Lecture Notes on Growth and Firm Heterogeneity

Erzo G.J. Luttmer

University of Minnesota

December 11, 2018

Peking University

growth and firm heterogeneity

1. a model of blueprint capital accumulation based on my
   - "On the Mechanics of Firm Growth"
     - *Review of Economic Studies (2011)*
2. a model of productivity growth based on my
   - "Selection, Growth, and the Size Distribution of Firms"
     - *Quarterly Journal of Economics (2007)*
   - "Technology Diffusion and Growth"
     - *Journal of Economic Theory (2012)*

- ► for a survey see
  - "Models of Growth and Firm Heterogeneity"
    - *Annual Review of Economics (2010)*
- ► on the potential multiplicity of stationary densities, see
  - "Four Models of Knowledge Diffusion and Growth"
    - *Federal Reserve Bank of Minneapolis, w.p. 724 (2015)*

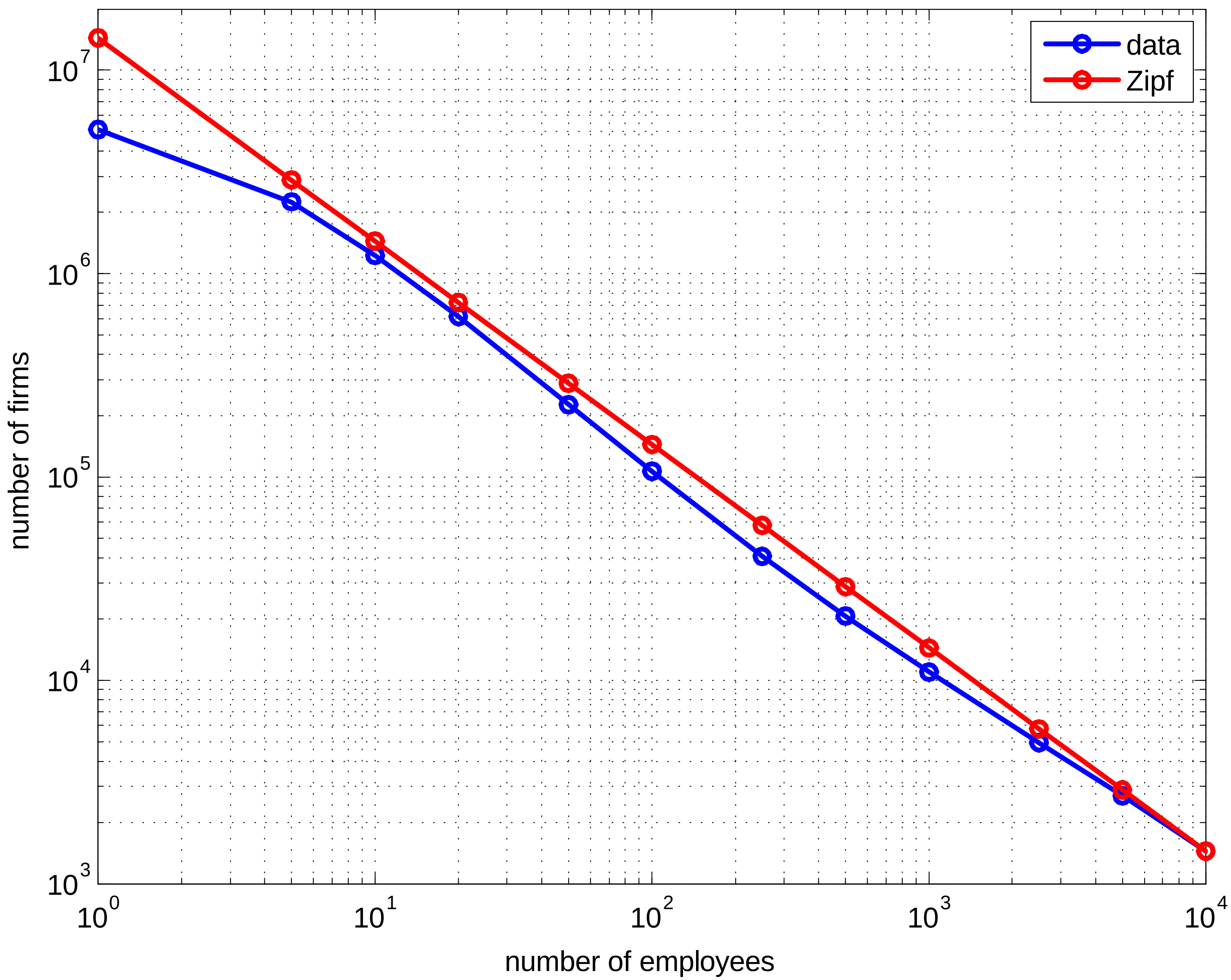
Zipf's Law

$$\Pr\left[N \geq n\right] = \frac{1}{n}$$

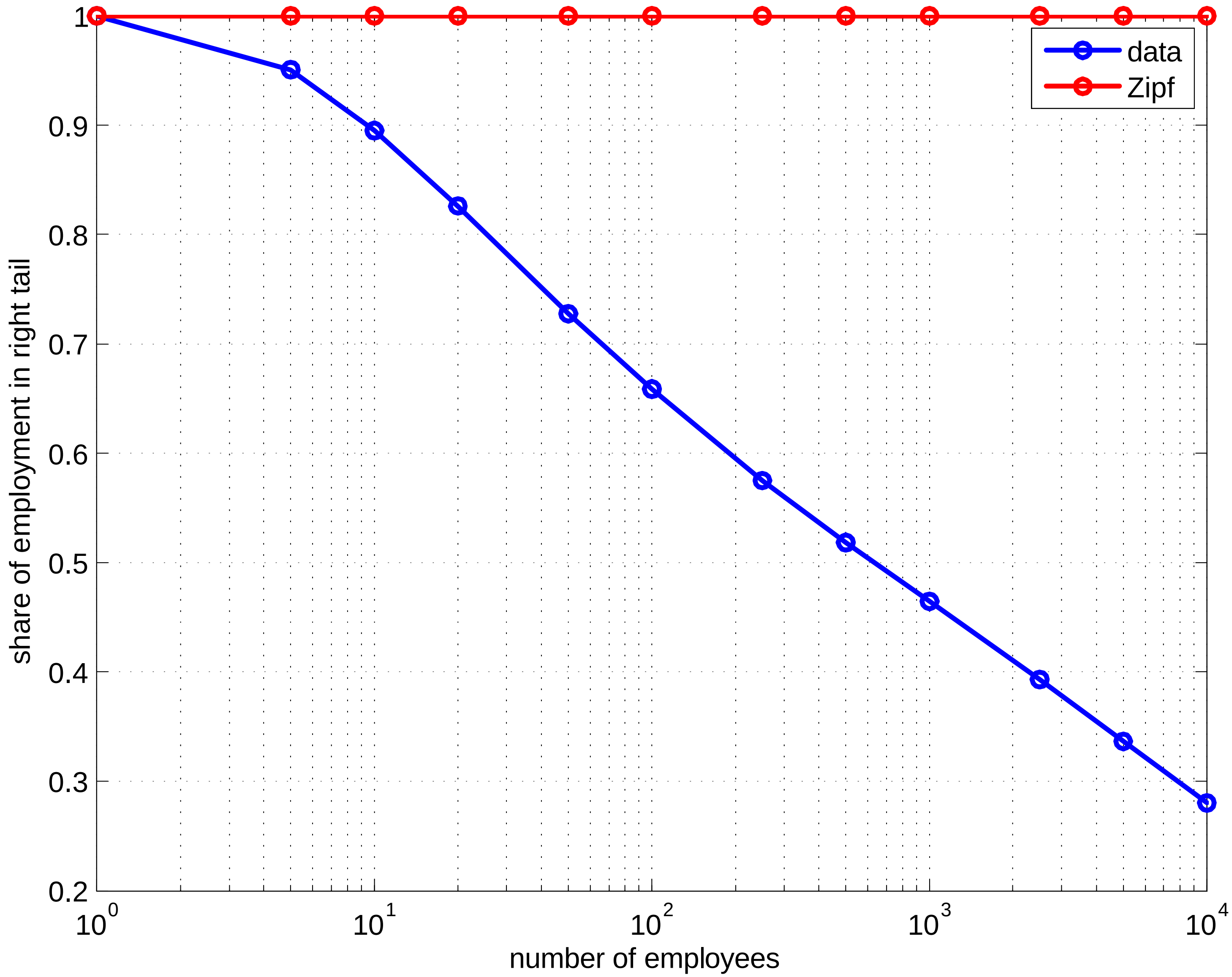
$$\sum_{n=1}^{M} n \Pr\left[N = n\right] = \sum_{n=1}^{M} \frac{n}{n(n+1)} = \sum_{n=1}^{M} \frac{1}{n+1} \sim \ln(M)$$

$$\text{since } \sum_{n=1}^{M} \frac{1}{n+1} \text{ behaves like } \int_{1}^{M} \frac{\mathrm{d}x}{x} \text{ for large } M$$

