Starting from $k_0 < \kappa$, the present value of all consumption produced in the small open economy equals

$$\begin{aligned} a_0 &= \int_0^\infty e^{-rt} f(x_t)\mathrm{d}t \\ &= \int_0^T e^{-rt} f(0)\mathrm{d}t + e^{-rT} \int_T^\infty e^{-(r-g)(t-T)} f(\xi)\mathrm{d}t = \frac{f(0)}{r} + e^{-rT} \times \left[\frac{f(\xi)}{r-g} - \frac{f(0)}{r} \right]. \end{aligned}$$

The first term is the present value of consumption associated with permanent stagnation, discounted at the world interest rate. The second term is the difference between the present values associated with steady-state growth and with permanent stagnation, discounted to account for the delayed arrival of balanced growth. Since $g \in (0, r)$ and $f(\xi) > f(0)$, this second term is positive.

To determine initial wealth as a function of the initial capital stock, use the expression (33) for T to write

$$a_0 = \frac{f(0)}{r} + \left(\frac{k_0}{\kappa} \right)^{r/A} \times \left[\frac{f(\xi)}{r-g} - \frac{f(0)}{r} \right], \quad k_0 \in (0, \kappa]. \tag{34}$$

In the high-capital stock regime, $x_t/k_t = (r-g)/A$ and output of consumption grows at the rate g. Hence

$$a_0 = \frac{f(x_0)}{r-g}, \quad x_0 = \frac{(r-g)k_0}{A}, \quad k_0 \in (\kappa, \infty). \tag{35}$$

[4] This delayed rise in unskilled wages opens up the possibility that households who can supply only unskilled labor might prefer the economy to be closed.

This is continuous at κ. Recall that $r \in (\rho, A)$, and so $r/A < 1$. More specifically, note from (25) that

$$\frac{r}{A} = 1 - \left(\frac{1 - \frac{\rho}{A}}{\alpha + (1-\alpha)\sigma} \right) \times \alpha > 1 - \alpha,$$

since $\rho > (1-\alpha)(1-\sigma)A$ implies that the factor multiplying α is in $(0,1)$. Thus the slope $\partial a_0 / \partial k_0$ drops as k_0 passes through κ from below. Wealth is an increasing and concave function of the initial capital stock $k_0 > 0$, and bounded below by $f(0)/r$.

It is useful to note that a_0 is continuous at $k_0 = 0$. A small open economy with no initial capital cannot accumulate capital, and hence will be producing $f(0)$ forever. At world interest rates, this has the present value $f(0)/r$, which is exactly what (34) converges to as k_0 goes to zero. Wealth in a small open economy with very little capital is pretty much what it would be if the economy could not accumulate capital at all.

5.2.2 The Trade Balance

Suppose the initial capital stock is in $(0, \kappa)$ so that all capital is initially used to produce more capital and the small open economy initially produces only a flow of $f(0)$ units of consumption. Suppose further that the net foreign asset position of the small open economy is initially zero. Thus $b_0 = 0$ and initial wealth is given by (34). Then $c_0 = (r-g)a_0$ gives

$$c_0 = (r-g) \left(\frac{1 - e^{-rT}}{r} \right) f(0) + e^{-rT} f(\xi). \tag{36}$$

The small open economy will run an initial trade surplus if $c_0 < f(0)$. This is equivalent to

$$\left[r f(\xi) - (r-g) f(0) \right] e^{-rT} < g f(0).$$

Note that $f(0) = \alpha f(\xi)$ and $r - \alpha(r-g) = (1-\alpha)r + \alpha g$ is positive. Thus this condition can be written as

$$e^{-rT} < \frac{\alpha g}{(1-\alpha)r + \alpha g},$$

or, using the definition of T given in (33),

$$\left(\frac{k_0}{\kappa} \right)^{r/A} < \frac{\alpha g}{(1-\alpha)r + \alpha g}. \tag{37}$$

Irrespective of the parameters of the economy, if the world economy is on its balanced growth path, then there will be a k_0 low enough so that this condition holds. The small open economy runs a trade surplus during its initial stages of growth if its initial capital stock is low enough.

