Permanent Primary Deficits in an Economy with Aggregate and Idiosyncratic Risk

Amol Amol and Erzo G.J. Luttmer

University of Minnesota

May 2024

introduction

- we left aggregate uncertainty out of Amol and Luttmer [2022]
  - that is, Minneapolis Fed working paper 794
  - there was already a lot going on in that paper
  - and we knew it did not affect our results
  - see for example footnote 20
- still, it may be useful to know what the risk premium on nominal government debt will be when there is aggregate risk
- so here it goes
  - the notation is as in Amol and Luttmer [2022], except,
  - there we wrote σY_t for the flow of baby bonds transferred to newborn consumers,
  - while here we write ϑY_t for that flow,
  - and use σ instead as a parameter that governs aggregate risk
  - we now also allow individual labor supplies to depreciate, following Blanchard [1985]

demographics and labor supply

- there is a flow $\delta > 0$ of newborn consumers
  - they die randomly at the same rate δ
  - the age distribution is stationary
  - so the measure of consumers is 1
- everyone at age $a \geq 0$ supplies $L_y e^{-\lambda a} > 0$ units of labor
  - inelastically, conditional on survival
  - we assume $\delta + \lambda > 0$
  - it is easy to add a transitory episode during which young workers gain experience
- the aggregate labor supply is

$$L = \int_0^\infty \delta e^{-\delta a} L_y e^{-\lambda a} \mathrm{d}a = \frac{\delta L_y}{\delta + \lambda}$$

preferences

- given consumption flows $C_{j,t}$ and a Brownian motion $\widetilde{Z}_{j,t}$,
  - the utility process $U_{j,t}$ of consumer j evolves according to

$$\mathrm{d}U_{j,t} = U_{j,t}\left(\mathcal{A}_t U_j \mathrm{d}t + \mathcal{S}_t U_j' \mathrm{d}\widetilde{Z}_{j,t}\right)$$

  where $\mathcal{A}_t U_j$ and $\mathcal{S}_t U_j$ satisfy

$$(\rho+\delta)U_{j,t}^{1-1/\varepsilon} = (\rho+\delta)C_{j,t}^{1-1/\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)U_{j,t}^{1-1/\varepsilon}\left(\mathcal{A}_t U_j - \frac{1}{2}\xi\|\mathcal{S}_t U_j\|^2\right)$$

  - utility is homogeneous of degree 1 in consumption trajectories
- we will have $\widetilde{Z}_{j,t} = [Z_t, Z_{j,t}]'$
- Epstein-Zin
  - the intertemporal elasticity of substitution equals $\varepsilon \in (0,1)\cup(1,\infty)$
  - the $\varepsilon \to 1$ limit is well defined
  - the coefficient of relative risk aversion equals $\xi > 0$

the technology

- standard Brownian motions
  - a common shock Z_t
  - idiosyncratic shocks $Z_{j,t}$
- capital stock $K_{j,t}$ of a consumer j evolves according to

$$\mathrm{d}K_{j,t} = (\mu K_{j,t} - X_{j,t})\,\mathrm{d}t + K_{j,t}\,(\sigma \mathrm{d}Z_t + \varsigma \mathrm{d}Z_{j,t}) + \mathrm{d}I_{j,t},$$

  - here, $X_{j,t} \geq 0$ is a flow of capital used up to produce consumption
  - and $I_{j,t}$ represents cumulative additions of capital
- in the aggregate,

$$K_t = \int_0^1 K_{j,t}\mathrm{d}j, \quad 0 = \int_0^1 I_{j,t}\mathrm{d}j, \quad X_t = \int_0^1 X_{j,t}\mathrm{d}j,$$

- the technology in the consumption sector is Cobb-Douglas,

$$Y_t = X_t^{1-\alpha} L^{\alpha},$$

- the aggregate stock of capital K_t evolves according to

$$\mathrm{d}K_t = K_t\,(\mu \mathrm{d}t + \sigma \mathrm{d}Z_t) - X_t \mathrm{d}t$$

the consumption sector

- two prices in units of consumption
  - the wage is w_t
  - the price capital is q_t
- so the market value of the aggregate capital stock is $q_t K_t$
- recall that X_t is the *flow* of capital used up to produce consumption
  - the Cobb-Douglas technology therefore gives

$$\begin{bmatrix} q_t X_t \\ w_t L \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} Y_t, \quad Y_t = X_t^{1-\alpha} L^{\alpha}$$

- we will construct a steady state equilibrium with

$$\frac{X_t}{K_t} = x \in (0, \infty)$$

so that

$$\mathrm{d}K_t = K_t \left((\mu - x)\mathrm{d}t + \sigma \mathrm{d}Z_t \right)$$

- so K_t, X_t, Y_t, w_t, and q_t will all be geometric Brownian motions

output and the price of capital

- recall that
$$\mathrm{d}K_t = K_t\left((\mu - x)\mathrm{d}t + \sigma \mathrm{d}Z_t\right),$$
together with $X_t = xK_t$, $Y_t = X_t^{1-\alpha}L^{\alpha}$ and $q_t X_t = (1-\alpha)Y_t$