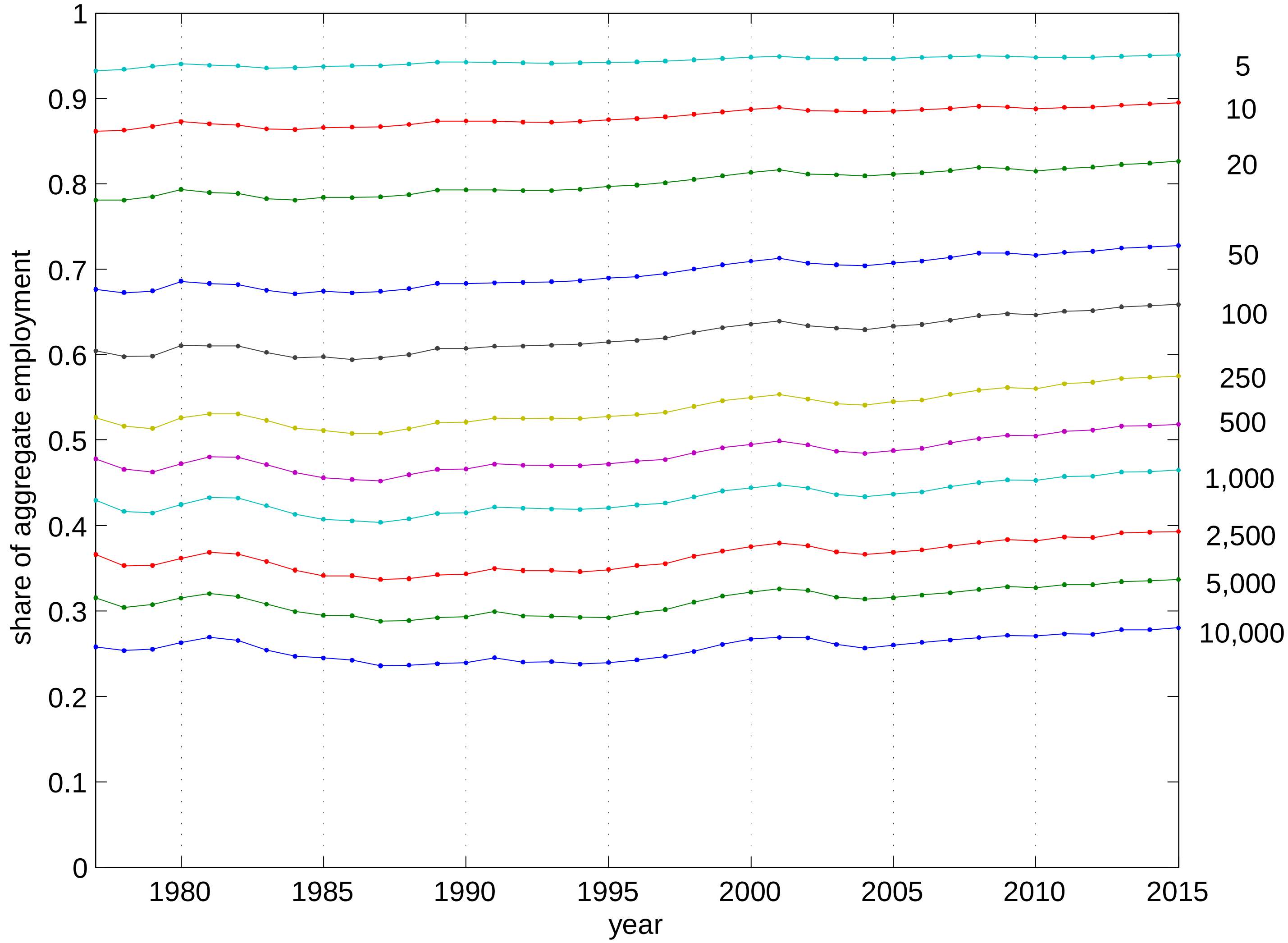
right tail of the firm size distribution (BDS, 2015)



. . . cannot be literally Zipf (BDS, 2015)