Proposition *Suppose the initial net foreign asset position of the small open economy is zero. The small open economy will not trade with the rest of the world if $k_0 \geq \kappa$. Define*

$$\frac{k_*}{\kappa} = \left(\frac{\alpha g}{(1-\alpha)r + \alpha g} \right)^{A/r}.$$

Then $k_0 \in [0, k_*)$ implies an initial trade surplus and $k_0 \in (k_*, \kappa)$ an initial trade deficit.

An initial trade surplus may persist in the long run or turn into a deficit over time, depending on the initial level of the capital stock. To see this, observe that $b_0 = 0$ implies a net foreign asset position

$$b_t = e^{rt} \int_0^t e^{-rs}[f(x_s) - c_s]\mathrm{d}s.$$

Consumption is $c_t = c_0 e^{gt}$, where c_0 is given in (36). Output of consumption goods is $f(x_t) = f(0)$ as long as $t < T$, while $f(x_t) = f(\xi)e^{g(t-T)}$ for all $t \geq T$. The net foreign asset position of the small open economy is therefore

$$b_t = e^{rt} \left[\left(\frac{1 - e^{-rt}}{r} \right) f(0) - \left(\frac{1 - e^{-(r-g)t}}{r - g} \right) c_0 \right]$$

for $t \in [0, T]$ and,

$$b_t = e^{rt} \left[\left(\frac{1 - e^{-rT}}{r} \right) f(0) + e^{-rT} \left(\frac{1 - e^{-(r-g)(t-T)}}{r - g} \right) f(\xi) - \left(\frac{1 - e^{-(r-g)t}}{r - g} \right) c_0 \right].$$

for $t \in [T, \infty)$. As expected, (36) ensures that $e^{-rt}b_t$ converges to zero as t becomes large. At the threshold $t = T$, (36) and then $f(0) = \alpha f(\xi)$ yields

$$b_T = e^{gT} \left[\left(\frac{1 - e^{-rT}}{r} \right) \alpha - \left(\frac{e^{-gT} - e^{-rT}}{r - g} \right) \right] f(\xi).$$

This is the net foreign asset position of the small open economy when the country reaches its balanced growth path. Observe now that

$$\lim_{T \to \infty} e^{-gT} b_T = \frac{\alpha}{r} f(\xi)$$

and

$$\lim_{T \downarrow 0} \frac{b_T}{T} = \lim_{T \downarrow 0} \frac{1}{T} \left[\left(\frac{1 - e^{-rT}}{r} \right) \alpha - \left(\frac{e^{-gT} - e^{-rT}}{r - g} \right) \right] f(\xi) = -(1-\alpha) f(\xi).$$

Thus the small open economy will be a net creditor to the rest of the world at time T if T is large, and a net debtor if T is small. Recall from (33) that T ranges from 0 to

∞ as k_0 varies from κ to 0. If the country has a very low initial capital stock, it will reach its balanced growth path as a net creditor. If its initial capital stock is close to the threshold κ, it will reach balanced growth as a net debtor.

From date T on, the fact that consumption and output grow at the common rate g implies that $b_t = e^{g(t-T)} b_T$ for all $t > T$. The sign of the net foreign asset position will forever remain what it was at date T, and the size relative to output will be constant. Initially very poor countries will be creditors in the long run.

The basic intuition for these results follows immediately from the fact that wealth is a continuous function of the initial capital stock, both at zero capital, and at the level of capital that implies balanced growth. An economy with no capital and no initial claims on the rest of the world has wealth equal to the present value of a constant consumption flow and will choose an increasing path of consumption when faced with a world interest rate $r > \rho$. This implies initial trade surpluses, and an economy with very little initial capital is no different in this respect. Conversely, and economy with initial capital close to the threshold κ will have wealth close to that of an economy with κ units of capital, and thus c_0 close to $f(\xi)$. But it is still specializing as much as it can in capital accumulation and so produce only $f(0) < f(\xi)$ units of consumption, implying a deficit.