- Ito's lemma implies
$$\begin{aligned} \mathrm{d}Y_t &= Y_t\left(g\mathrm{d}t + \sigma_C \mathrm{d}Z_t\right) \\ \mathrm{d}q_t &= q_t\left(\mu_q \mathrm{d}t + \sigma_q \mathrm{d}Z_t\right) \end{aligned}$$
where
$$\begin{bmatrix} g - \frac{1}{2}\sigma_C^2 \\ \mu_q - \frac{1}{2}\sigma_q^2 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ -\alpha \end{bmatrix}\left(\mu - x - \frac{1}{2}\sigma^2\right), \quad \begin{bmatrix} \sigma_C \\ \sigma_q \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ -\alpha \end{bmatrix}\sigma$$

- note that this implies $\sigma_C = \sigma + \sigma_q$ and
$$x = \mu + \mu_q + \sigma\sigma_q - g$$
  - alternatively, apply Ito to $q_t K_t = (1-\alpha)Y_t/x$
  - note that $x = q_t X_t/(q_t K_t)$ is a dividend-price ratio

▶ the expected return on capital $\mu + \mu_q + \sigma\sigma_q$ exceeds g by $x \in (0, \infty)$

the government

- aggregate household consumption is C_t
- the government consumes
$$G_t = \gamma C_t$$
and so $Y_t = C_t + G_t = (1+\gamma)C_t$
- consumption tax revenues are $T_t = \tau C_t$ and so
$$E_t = (1+\tau)C_t$$
gives real consumption expenditures
- the flow of transfers to newborn consumers at time t is $\vartheta Y_t \geq 0$

▶ the *surplus ratio* of the government is
$$\mathcal{S} = 1 - \frac{(1+\gamma)(1+\vartheta)}{1+\tau} = \frac{T_t - (G_t + \vartheta Y_t)}{E_t} \leq 1$$
and this may well be negative

- the primary surpluses of the government are $\mathcal{S} \times E_t$
  - always of the same sign...
  - and E_t follows the same geometric Brownian motion as Y_t and C_t

treasury direct

- the government issues deposits
  - at time $t = 0$, the supply of these deposits is $D_0 > 0$
  - the government also pays interest, by issuing more deposits
  - the time-t nominal interest rate is $i_t \geq 0$
  - the price of consumption in units of government deposits is P_t
- as long as $1/P_t$ is positive, the supply D_t of government deposits evolves according to

$$\mathrm{d}D_t = i_t D_t \mathrm{d}t - \mathcal{S} P_t E_t \mathrm{d}t$$

- two *conjectures* about equilibrium

1. the price level satisfies

$$\mathrm{d}P_t = P_t \left(\pi_t \mathrm{d}t + \sigma_{P,t} \mathrm{d}Z_t \right)$$

   for some π_t and $\sigma_{P,t}$ to be determined

2. the real value of government securities satisfies

$$\frac{D_t}{P_t E_t} = \texttt{constant}$$

applying Ito's lemma

- government deposits follow

$$\mathrm{d}D_t = i_t D_t \mathrm{d}t - \mathcal{S} P_t E_t \mathrm{d}t,$$

and nominal consumption expenditures follow

$$\mathrm{d}\left(P_t E_t\right) = \left(P_t E_t\right)\left(\left(\pi_t + g + \sigma_C \sigma_{P,t}\right)\mathrm{d}t + (\sigma_{P,t} + \sigma_C)\mathrm{d}Z_t\right)$$

- this implies

$$\mathrm{d}\left(\frac{D_t}{P_t E_t}\right) = \frac{D_t}{P_t E_t} \times \left(i_t - (\pi_t + g + \sigma_C \sigma_{P,t}) + (\sigma_{P,t} + \sigma_C)^2\right)\mathrm{d}t$$

$$-\frac{D_t}{P_t E_t} \times (\sigma_{P,t} + \sigma_C)\mathrm{d}Z_t - \mathcal{S}\mathrm{d}t.$$

▶ forcing $D_t/(P_t E_t)$ to be constant gives

$$\left(i_t - \pi_t + \sigma_C^2 - g\right) \times \frac{D_t}{P_t E_t} = \mathcal{S}, \quad \sigma_{P,t} = -\sigma_C \tag{1}$$

▶ the real cumulative returns R_t on government deposits are

$$\mathrm{d}R_t = R_t\left(\mu_R \mathrm{d}t + \sigma_R \mathrm{d}Z_t\right) \tag{2}$$

where

$$\mu_R = i_t - \pi_t + \sigma_C^2, \quad \sigma_R = \sigma_C \tag{3}$$

newborn wealth

- recall that, in the aggregate,

$$w_t L = (1-\alpha)Y_t, \quad \mathrm{d}Y_t = Y_t\left(g\mathrm{d}t + \sigma_C \mathrm{d}Z_t\right)$$

  and the cumulative returns on government deposits satisfy

$$\mathrm{d}R_t = R_t\left(\mu_R \mathrm{d}t + \sigma_R \mathrm{d}Z_t\right), \quad \sigma_R = \sigma_C.$$