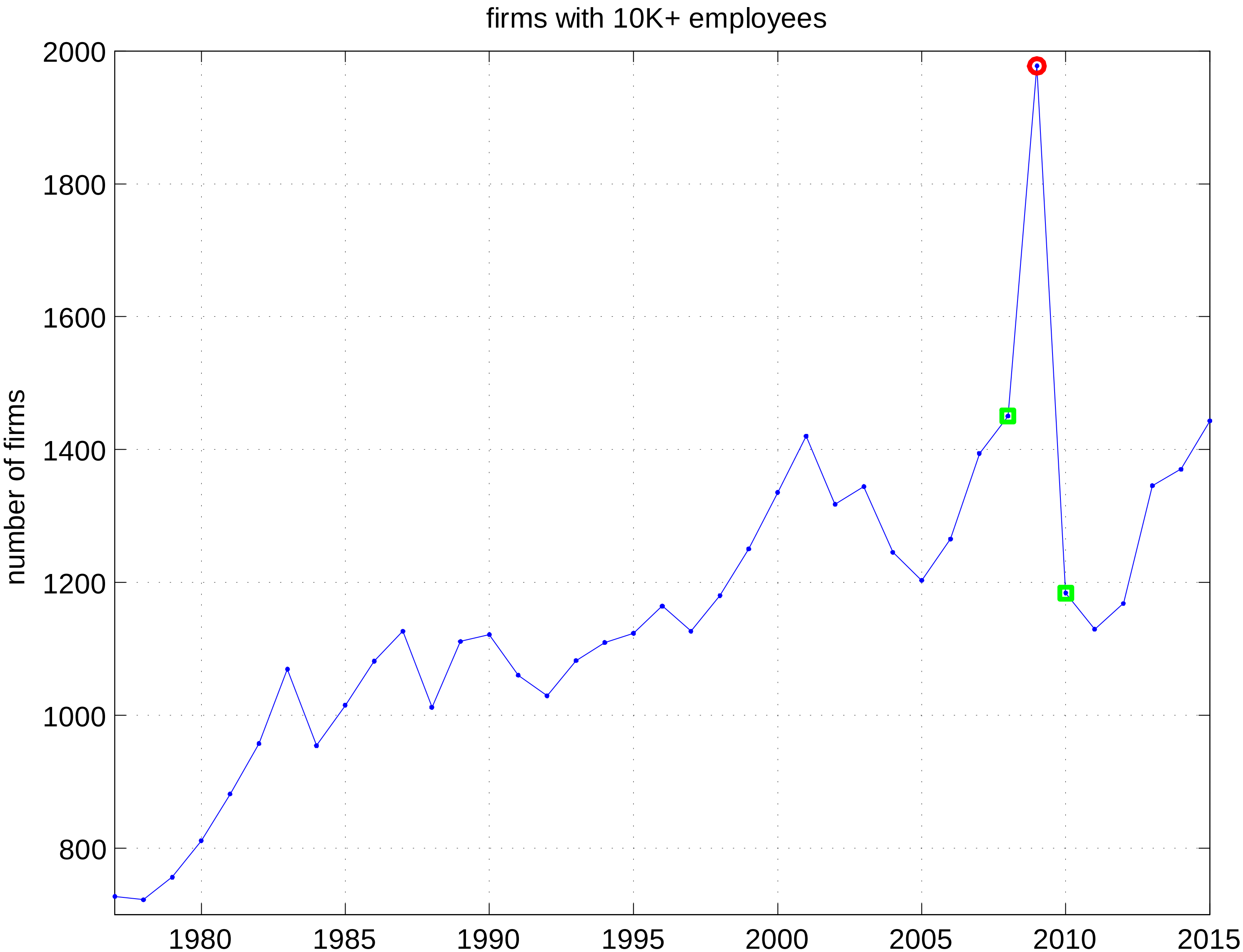


some movement over time, but still quite stable

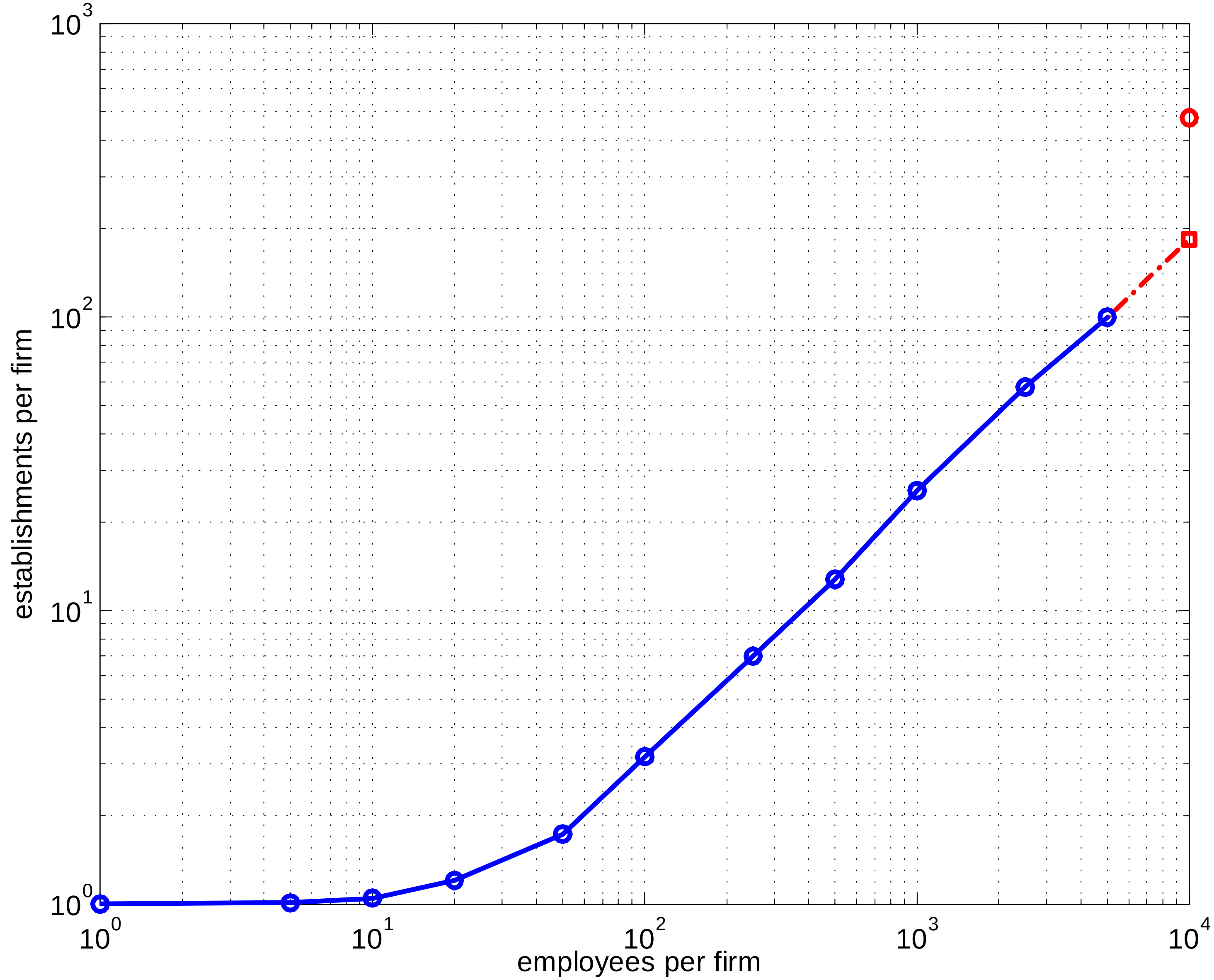


(updated from Luttmer [2010], *Annual Review of Economics*)

public service announcement: BDS data does have issues



large firms have many establishments (BDS, 2015)



the simplest example

- deterministic growth, conditional on survival

$$p(a) \;=\; \delta e^{-\delta a}, \;\; S(a) = e^{\gamma a}$$

this implies

$$\Pr\left[S(a) \geq s\right] \;=\; \Pr\left[a \geq \frac{1}{\gamma} \times \ln(s)\right] = e^{-\delta \times \frac{1}{\gamma} \times \ln(s)} = s^{-\delta/\gamma}$$

- deterministic growth and population growth
  - size of entering cohort at time t is $E_t = Ee^{\eta t}$
  - relative size of age-a cohort is $\eta e^{-\eta a}$
  - adding up over all cohorts

$$\int_0^\infty \iota\left[e^{\gamma a} > s\right] \eta e^{-\eta a} \mathrm{d}a = e^{-\eta \times \frac{\ln(s)}{\gamma}} = s^{-\eta/\gamma}$$

the Beta and Gamma functions

- the Gamma function, for $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$$

- implies a recursion

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u} \mathrm{d}u = -u^x e^{-u}\Big|_0^\infty + x \int_0^\infty u^{x-1} e^{-u} \mathrm{d}u = x\Gamma(x).$$

  – Clearly, $\Gamma(1) = 1$ and hence $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

- the Beta function for $x > 0$ and $y > 0$ is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t$$

- note that $u = e^{-t}$ gives $\mathrm{d}u = -e^{-t}\mathrm{d}t = -u\mathrm{d}t$ and thus

$$\int_0^1 u^{x-1}(1-u)^{y-1} \mathrm{d}u = \int_0^1 u^x (1-u)^{y-1} \left[u^{-1}\mathrm{d}u\right] = \int_0^\infty e^{-xt}(1-e^{-t})^{y-1} \mathrm{d}t.$$

- can show

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Stirling's formula

- for large x,

$$\Gamma(x) \approx \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$$

- hence

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \sim \left(\frac{x}{x+y}\right)^x \frac{1}{(x+y)^y}$$

  for large x. Now

$$\lim_{x\to\infty} \left(\frac{x}{x+y}\right)^x = \lim_{x\to\infty} \left(1 - \frac{y}{x+y}\right)^x = e^{-y},$$

  and so

$$B(x,y) \sim \frac{1}{x^y}$$

  for large x.

- in other words,

$$\lim_{x\to\infty} \frac{\Gamma(x)x^y}{\Gamma(x+y)} = 1$$

  for any $y > 0$.

a birth-death example

- existing projects beget new projects, randomly, at the rate $\mu > 0$
- cohort distribution $\{p_{n,t}\}_{n=1}^{\infty}$, starting from $p_{1,0} = 1$,

$$\mathrm{D}p_{1,t} = -\mu p_{1,t},$$

and

$$\mathrm{D}p_{n,t} = \mu(n-1)p_{n-1,t} - \mu n p_{n,t}, \qquad n-1 \in \mathbb{N}$$

- first

$$p_{1,t} = e^{-\mu t}$$

and then

$$\mathrm{D}\left[e^{\mu n t} p_{n,t}\right] = e^{\mu n t}\mu(n-1)p_{n-1,t}, \qquad n-1 \in \mathbb{N}$$

so that

$$p_{n,t} = \mu(n-1)\int_0^t e^{\mu n(s-t)} p_{n-1,s}\mathrm{d}s, \qquad n-1 \in \mathbb{N}.$$

- iterate to construct the geometric solution

$$p_{n,t} = e^{-\mu t}(1 - e^{-\mu t})^{n-1}, \;\; n \in \mathbb{N}$$

verification

- for $n-1 \in \mathbb{N}$, observe that

$$p_{n,t} = e^{-\mu t}(1-e^{-\mu t})^{n-1}$$

implies

$$\begin{aligned}
\mathrm{D}p_{n,t} &= -\mu e^{-\mu t}(1-e^{-\mu t})^{n-1} + \mu(n-1)e^{-2\mu t}(1-e^{-\mu t})^{n-2} \\
&= \mu(n-1)e^{-\mu t}(1-e^{-\mu t})^{n-2} \\
&\quad +\mu(n-1)(e^{-2\mu t} - e^{-\mu t})(1-e^{-\mu t})^{n-2} - \mu e^{-\mu t}(1-e^{-\mu t})^{n-1} \\
&= \mu(n-1)e^{-\mu t}(1-e^{-\mu t})^{n-2} - \mu n e^{-\mu t}(1-e^{-\mu t})^{n-1} \\
&= \mu(n-1)p_{n-1,t} - \mu n p_{n,t}
\end{aligned}$$

as required.

combine with random firm exit at rate $\delta > 0$

- implied age distribution of firms has a density $\delta e^{-\delta t}$
- the stationary size distribution is then given by

$$
\begin{aligned}
s_n &= \int_0^\infty \delta e^{-\delta t} p_{n,t} \mathrm{d}t \\
&= \int_0^\infty \delta e^{-\delta t} e^{-\mu t} (1 - e^{-\mu t})^{n-1} \mathrm{d}t \\
&= \frac{\delta}{\mu} \int_0^\infty e^{-(1+\delta/\mu)[\mu t]} (1 - e^{-[\mu t]})^{n-1} \mathrm{d}[\mu t] \\
&= \frac{\delta}{\mu} \int_0^\infty e^{-(1+\delta/\mu)s} (1 - e^{-s})^{n-1} \mathrm{d}s = \frac{\delta}{\mu} \frac{\Gamma(n)\Gamma(1+\delta/\mu)}{\Gamma(n+1+\delta/\mu)}
\end{aligned}
$$

- the right tail probabilities are

$$
R_n = \sum_{k=n}^{\infty} s_k = \sum_{k=n}^{\infty} \frac{\delta}{\mu} \frac{\Gamma(k)\Gamma(1+\delta/\mu)}{\Gamma(k+1+\delta/\mu)} = \frac{\delta}{\mu} \frac{\Gamma(n)\Gamma(\delta/\mu)}{\Gamma(n+\delta/\mu)}.
$$

for all $n \in \mathbb{N}$.

doing the sum

- the claim is that

$$R_n = \sum_{k=n}^{\infty} \frac{\delta}{\mu} \frac{\Gamma(k)\Gamma(1+\delta/\mu)}{\Gamma(k+1+\delta/\mu)} = \frac{\delta}{\mu} \frac{\Gamma(n)\Gamma(\delta/\mu)}{\Gamma(n+\delta/\mu)}.$$

▶ the summation follows from

$$\frac{\Gamma(n)\Gamma(x)}{\Gamma(n+x)} - \frac{\Gamma(n+1)\Gamma(x)}{\Gamma(n+1+x)} = \left(1 - \frac{n}{n+x}\right) \frac{\Gamma(n)\Gamma(x)}{\Gamma(n+x)} = \frac{\Gamma(n)\Gamma(1+x)}{\Gamma(n+1+x)}$$

the mean

- finite if $\delta > \mu$
  - summation by parts implies

$$\sum_{n=1}^{\infty} n s_n = \sum_{n=1}^{\infty} R_n = \sum_{n=1}^{\infty} \frac{\delta}{\mu} \frac{\Gamma(n)\Gamma(\delta/\mu)}{\Gamma(n+\delta/\mu)} = \frac{\delta}{\mu} \frac{\Gamma(\delta/\mu - 1)}{\Gamma(\delta/\mu)} = \frac{1}{1-\mu/\delta}.$$

  - to verify: consider $\sum_{k=n}^{\infty} R_k$ and use the same result as for R_n itself.
- infinite if $\delta \leq \mu$
  - may be fine if there is a finite number of firms
  - problematic in models with a continuum of firms
- key
  - cannot have μ exogenous if n is employment
  - firms cannot grow at just any rate—workers come from somewhere
  - must respect labor market clearing

the tail index, Zipf's law

- Stirling's approximation, for x large

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$$

- hence

$$R_n = \frac{\delta}{\mu} \frac{\Gamma(n)\Gamma(\delta/\mu)}{\Gamma(n+\delta/\mu)} \sim n^{-\delta/\mu}.$$

So $\ln(R_n)$ behaves like $-(\delta/\mu)\ln(n)$, and the slope is greater than 1 in absolute value if we assume $\mu < \delta$ to ensure a finite mean. In US data, δ/μ appears to be about 1.05.