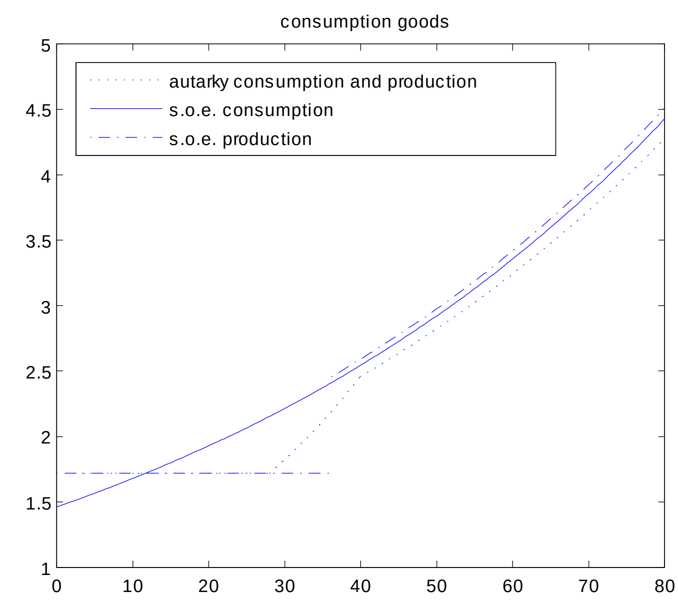

FIGURE 3 Autarky and Small Open Economy Transitions

The transition of a small open economy with an initial capital stock below the threshold k_* is shown in Figures 3-5. Figure 3 shows consumption, and the closed-economy consumption path is included for comparison. As explained earlier, there will be an instantaneous jump in output of consumption goods, from $f(0) = \alpha f(\xi)$ to $f(\xi)$, when k_t reaches κ. Output of consumption goods in the closed economy will reach $f(\xi)$ later, and along a smooth path. From then on, both economies grow at the rate g.

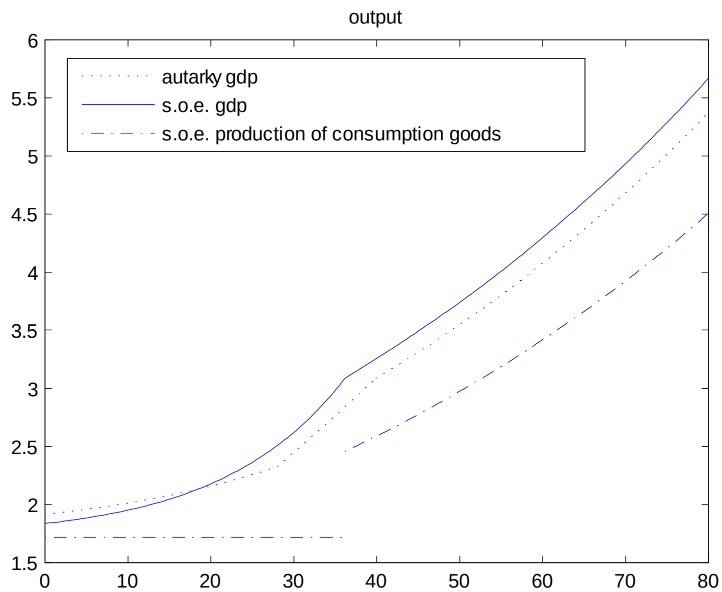

FIGURE 4 Autarky and Small Open Economy Transitions

The value of aggregate output of both consumption goods and capital, measured as $y_t = f(x_t) + q_t A(k_t - x_t)$, is shown in Figure 4, again with the closed economy path included for comparison. Since $f(0) = \alpha f(\xi)$ and $Aq_T = \mathrm{D}f(\xi) = (1-\alpha)f(\xi)/\xi$ the jump in GDP is $f(\xi) - f(0) + q_T A(0 - \xi) = 0$ at date T. At the initial date $y_0 = f(0) + q_0 A k_0$ in both the closed economy and the small open economy. But $q_0 = e^{(A-r)T}\mathrm{D}f(\xi)/A$ in the small open economy, and $q_0 = e^{(A-\rho)\tau}\mathrm{D}f(\xi)/A$ in the closed economy. As argued earlier, the closed economy begins to use capital to produce consumption earlier than the small open economy, and so $T > \tau$. But $A - r < A - \rho$. The low interest rate in the closed economy implies that the price of capital declines more quickly in the closed economy than in the small open economy. In Figure 5, this second effect dominates, resulting in a higher initial price of capital in the closed economy, and hence a higher market value of output in the initial stages. Over time, the higher rate at which the price of capital declines and the slower rate at which the capital stock is built up during the transition phase imply that the closed economy will eventually lag behind the small open economy.