- individual labor supply
  - depreciates gradually at the rate $\lambda \geq 0$
  - drops to zero randomly at the rate $\delta > 0$
  - assume perfect annuity markets
- the present value of a time-t unit of labor is

$$s_t = \frac{w_t}{\delta + \lambda + \mu_R - g}$$

- the newborn cohort at time t starts with $s_t L_y$ and baby bonds,
  - their initial wealth $W_{y,t}$ is determined by

$$\delta\left(W_{y,t} - s_t L_y\right) = \vartheta Y_t.$$

aggregate wealth

- the aggregate wealth of consumers alive at time t is

$$W_t = q_t K_t + \frac{w_t L}{\delta + \lambda + \mu_R - g} + \frac{D_t}{P_t}$$

  – none of this is risk-free

- portfolio shares will be constant in equilibrium

  – define $\psi = q_t K_t / W_t$ and note that

$$\psi = \frac{q_t X_t}{E_t} \times \frac{1}{X_t / K_t} \times \frac{E_t}{W_t}, \qquad 1 - \psi = \left(\frac{w_t L / E_t}{\delta + \lambda + \mu_R - g} + \frac{D_t}{P_t E_t} \right) \frac{E_t}{W_t}$$

  – now use

$$\frac{X_t}{K_t} = x = \mu + \mu_q + \sigma \sigma_q - g, \qquad \begin{bmatrix} q_t X_t \\ w_t L \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 - \alpha \end{bmatrix} Y_t, \quad \frac{Y_t}{E_t} = \frac{1 + \gamma}{1 + \tau}$$

- this gives

$$\psi = \frac{\alpha}{\mu + \mu_q + \sigma \sigma_q - g} \frac{1 + \gamma}{1 + \tau} \frac{E_t}{W_t}, \qquad 1 - \psi = \left(\frac{1 - \alpha}{\delta + \lambda + \mu_R - g} \frac{1 + \gamma}{1 + \tau} + \frac{D_t}{P_t E_t} \right) \frac{E_t}{W_t}$$

consumer j wealth

- for consumer j at time t,

$$W_{j,t} = q_t K_{j,t} + \frac{D_{j,t}}{P_t} + B_{j,t}$$

  - $K_{j,t}$ is capital subject to idiosyncratic and aggregate risk
  - $D_{j,t}$ is the nominal value of government deposits
  - $B_{j,t}$ is the real value of a real risk-free bank account

- the respective cumulative return processes are

$$\mathrm{d}\begin{bmatrix} R_{j,t} \\ R_t \\ R_{B,t} \end{bmatrix} = \begin{bmatrix} R_{j,t}\left(\mu + \mu_q + \sigma\sigma_q\right) \\ R_t\mu_R \\ R_{B,t}r \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \sigma + \sigma_q \\ \sigma_R \\ 0 \end{bmatrix}\mathrm{d}Z_t + \begin{bmatrix} \varsigma \\ 0 \\ 0 \end{bmatrix}\mathrm{d}Z_{j,t}$$

  where

$$\sigma_R = \sigma_C = (1-\alpha)\sigma = \sigma + \sigma_q$$

▶ the drift parameters to be determined are

$$\mu_q, \qquad \mu_R = i_t - \pi_t + \sigma_C^2, \qquad r$$

Merton portfolios

- consumers solve a Merton problem with a risk-aversion coefficient ξ
  - everyone faces the same risk-return trade-offs
- in equilibrium, they must choose ψ so that

$$\begin{bmatrix} \sigma_R^2 & \sigma_R^2 \\ \sigma_R^2 & \sigma_R^2 + \varsigma^2 \end{bmatrix} \begin{bmatrix} 1-\psi \\ \psi \end{bmatrix} = \frac{1}{\xi} \begin{bmatrix} \mu_R - r \\ \mu + \mu_q + \sigma\sigma_q - r \end{bmatrix}$$

  - note that

$$\begin{bmatrix} \sigma_R^2 & \sigma_R^2 \\ \sigma_R^2 & \sigma_R^2 + \varsigma^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \sigma_R^2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \varsigma^2$$

- this implies

$$1 = \frac{\mu_R - r}{\xi \sigma_R^2}, \quad \psi = \frac{\mu + \mu_q + \sigma\sigma_q - \mu_R}{\xi \varsigma^2}$$

  - recall that $x = \mu + \mu_q + \sigma\sigma_q - g$, and so

$$x = \xi\varsigma^2\psi + \mu_R - g$$

what we have so far

- given E_t/W_t,

$$(\mu_R - g) \times \frac{D_t}{P_t E_t} = 1 - \frac{(1+\gamma)(1+\vartheta)}{1+\tau} \quad (1)$$

$$\psi = \frac{\alpha}{\xi \varsigma^2 \psi + \mu_R - g} \frac{1+\gamma}{1+\tau} \frac{E_t}{W_t} \quad (2)$$

$$1 - \psi = \left(\frac{1-\alpha}{\delta + \lambda + \mu_R - g} \frac{1+\gamma}{1+\tau} + \frac{D_t}{P_t E_t} \right) \frac{E_t}{W_t} \quad (3)$$

can be solved for $\mu_R - g \in (-(\delta+\lambda), \infty)$, $\psi \in (0,1)$, and $D_t/(P_t E_t) \geq 0$