- note that $\mu \uparrow \delta$ gives

$$s_n = \frac{\Gamma(n)\Gamma(2)}{\Gamma(n+2)} = \frac{1}{n(n+1)}$$

and thus

$$R_n = \sum_{k=n}^{\infty} s_k = \sum_{k=n}^{\infty} \frac{1}{k(k+1)} = \frac{1}{n}$$

since $(1/n) - 1/(1+n) = 1/[n(n+1)]$. This is Zipf's law.

alternative derivation

- a unit measure of firms
  - exit at the rate δ
  - replaced by a new entrant with $n = 1$
- hence

$$0 = -(\delta + \mu)s_1 + \delta$$

and

$$0 = \mu(n-1)s_{n-1} - (\delta + \mu n)s_n, \qquad n - 1 \in \mathbb{N}.$$

- this yields

$$s_n = \frac{\mu(n-1)}{\delta + \mu n} \times s_{n-1}, \qquad n - 1 \in \mathbb{N}.$$

- combined with $s_1 = \delta/(\delta + \mu)$ this yields

$$s_{n+1} = \frac{\delta}{\delta + \mu} \prod_{k=1}^{n} \frac{\mu k}{\delta + \mu(k+1)} = \frac{\delta}{\mu} \frac{\Gamma(n+1)\Gamma\left(\frac{\mu+\delta}{\mu}\right)}{\Gamma\left(n+1+\frac{\mu+\delta}{\mu}\right)}$$

which holds for all $n + 1 \in \mathbb{N}$.

model 1

the homogeneous blueprints model in

Luttmer [*Review of Economic Studies*, 2011]

dynastic households

- preferences

$$\int_0^\infty e^{-\rho t} H_t \ln(c_t) \mathrm{d}t,$$

household consumption is

$$C_t = H_t c_t$$

$$H_t = H e^{\eta t}, \qquad \rho > \eta > 0$$

- c_t is a CES composite good of differentiated commodties

$$c_t = \left[\int_0^{N_t} c_{\omega,t}^{1-1/\varepsilon} \mathrm{d}\omega \right]^{1/(1-1/\varepsilon)}$$

where $\varepsilon > 1$

household choices

- the dynastic present-value budget constraint

$$\int_0^\infty \exp\left(-\int_0^t r_s \mathrm{d}s\right) H_t c_t \mathrm{d}t \leq \texttt{wealth}$$

implies the first-order condition

$$e^{-\rho t} H_t \times \frac{1}{c_t} = \lambda \exp\left(-\int_0^t r_s \mathrm{d}s\right) H_t$$

or simply

$$\frac{e^{-\rho t}}{c_t} = \lambda \pi_t$$

- differentiating yields the Euler condition

$$r_t = \rho + \frac{\mathrm{D}c_t}{c_t}$$

household choices

the differentiated commodity demands are

$$c_{\omega,t} = \left(\frac{p_{\omega,t}}{P_t}\right)^{-\varepsilon} H_t c_t$$

where P_t is the price index

$$P_t = \left(\int_0^{N_t} p_{\omega,t}^{1-\varepsilon} \mathrm{d}\omega\right)^{1/(1-\varepsilon)}$$

producers

- blueprint + linear labor-only technology yields output $y_{\omega,t} = zl_{\omega,t}$
- the time-t wage in units of the composite consumption good $= w_t$
- to maximize $P_t v_{\omega,t} = (p_{\omega,t} - P_t w_t/z)c_{\omega,t}$ subject to $c_{\omega,t} = (p_{\omega,t}/P_t)^{-\varepsilon} H_t c_t$ set

$$\frac{p_{\omega,t}}{P_t} = \frac{w_t/z}{1 - 1/\varepsilon}$$

- eliminating $p_{\omega,t}/P_t$ from the price index gives

$$w_t = \left(1 - \frac{1}{\varepsilon}\right) z N_t^{\frac{1}{\varepsilon-1}}$$

▶ implied employment and profits per blueprint

$$\begin{bmatrix} w_t l_{\omega,t} \\ v_{\omega,t} \end{bmatrix} = \begin{bmatrix} w_t l_t \\ v_t \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} \frac{H_t c_t}{N_t}$$

▶ in particular

$$\frac{v_t}{w_t} = \frac{l_t}{\varepsilon - 1}$$

entrants and incumbents

- ▶ two technologies for developing new blueprints

  - skilled entrepreneurial time only

    - · new blueprints from scratch

  - existing blueprints and labor

    - · new blueprint codes for distinct differentiated commodity

- ▶ firm = collection of blueprints derived from the same initial blueprint

  - no reason to trade blueprints—any positive cost forces no trade

costly blueprint replication

- recall that profits per blueprint are

$$v_t = \frac{w_t l_t}{\varepsilon - 1}$$

- the price of a blueprint in units of consumption is q_t
- a flow of m_t units of labor can be used to replicate an existing blueprint randomly at the rate $g(m_t)$

▶ therefore

$$r_t q_t = \max_m \left\{ w_t \left(\frac{l_t}{\varepsilon - 1} - m \right) + q_t g(m) + \mathrm{D} q_t \right\}$$

▶ the first-order condition for replication is

$$1 = \frac{q_t}{w_t} \times \mathrm{D} g(m_t)$$

a Roy model of primary factor supplies

- talent distribution $T \in \Delta\left(\mathbb{R}^2_{++}\right)$

per capita supply of entrepreneurial services

$$E\left(\frac{q}{w}\right) = \int_{qx>wy} x \mathrm{d}T(x,y)$$

per capita supply of labor

$$L\left(\frac{q}{w}\right) = \int_{wy>qx} y \mathrm{d}T(x,y)$$

aggregate blueprint accumulation

the number of blueprints evolves according to

$$\mathrm{D}N_t = g(m_t)N_t + H_t E\left(\frac{q_t}{w_t}\right)$$

- $N_0 > 0$ is a given initial value
- this will be non-stationary

► in per-capita terms

$$\mathrm{D}\left(\frac{N_t}{H_t}\right) = -\left(\eta - g(m_t)\right) \times \frac{N_t}{H_t} + E\left(\frac{q_t}{w_t}\right)$$

- a steady state requires

$$\eta > g(m)$$

  – this will be an equilibrium outcome
  – but individual firm histories are non-stationary

the number of firms

- blueprints, not firms, matter for aggregate dynamics
  - but very relevant for observables

▶ entrepreneurs set up new firms

$$\mathrm{D}M_t = H_t E\left(\frac{q_t}{w_t}\right)$$

- per capita

$$\mathrm{D}\left(\frac{M_t}{H_t}\right) = -\eta \times \frac{M_t}{H_t} + E\left(\frac{q_t}{w_t}\right)$$

▶ in the steady state

$$\frac{M_t}{H_t} = \frac{1}{\eta} E\left(\frac{q_t}{w_t}\right)$$

the dynamic equilibrium

- use the N_t/H_t as the state, with q_t/c_t as the co-state
  - the marginal utility weighted price q_t/c_t removes r_t from the system