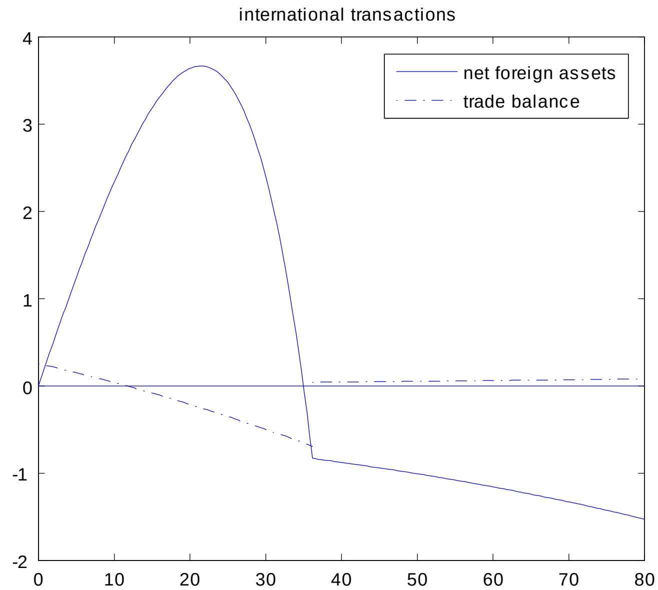

FIGURE 5 Small Open Economy—Low Initial Capital Stock

The trade surplus and the net foreign asset position for this small open economy are shown in Figure 5. As we already know from Figure 3, the initial capital stock is low enough that this economy will initially run a surplus. Figure 5 shows that the initial capital stock is close enough to κ to imply that the net foreign asset position of this economy is negative by the time it reaches its balanced growth path. A sufficiently low initial capital stock would have implied a positive net foreign asset position forever.

6. Concluding Remarks

China is not a small open economy and its rapid growth has real consequences in the rest of the world. But its recent pattern of development is remarkably similar to what is predicted here for an economy with a small initial capital stock. China is running a trade surplus, investment accounts for a very large share of its GDP, and unskilled wages have remained low. Of course, it will not make a great leap onto a balanced growth path at some specific date T, but this instantaneous adjustment is not in any case a robust prediction of the small open economy described here.

Knowledge can be embodied in skilled workers, managers or entrepreneurs. It can also be embodied in organizations, in the sense that no individual associated with the organization fully comprehends why a particular organization is successful, but experience simply shows that it is. Such organization capital can be accumulated through a process of experimentation and selection (Luttmer [2007]). Making this fully explicit and showing, without unmotivated functional form assumptions, how it can lead to persistent growth in the presence of a fixed factor, remains a subject for further research.

A Labor-Only and Cobb-Douglas Technologies

Suppose there is a labor-only technology that requires B units of effort to produce one unit of consumption. Alternatively, m units of labor and x units of capital can be used to produce $m^{\alpha}x^{1-\alpha}$ units of consumption. Given a capital stock x, output of consumption goods will be

$$f(x) = \max_{h,m \geq 0} \left\{ \frac{h}{B} + x^{1-\alpha}m^{\alpha} : h + m \leq 1 \right\}.$$

Consider a Lerner diagram for the two technologies. If both technologies are used, then there must be factor prices v and w such that

$$k^{1-\alpha}m^{\alpha} = 1, \quad \frac{vk}{wm} = \frac{1-\alpha}{\alpha}, \quad \frac{vk}{w} + m = B.$$