- then $r - g$, x, and g follow from

$$\begin{bmatrix} r - g \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\mu_R - g) + \begin{bmatrix} -\sigma_C^2 \\ \psi \varsigma^2 \end{bmatrix} \xi \quad (4)$$

and

$$g - \frac{1}{2}\sigma_C^2 = (1-\alpha)\left(\mu - x - \frac{1}{2}\sigma^2\right), \quad (5)$$

where $\sigma_C = (1-\alpha)\sigma$

the Epstein-Zin decision rule

- the budget constraint implies

$$\mathrm{d}C_{j,t} = C_{j,t}\left(g_y \mathrm{d}t + \sigma_R \mathrm{d}Z_t + \varsigma \mathrm{d}Z_{j,t}\right)$$

where

$$g_y = \mu_R - \rho + \psi(\mu + \mu_q + \sigma\sigma_q - \mu_R) - \frac{E_t}{W_t}, \qquad \psi = \frac{\mu + \mu_q + \sigma\sigma_q - \mu_R}{\xi\varsigma^2}$$

- the optimal consumption-wealth ratio is

$$\frac{E_t}{W_t} = \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\left(\sigma_R^2 + \varsigma^2\psi^2\right)\right)$$

  - so this uses risk-adjusted consumption growth

- the results for x and g give E_t/W_t as a function of $\mu_R - g$ and ψ
  - see Amol and Luttmer [2022] for some magic possibilities…

- utility for a consumer j with wealth $W_{j,t}$ is determined by

$$U_{j,t} = C_{j,t}\left(\frac{E_t/W_t}{\rho + \delta}\right)^{-1/(1-1/\varepsilon)}, \quad C_{j,t} = \frac{W_{j,t}}{1+\tau}\frac{E_t}{W_t}$$

the equilibrium conditions for $\varepsilon = 1$

- the conditions for $\mu_R - g \in (-\delta, \infty)$ and $\psi \in (0,1)$,

$$\psi \;=\; \left(\frac{1-\alpha}{\xi\varsigma^2\psi + \mu_R - g}\frac{1+\gamma}{1+\tau}\right)(\rho+\delta), \tag{1}$$

$$1-\psi \;=\; \left(\frac{\alpha}{\delta+\lambda+\mu_R - g}\frac{1+\gamma}{1+\tau} + \frac{D}{PE}\right)(\rho+\delta), \tag{2}$$

where

$$(\mu_R - g) \times \frac{D}{PE} = 1 - \frac{(1+\gamma)(1+\vartheta)}{1+\tau}, \quad \frac{D}{PE} \geq 0, \tag{3}$$

- then $r-g$, x, and g follow from

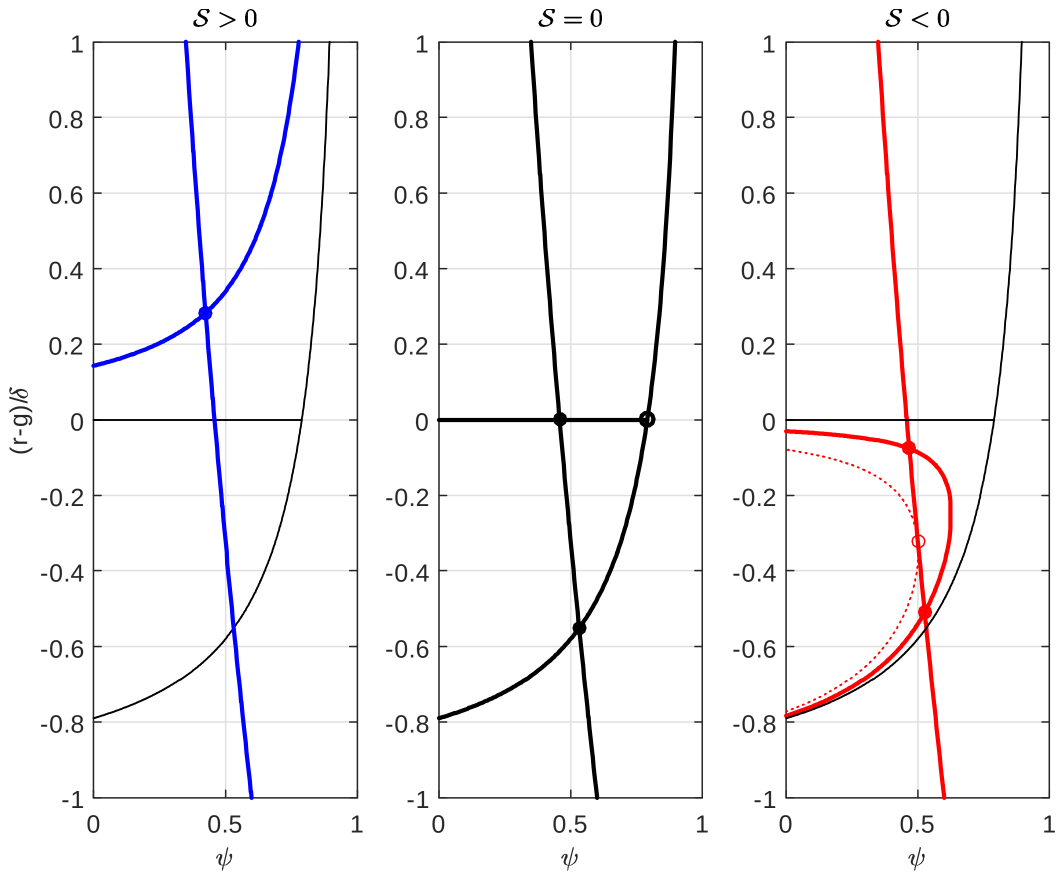
$$\begin{bmatrix} r-g \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\mu_R - g) + \begin{bmatrix} -\sigma_C^2 \\ \psi\varsigma^2 \end{bmatrix}\xi \tag{4}$$