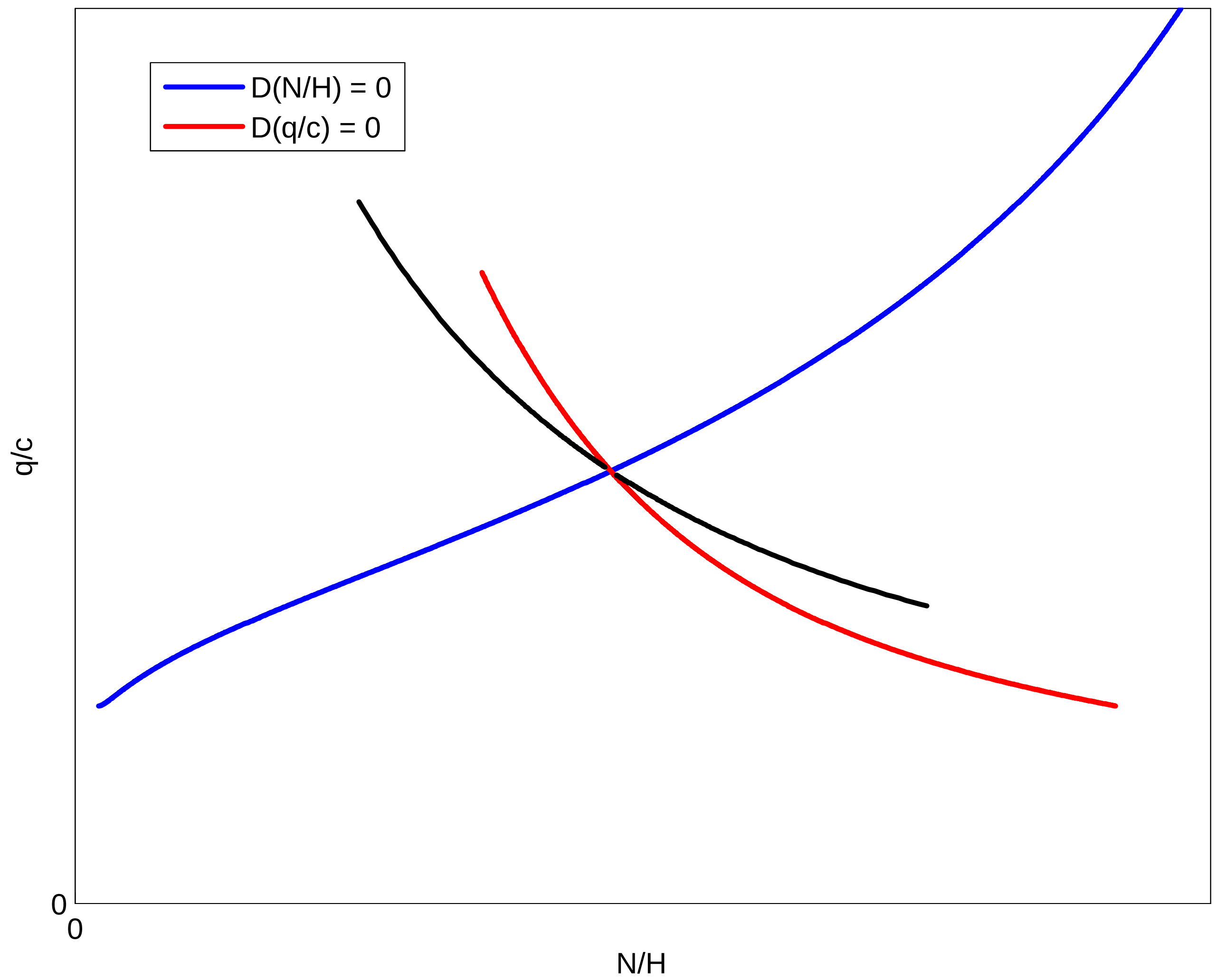
▶ the differential equation is

$$\begin{aligned} \mathrm{D}\left(\frac{N_t}{H_t}\right) &= -\left(\eta - g(m_t)\right) \times \frac{N_t}{H_t} + E\left(\frac{q_t}{w_t}\right) \\ \mathrm{D}\left(\frac{q_t}{c_t}\right) &= \left(\rho - \left(g(m_t) - \mathrm{D}g(m_t)m_t\right)\right) \times \frac{q_t}{c_t} - \frac{1}{\varepsilon}\frac{1}{N_t/H_t} \end{aligned}$$

where

$$\begin{aligned} 1 - \frac{1}{\varepsilon} &= \frac{w_t}{c_t} \times l_t \times \frac{N_t}{H_t} \\ 1 &= \frac{q_t}{w_t} \times \mathrm{D}g(m_t) \\ L\left(\frac{q_t}{w_t}\right) &= (l_t + m_t) \times \frac{N_t}{H_t} \end{aligned}$$

the phase diagram



balanced growth

- the per capita number of blueprints is constant

$$N_t = Ne^{\eta t}$$

- wages are

$$w_t = \left(1 - \frac{1}{\varepsilon}\right) z N_t^{\frac{1}{\varepsilon - 1}}$$

- implied growth from variety

$$\kappa \ = \ \frac{\mathrm{D}w_t}{w_t} = \frac{\eta}{\varepsilon - 1}$$

- familiar implications
  – integrating the world improves welfare, a level effect
  – persisent growth from variety depends on population growth

steady state equilibrium for $s = q/w$

let $m[s]$ and $l[s]$ solve

$$\frac{1}{\mathrm{D}g(m)} = s = \frac{1}{\rho - g(m)}\left(\frac{l}{\varepsilon - 1} - m\right)$$

- steady state demand for blueprints

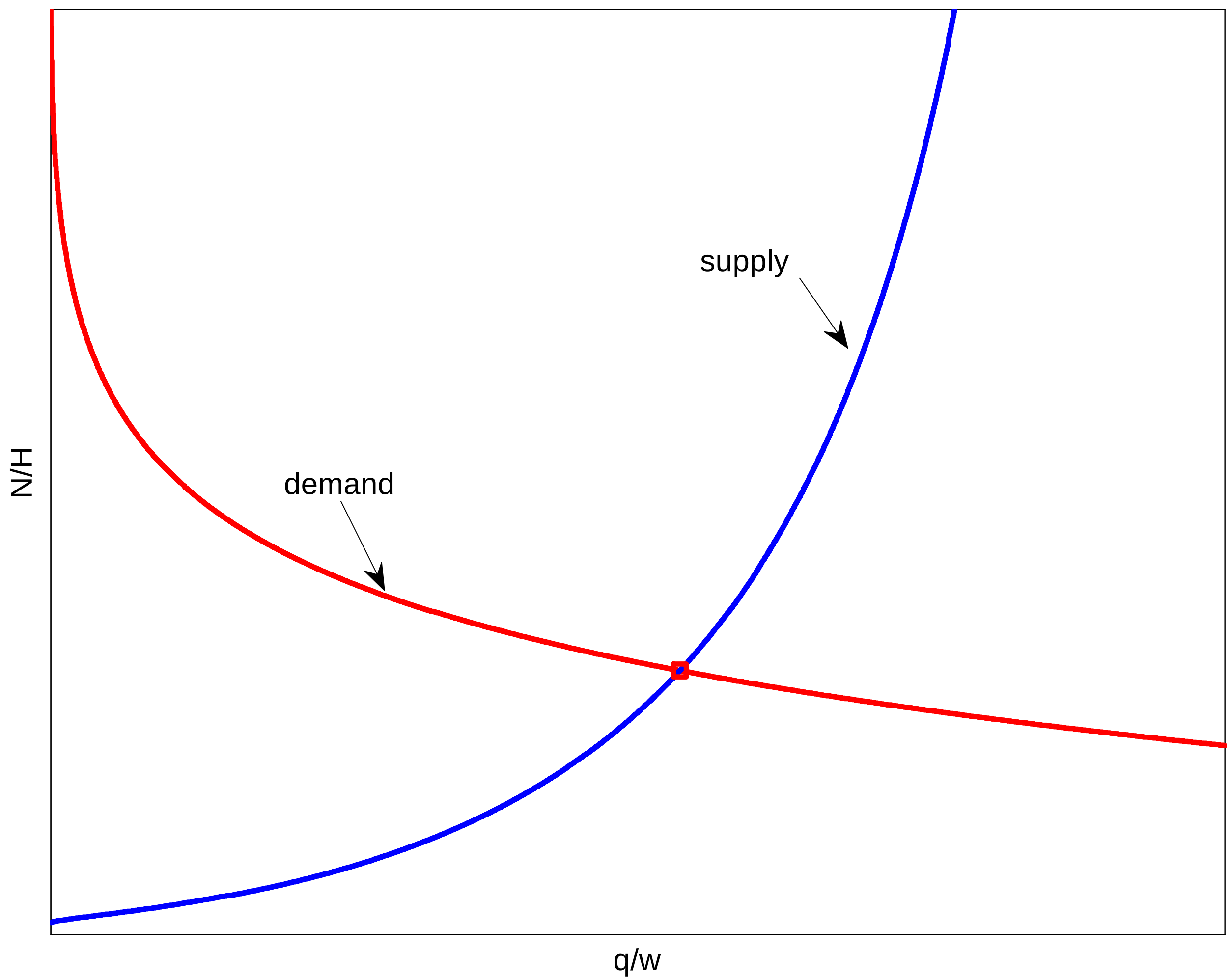
$$\frac{N}{H} = \frac{L(s)}{l[s] + m[s]}$$

- steady state supply of blueprints

$$\frac{N}{H} = \frac{E(s)}{\eta - g(m[s])}$$

- notice that this has an asymptote as $g(m[s]) \uparrow \eta$
- now clear the market
- the assumption $\rho > \eta$ implies that $\eta > g(m)$ guarantees $\rho > g(m)$

equilibrium



what if the skill distribution is degenerate at (x, y)?

- demand for blueprints

$$\frac{N}{H} = \frac{(1-a)y}{l[s] + m[s]}.$$

- supply of blueprints

$$\frac{N}{H}(\eta - g(m[s])) = ax$$

and

$$sx \leq y, \text{ w.e. if } a > 0.$$

where a = fraction of entrepreneurs.

- can have an equilibrium with $a = 0$ and $\eta = g(m[s])$
  - but then the size distribution of firms fans out forever

the Zipf limit

- a fraction $1 - 1/\Lambda \in (0,1)$ of the population can only supply labor,

$$L_\Lambda(s) = \left(1 - \frac{1}{\Lambda}\right)\ell + \frac{\mathcal{L}(s)}{\Lambda}, \qquad E_\Lambda(s) = \frac{\mathcal{E}(s)}{\Lambda}$$

▶ demand for blueprints

$$\frac{N}{H} = \frac{1}{l[s] + m[s]}\left(\left(1 - \frac{1}{\Lambda}\right)\ell + \frac{\mathcal{L}(s)}{\Lambda}\right)$$

▶ supply of blueprints

$$\frac{N}{H} = \frac{1}{\eta - g(m[s])}\frac{\mathcal{E}(s)}{\Lambda}$$

where $l[s]$ and $m[s]$ solve

$$\frac{1}{\mathrm{D}g(m)} = s = \frac{1}{\rho - g(m)}\left(\frac{l}{\varepsilon - 1} - m\right)$$

the Zipf limit

- the steady state $(m_\Lambda, l_\Lambda, s_\Lambda)$ solves

$$1 = s\mathrm{D}g(m), \qquad s = \frac{1}{\rho - g(m)}\left(\frac{l}{\varepsilon - 1} - m\right)$$

$$\frac{\mathcal{E}(s)}{(\eta - g(m))\Lambda} = \frac{1}{l+m}\left(\left(1 - \frac{1}{\Lambda}\right)\ell + \frac{1}{\Lambda} \times \mathcal{L}(s)\right)$$