This implies $m = \alpha B$ and $k = 1/(\alpha B)^{\alpha/(1-\alpha)}$. The resulting capital-labor ratio is $k/m = \xi$, where

$$\xi = \frac{1}{(\alpha B)^{1/(1-\alpha)}}.$$

Both technologies will be used if x is in the diversification cone for this economy, $x \in (0, \xi)$. In that case

$$m = \frac{x}{\xi}, \quad h = 1 - \frac{x}{\xi},$$

and the resulting output is linear in x,

$$f(x) = \alpha\xi^{1-\alpha} + (1-\alpha)\xi^{-\alpha}x.$$

On the other hand, if $x \geq \xi$, then the labor-only technology will not be used, and so $f(x) = x^{1-\alpha}$. Observe that f is continuously differentiable at $x = \xi$.

B The Limit

Recall

$$F(k,1) = \frac{(A_{n+1} - A_n) + (B_n - B_{n+1})k}{A_{n+1}B_n - A_nB_{n+1}}, \quad k \in \left[\frac{A_n}{B_n}, \frac{A_{n+1}}{B_{n+1}}\right]$$

This means that if n such that

$$\frac{A_n}{B_n} \leq k < \frac{A_{n+1}}{B_{n+1}}$$

then

$$\frac{1}{B_n} \leq F(k,1) < \frac{1}{B_{n+1}}.$$

Moreover, for any k there will be a unique $n = N(k)$ so that the above inequalities hold.

Now consider the geometric case $A_n = \alpha^n$, $B_n = \beta^n$, with $\alpha > 1 > \beta$. Take any $k > \alpha/\beta$. Then there is a unique $N(k) \in \mathbb{N}$ that satisfies $(\alpha/\beta)^{N(k)} \leq k < (\alpha/\beta)^{N(k)+1}$, or

$$-1 + \frac{\ln(k)}{\ln(\alpha/\beta)} \leq N(k) < \frac{\ln(k)}{\ln(\alpha/\beta)}.$$

For these $N(k)$, we have

$$\frac{1}{\beta^{N(k)}} \leq F(k,1) < \frac{1}{\beta^{N(k)+1}}.$$

Now take

$$\alpha = e^{a\Delta}, \quad \beta = e^{-b\Delta}$$

for some $a, b > 0$. Write F_Δ for the resulting production function and N_Δ for the associated grid. Then

$$-\Delta + \frac{\ln(k)}{a+b} \leq N_\Delta(k)\Delta < \frac{\ln(k)}{a+b}$$

and

$$e^{bN_\Delta(k)\Delta} \leq F_\Delta(k,1) < e^{b(N_\Delta(k)+1)\Delta}$$

for all $k \geq e^{(a+b)\Delta}$. Taking $\Delta \downarrow 0$ gives

$$\lim_{\Delta \downarrow 0} N_\Delta(k)\Delta = \frac{\ln(k)}{a+b}$$

uniformly in k. Hence

$$\lim_{\Delta \downarrow 0} F_\Delta(k,1) = k^{\frac{a}{a+b}}$$

and the convergence is uniform for $\ln(F_\Delta(k,1))$ and $k > 1$.

We can summarize this as

$$F_\Delta(K,L) = \max_{\{k_n, l_n\}_{n=1}^\infty} \left\{ \sum_{n=1}^\infty e^{\theta\Delta n} \min\{k_n, l_n\} : \sum_{n=1}^\infty e^{\gamma\Delta n} k_n \leq K, \ \sum_{n=1}^\infty l_n \leq L \right\} \to K^{1-\theta/\gamma} L^{\theta/\gamma}$$

as $\Delta \downarrow 0$. So, if from one technology to the next, the amount of human capital required to employ one unskilled worker grows by a factor $e^{\gamma\Delta}$, and the resulting output grows by a factor $e^{\theta\Delta}$, then the resulting technology is $K^{1-\theta/\gamma} L^{\theta/\gamma}$.

A shift that causes a decrease in θ and an increase in γ will lower the factor share of unskilled labor. This says that the factor share of unskilled labor will decline if K is increasing over time and it takes larger increases in capital to adopt new technologies that have smaller gains in labor productivity than before.

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