and

$$g - \frac{1}{2}\sigma_C^2 = (1-\alpha)\left(\mu - x - \frac{1}{2}\sigma^2\right), \tag{5}$$

where $\sigma_C = (1-\alpha)\sigma$

equations (1) and (2), with $D/(PE)$ eliminated using (3)



- this is for $\sigma_C^2 = 0$
  - to do $\sigma_C^2 > 0 \ldots$
  - just replace $(r - g)/\delta$ with $(\mu_R - g)/\delta$ on the vertical axes

an alternative statement of the equilibrium conditions

- condition (1) is the same as

$$\mu_R - g = -\xi\varsigma^2\psi + \frac{1-\alpha}{\psi}\frac{1+\gamma}{1+\tau} \times (\rho + \delta)$$

  – so ψ is decreasing in $\mu_R - g$,

  – and hence $\mu_R - g + \xi\varsigma^2\psi$ and $\mu_R - g$ must co-move

- the sum of (1) and (2) gives

$$\frac{1}{\rho+\delta} = \left(\frac{1-\alpha}{\xi\varsigma^2\psi + \mu_R - g} + \frac{\alpha}{\delta + \lambda + \mu_R - g}\right)\frac{1+\gamma}{1+\tau} + \frac{D}{PE}$$

  – the private-sector present values are decreasing in $\mu_R - g$

  – so the demand for government securities is increasing in $\mu_R - g$

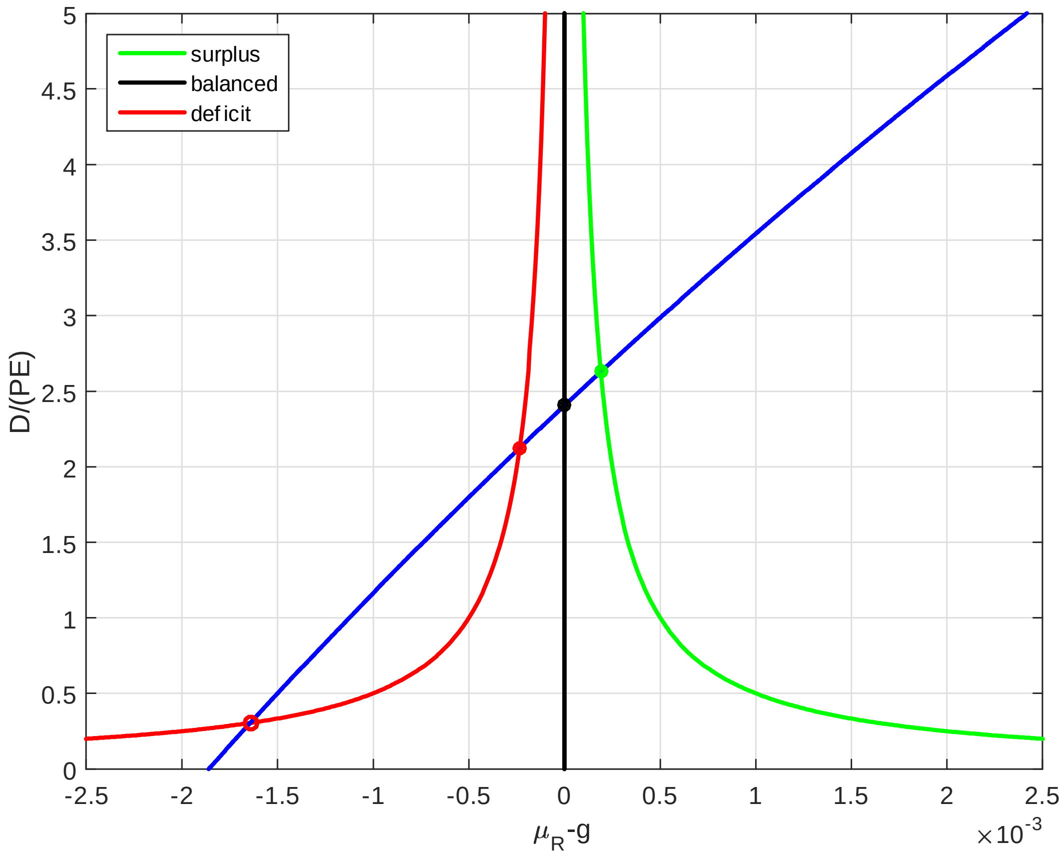
- the remaining equilibrium condition is (3),