- construct the $\Lambda \to \infty$ limit

$$\eta = g(m_\infty)$$

$$1 = s_\infty \mathrm{D}g(m_\infty), \qquad s_\infty = \frac{1}{\rho - \eta}\left(\frac{l_\infty}{\varepsilon - 1} - m_\infty\right)$$

$$\frac{\mathcal{E}(s_\infty)}{\lim_{\Lambda\to\infty}(\eta - g(m_\Lambda))\Lambda} = \frac{\ell}{l_\infty + m_\infty}$$

and

$$\frac{N_\infty}{H} = \frac{\ell}{l_\infty + m_\infty}$$

the Zipf limit

- employment per blueprint

$$\left(\left(1-\frac{1}{\Lambda}\right)\ell+\frac{\mathcal{L}(s_\Lambda)}{\Lambda}\right)\frac{H_t}{N_t}=l[s_\Lambda]+m[s_\Lambda]\to l_\infty+m_\infty$$

- number of firms per capita

$$\frac{1}{\eta}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}\to 0$$

- number of blueprints per firm

$$\frac{\frac{1}{\eta-g(m_\Lambda)}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}}{\frac{1}{\eta}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}}=\frac{1}{1-\frac{g(m_\Lambda)}{\eta}}\to\infty$$

- employment per firm

$$\frac{\left(1-\frac{1}{\Lambda}\right)\ell+\frac{\mathcal{L}(s_\Lambda)}{\Lambda}}{\frac{1}{\eta}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}}\to\frac{\ell}{0}=\infty$$

the Zipf limit

- the entry rate

$$\frac{\frac{\mathcal{E}(s_\Lambda)}{\Lambda}}{\frac{1}{\eta}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}} = \eta$$

- contribution of entry flow to employment

$$\frac{(l_\Lambda + m_\Lambda) \times \frac{\mathcal{E}(s_\Lambda)}{\Lambda}}{\left(1-\frac{1}{\Lambda}\right)\ell + \frac{\mathcal{L}(s)}{\Lambda}} \rightarrow \frac{(l_\infty + m_\infty) \times \lim_{\Lambda\to\infty}\frac{\mathcal{E}(s_\Lambda)}{\Lambda}}{\ell + \lim_{\Lambda\to\infty}\frac{\mathcal{L}(s_\Lambda)}{\Lambda}} = \frac{(l_\infty + m_\infty)\times 0}{\ell + 0} = 0$$

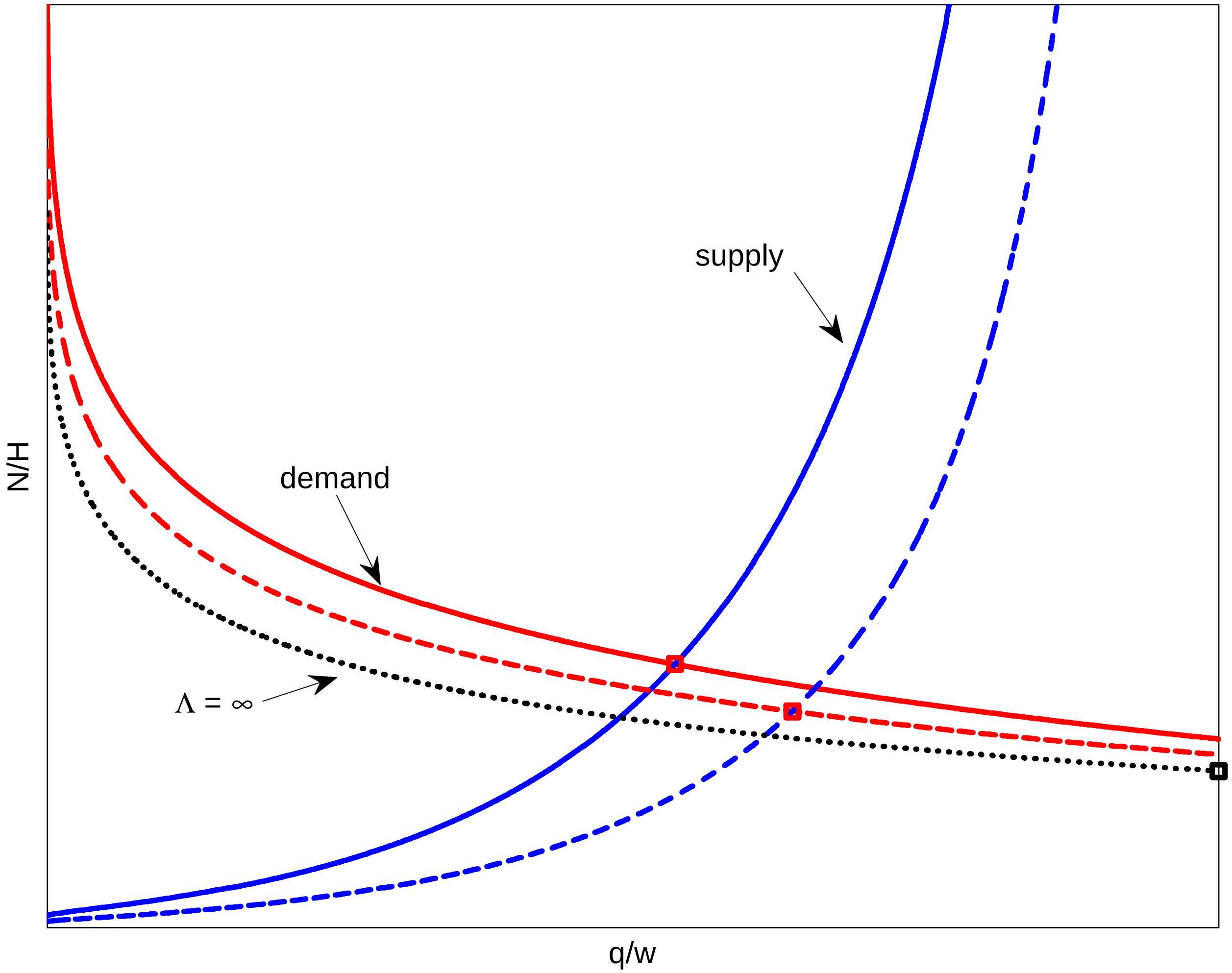
▶ to summarize

– robust entry

– average firm size explodes

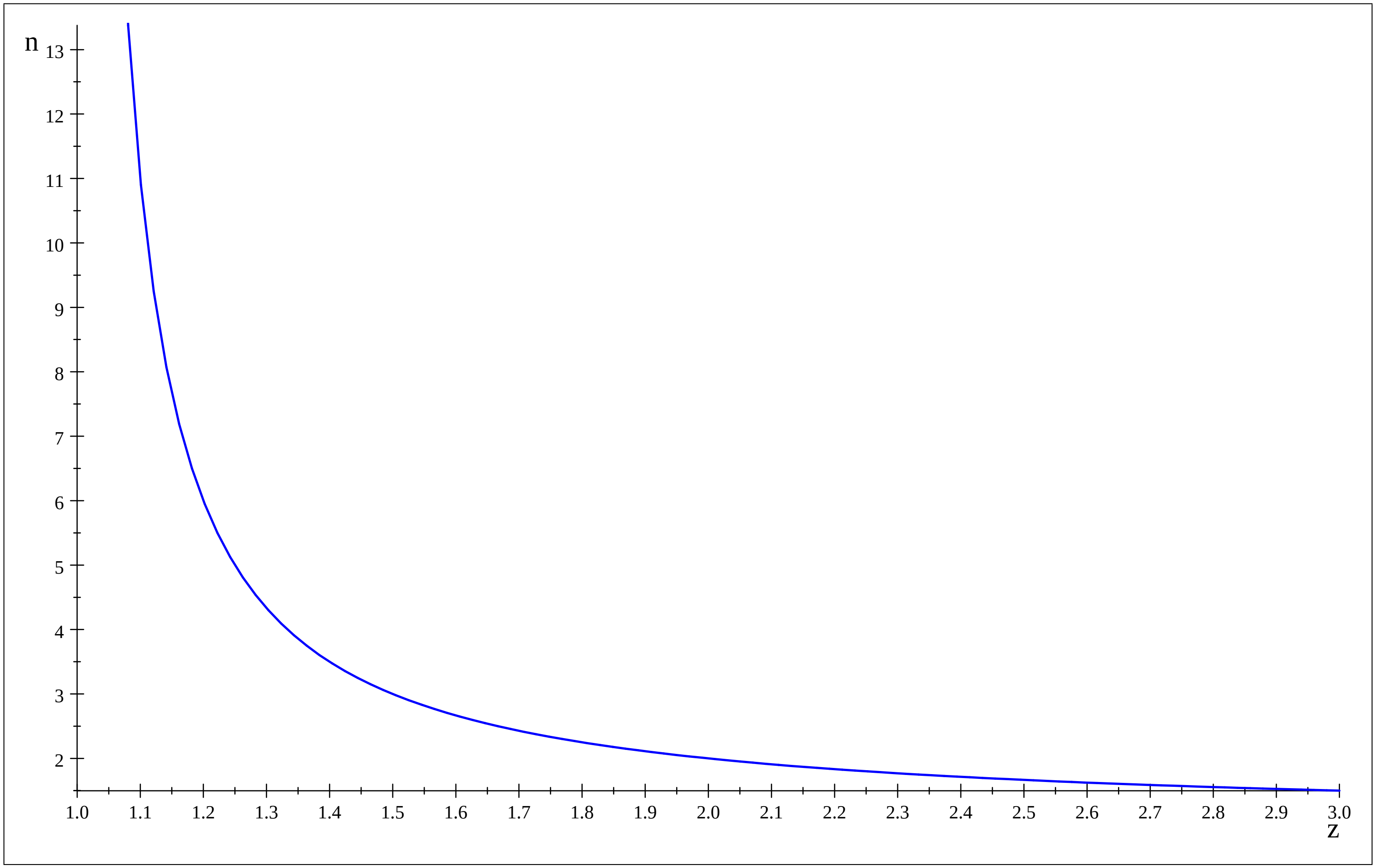
– contribution of entrants to employment growth negligible

increasing Λ

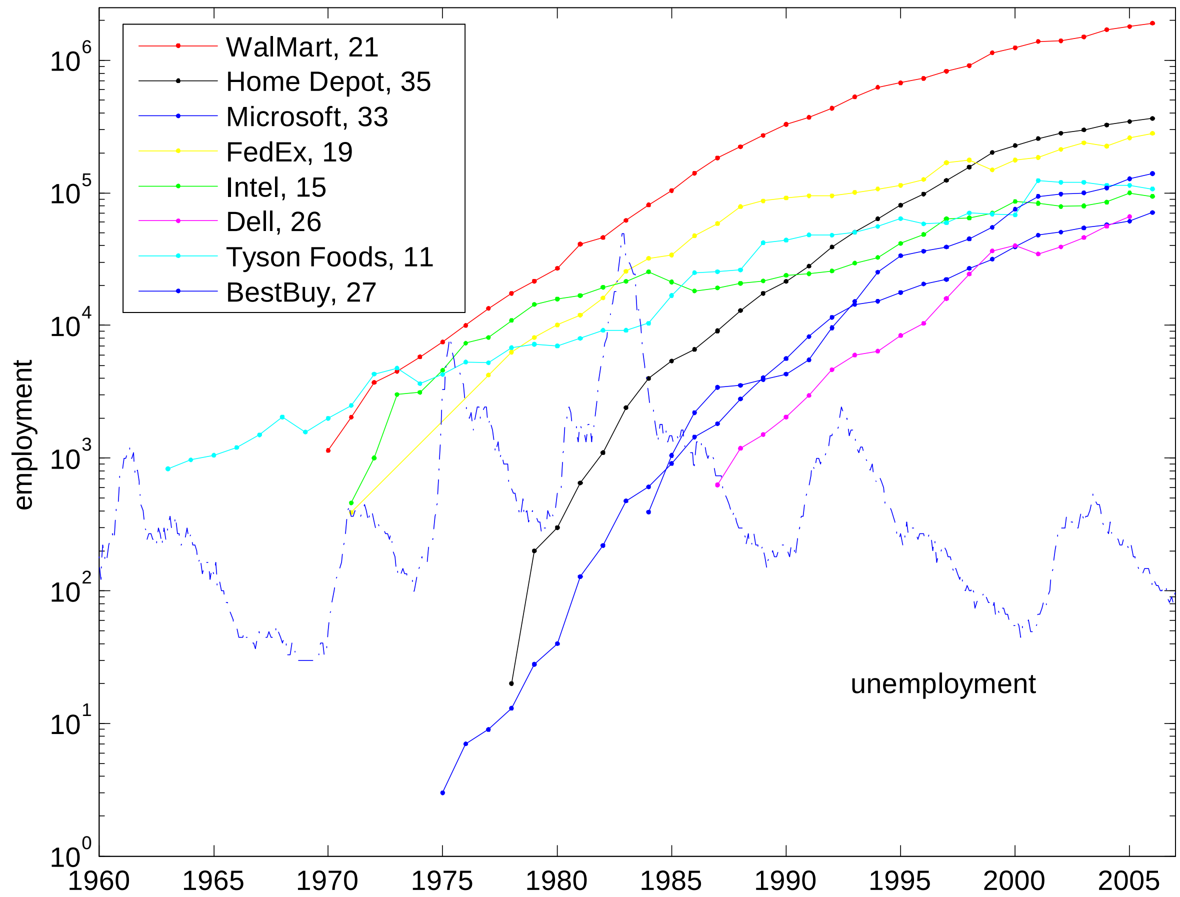


$$z = \texttt{tail index}$$

$$\frac{1}{1-\frac{1}{z}} = \texttt{number of blueprints per firm}$$



some firms grow much faster than $g(m) < \eta = 0.01$



- and large firms are much younger then implied by this model
  - fix: two-type model with transitory rapid growth in Luttmer [2011]

transitory growth

- suppose

$$[N_t, E_t] = [N, E]\, e^{\eta t}, \quad p(b) = \delta e^{-\delta b}, \quad S(a,b) = e^{\gamma \min\{a,b\}}$$

- fix some age cohort,

$$\begin{aligned} \Pr[S_a \geq s] &= \Pr\left[e^{\gamma \min\{a,b\}} \geq s\right] \\ &= \Pr\left[\min\{a,b\} \geq \frac{1}{\gamma} \times \ln(s)\right] = \begin{cases} 0 & \text{if } a < \frac{1}{\gamma} \times \ln(s) \\ e^{-\delta \times \frac{1}{\gamma} \times \ln(s)} & \text{if } a \geq \frac{1}{\gamma} \times \ln(s) \end{cases} \end{aligned}$$

or

$$\Pr[S_a \geq s] = \begin{cases} 0 & \text{if } a < \frac{1}{\gamma} \times \ln(s) \\ s^{-\delta/\gamma} & \text{if } a \geq \frac{1}{\gamma} \times \ln(s) \end{cases}$$

- adding up over all cohorts

$$\int_0^\infty \eta e^{-\eta a} \Pr[S_a \geq s]\, \mathrm{d}a = \int_{\frac{1}{\gamma}\ln(s)}^\infty \eta e^{-\eta a} s^{-\delta/\gamma} \mathrm{d}a = s^{-\delta/\gamma} \times e^{-\eta \times \frac{1}{\gamma}\ln(s)} = s^{-(\delta+\eta)/\gamma}$$

▶ now we can have γ much larger than η

outside the steady state

- see the phase diagram—one aggregate state variable

▶ far below the steady state

  – q/w is very high
  – Roy model implies that "everyone" is an entrepreneur

▶ near the steady state

  – slow convergence when the firm size distribution is close to Zipf

- see my

  – "Slow Convergence in Economies with Organization Capital"
  – *Federal Reserve Bank of Minneapolis w.p. 748, 2018*
  – and further references therein

model 2

based on

Luttmer [*Quarterly Journal of Economics*, 2007]

and

Luttmer [*Journal of Economic Theory*, 2012]

a crash course on the KFE for $\mathrm{d}y_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$

- without noise, $f(t, y) = f(0, y - \mu t)$ implies

$$\mathrm{D}_t f(t, y) = -\mu \mathrm{D}_y f(0, y - \mu t) = -\mu \mathrm{D}_y f(t, y)$$

- without drift, random increments make population move downhill
  - CDF satisfies

$$\mathrm{D}_t F(t, y) = \frac{1}{2}\sigma^2 \mathrm{D}_y f(t, y)$$

  - differentiate

$$\mathrm{D}_t f(t, y) = \frac{1}{2}\sigma^2 \mathrm{D}_{yy} f(t, y)$$

▶ combine and add random death at rate δ

$$\mathrm{D}_t f(t, y) = -\mu \mathrm{D}_y f(t, y) + \frac{1}{2}\sigma^2 \mathrm{D}_{yy} f(t, y) - \delta f(t, y)$$

- real justification: take limit in binomial tree

the effect of exit at b

- number of firms

$$M_t = \int_b^\infty f(t,y)\mathrm{d}y$$

  – boundary conditions

$$f(t,b) = 0, \quad \lim_{y\to\infty} \left[\, f(t,y),\ \mathrm{D}_y f(t,y),\ \mathrm{D}_{yy} f(t,y) \,\right] = 0$$

▶ this yields

$$\begin{aligned}
\frac{\partial}{\partial t}\int_b^\infty f(t,y)\mathrm{d}y &= \int_b^\infty \mathrm{D}_t f(t,y)\mathrm{d}y \\
&= -\mu\int_b^\infty \mathrm{D}_y f(t,y)\mathrm{d}y + \frac{1}{2}\sigma^2\int_b^\infty \mathrm{D}_{yy} f(t,y)\mathrm{d}y - \delta\int_b^\infty f(t,y)\mathrm{d}y \\
&= -f(t,y)\Big|_b^\infty - \frac{1}{2}\sigma^2 \mathrm{D}_y f(t,b) - \delta\int_b^\infty f(t,y)\mathrm{d}y
\end{aligned}$$

▶ therefore

$$\mathrm{D}M_t = -\frac{1}{2}\sigma^2 \mathrm{D}_y f(t,b) - \delta M_t$$

  – a steep density at the exit thresholds implies a lot of exit

entry and exit

- flow of entrants

$$E_t = Ee^{\eta t}$$

– entry at $y_0 = x$, and then

$$\mathrm{d}y_a = \mu \mathrm{d}a + \sigma \mathrm{d}B_a$$

– exit when y_a hits $b < x$

- density of firms

$$m(t, y) = M_t f(t, y)$$

where

$$M_t = \int_b^\infty m(t, y)\mathrm{d}y$$

- conjecture that there is a stationary density

$$M_t = Me^{\eta t}, \quad f(t, y) = f(y)$$

– which implies $\mathrm{D}_t m(t, y) = \eta M_t f(y)$ and

$$\left[\, \mathrm{D}_y m(t, y) \;\; \mathrm{D}_{yy} m(t, y) \,\right] = M_t \left[\, \mathrm{D}f(y) \;\; \mathrm{D}^2 f(y) \,\right]$$

entry and exit

- the KFE simplifies to

$$\eta f(y) = -\mu \mathrm{D} f(y) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(y), \quad y \in (b, x) \cup (x, \infty)$$