$$(\mu_R - g) \times \frac{D}{PE} = \mathcal{S}, \quad \frac{D}{PE} \geq 0$$

  where

$$\mathcal{S} = 1 - \frac{(1+\gamma)(1+\vartheta)}{1+\tau}$$

the equilibrium conditions in terms of $\mu_R - g$ and $D/(PE)$



- the blue curve is obtained by eliminating ψ from (1)-(2)
  - it shifts down with an increase in α
  - it shifts up with an increase in τ
- the red and green curves are $(\mu_R - g)D/(PE) = \mathcal{S}$ for $\mathcal{S} \neq 0$

the balanced budget bubble

- suppose $(1+\tau)/(1+\gamma) = 1+\vartheta \geq 1$ so that $\mathcal{S} = 0$ and $\vartheta \geq 0$
- by (3), an equilibrium with $D/(PE) > 0$ requires $\mu_R - g = 0$
- use this to solve (1) for ψ
- and then plug ψ into (2) to solve for $D/(PE)$
- this gives

$$\frac{D}{PE} = \frac{1}{\rho+\delta}\left(1 - \left(\alpha \times \frac{\rho+\delta}{\delta+\lambda}\frac{1+\gamma}{1+\tau} + \sqrt{(1-\alpha) \times \frac{\rho+\delta}{\xi\varsigma^2}\frac{1+\gamma}{1+\tau}}\right)\right)$$

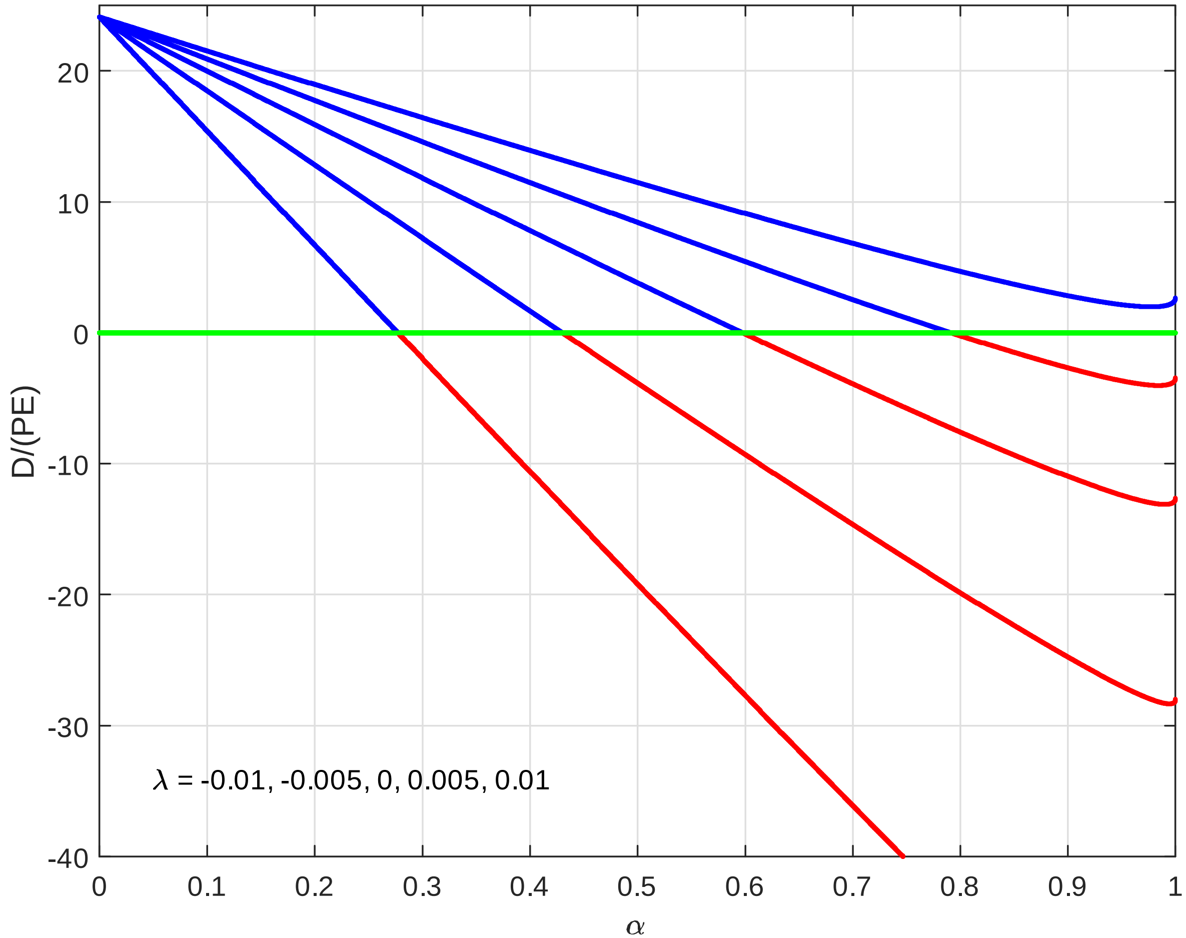
  - no balanced budget bubble unless this is positive
  - in that case, $D/(PE)$ is decreasing in ρ and increasing in

$$\lambda, \quad \xi\varsigma^2, \quad \frac{1+\tau}{1+\gamma} = 1+\vartheta$$

- consider the parameters

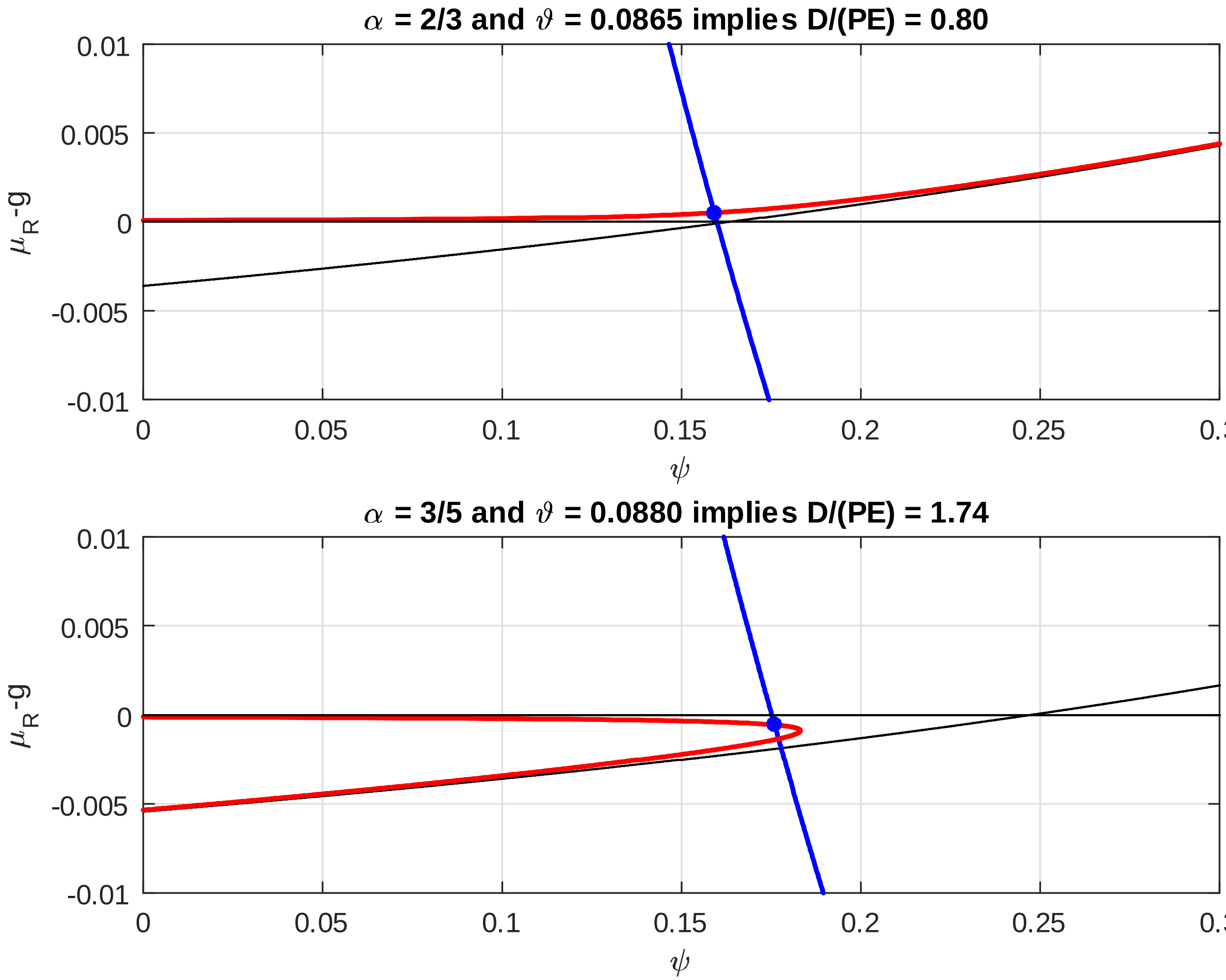
ρ	δ	ξ	ς	γ	τ
0.01	0.02	4	0.3	0.15	0.25

balanced budget bubble / consumption expenditures



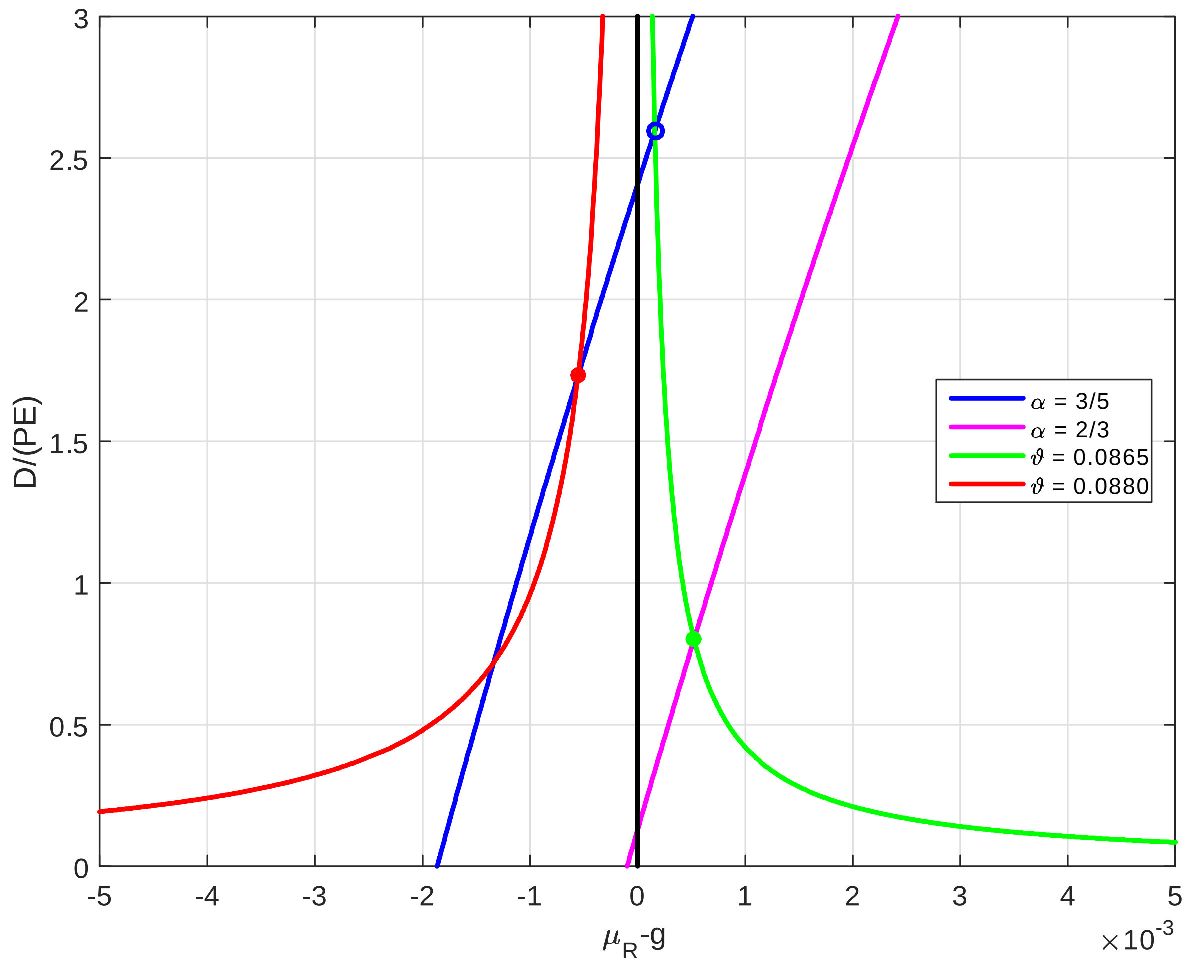
- extremely sensitive to $\alpha \in (0, 1)$ and $\lambda \in (-\delta, \infty)$
  - and also to $\rho > 0$
- red line segments: permanent primary deficits are not possible

a drop in the labor share + a tiny increase in baby bonds



- this is for $\lambda = 0.002$; a balanced budget would require $\vartheta = 0.0870$

a drop in the labor share + a tiny increase in baby bonds



- in this bare-bones model,
  - a large deflation upon the simultaneous decline in α and rise in ϑ
- could imagine gradual and protracted adjustments

conclusion

- in this economy
  - there is aggregate risk that makes claims to labor income and nominal government debt risky; but aggregate risk is small
  - capital is subject not only to aggregate risk but also to idiosyncratic risk that is large and uninsurable
- a reduction in the factor share of labor
  - reduces the supply of the relatively safe claims to labor income,
  - which increases the demand for the relatively safe asset supplied by the government
  - this lowers the rate of return at which the government can borrow
- in our example, this makes it possible
  - for the government to start running permanent primary deficits;
  - if the government does so by increasing transfers,
  - then these deficits temper the rise in the real value of government debt that comes with lower required returns