  – boundary conditions

$$f(b) = 0, \quad \lim_{x \uparrow y} f(x) = \lim_{x \downarrow y} f(x), \quad \lim_{x \to \infty} f(x) = 0$$

- try solutions of the form $e^{-\alpha y}$
- this implies a quadratic characteristic equation

$$\eta = \mu\alpha + \frac{1}{2}\sigma^2\alpha^2 \Rightarrow \alpha_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

▶ the solution for $f(y)$ is a linear combination of $e^{-\alpha_+ y}$ and $e^{-\alpha_- y}$

  – one for each of the two domains (b, x) and (x, ∞)

  – the boundary conditions pin down these linear combinations

the solution

- the density is

$$f(y) \;=\; \frac{\alpha e^{-\alpha(y-b)}}{(e^{\alpha_*(x-b)}-1)/\alpha_*} \times \min\left\{ \frac{e^{(\alpha+\alpha_*)(y-b)}-1}{\alpha+\alpha_*}, \frac{e^{(\alpha+\alpha_*)(x-b)}-1}{\alpha+\alpha_*} \right\},$$

where

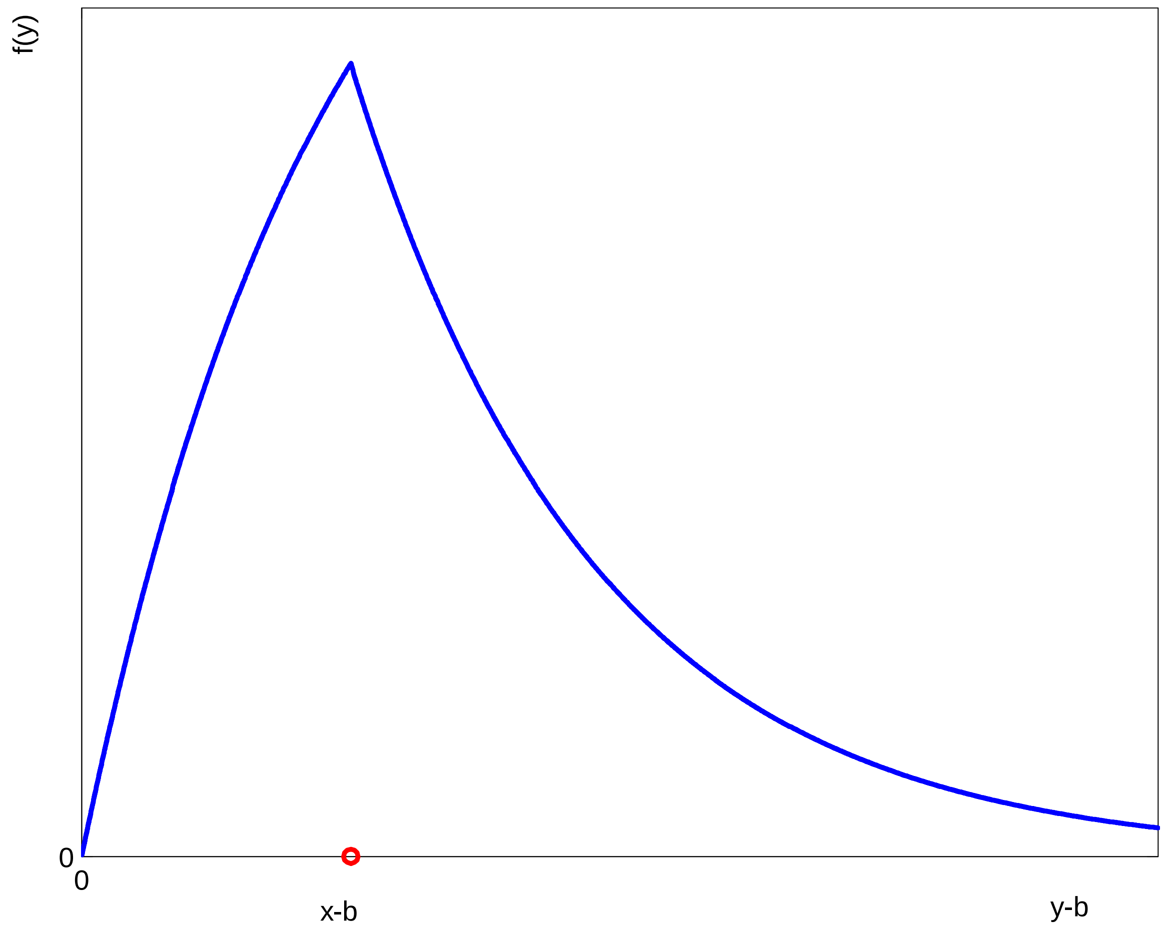
$$\alpha \;=\; -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \;\; \alpha_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- note that the right tail behaves like $e^{-\alpha y}$

- the implied entry rate $\epsilon = E_t/M_t$ is

$$\epsilon = \eta + \frac{1}{2}\sigma^2 \mathrm{D} f(b) = \eta + \frac{1}{2}\alpha\sigma^2 \left(\frac{e^{\alpha_*(x-b)}-1}{\alpha_*} \right)^{-1}$$

the stationary density



an economy with differentiated commodities

- preferences

$$\int_0^\infty e^{-\rho t} H_t \ln(c_t) \mathrm{d}t$$

household consumption is

$$C_t = H_t c_t$$

$$H_t = H e^{\eta t}, \qquad \rho > \eta > 0$$

- c_t is a CES composite good of differentiated commodties

$$c_t = \left[\int c_{\omega,t}^{1-1/\varepsilon} \mathrm{d}M_t(\omega) \right]^{1/(1-1/\varepsilon)}$$

where $\varepsilon > 1$

household choices (dynamic)

- the dynastic present-value budget constraint

$$\int_0^\infty \exp\left(-\int_0^t r_s \mathrm{d}s\right) H_t c_t \mathrm{d}t \;\leq\; \texttt{wealth}$$

implies the first-order condition

$$e^{-\rho t} H_t \times \frac{1}{c_t} \;=\; \lambda \exp\left(-\int_0^t r_s \mathrm{d}s\right) H_t$$

or simply

$$\frac{e^{-\rho t}}{c_t} \;=\; \lambda \pi_t$$

- differentiating yields the Euler condition

$$r_t = \rho + \frac{\mathrm{D}c_t}{c_t}$$

household choices (static)

- the differentiated commodity demands are

$$c_{\omega,t} = \left(\frac{p_{\omega,t}}{P_t}\right)^{-\varepsilon} H_t c_t$$

where P_t is the price index

$$P_t = \left(\int p_{\omega,t}^{1-\varepsilon} \mathrm{d}M_t(\omega)\right)^{1/(1-\varepsilon)}$$

producers

- blueprint + linear labor-only technology yields output $y_{\omega,t} = e^{z_{\omega,t}} l_{\omega,t}$
- the time-t wage in units of the composite consumption good $= w_t$

▶ max $P_t v_{\omega,t} = (p_{\omega,t} - P_t w_t e^{-z_{\omega,t}}) y_{\omega,t}$ s.t. $y_{\omega,t} = (p_{\omega,t}/P_t)^{-\varepsilon} H_t c_t$ gives

$$\frac{p_{\omega,t}}{P_t} = \frac{w_t e^{-z_{\omega,t}}}{1 - 1/\varepsilon}$$

▶ eliminating $p_{\omega,t}/P_t$ from the price index gives

$$w_t = \left(1 - \frac{1}{\varepsilon}\right) e^{Z_t}, \quad e^{Z_t} = \left(\int e^{(\varepsilon-1) z_{\omega,t}} \mathrm{d}M_t(\omega)\right)^{1/(\varepsilon-1)}$$

▶ implied employment and profits

$$\begin{bmatrix} w_t l_{\omega,t} \\ v_{\omega,t} \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} e^{(\varepsilon-1)(z_{\omega,t} - Z_t)} H_t c_t$$

- also: a firm continuation cost of $\phi > 0$ units of labor

aggregate variable labor and consumption

- define

$$L_t = \int l_{\omega,t} \mathrm{d}M_t(\omega)$$

▶ the CES aggregator applied to $y_{\omega,t} = e^{z_{\omega,t}} l_{\omega,t}$ gives

$$H_t c_t = e^{Z_t} L_t \tag{1}$$

where

$$e^{Z_t} = \left(\int e^{(\varepsilon-1) z_{\omega,t}} \mathrm{d}M_t(\omega) \right)^{1/(\varepsilon-1)}$$

- recall

$$w_t = \left(1 - \frac{1}{\varepsilon}\right) e^{Z_t} \tag{2}$$

▶ from (1) and (2)

$$w_t L_t = \left(1 - \frac{1}{\varepsilon}\right) H_t c_t$$

as expected.

incumbent productivity processes

- log productivity of firm ω

$$\mathrm{d}z_{\omega,t} = \theta_z \mathrm{d}t + \sigma_z \mathrm{d}W_{\omega,t}$$

- recall that variable profits are

$$\frac{v_{\omega,t}}{c_t} = \frac{1}{\varepsilon} \times e^{(\varepsilon-1)(z_{\omega,t}-Z_t)} H_t, \qquad c_t = \frac{e^{Z_t} L_t}{H_t},$$

where

$$e^{Z_t} = \left(\frac{\int e^{(\varepsilon-1)z_{\omega,t}} \mathrm{d}M_t(\omega)}{\int \mathrm{d}M_t(\omega)} \right)^{1/(\varepsilon-1)} \times \left(\int \mathrm{d}M_t(\omega) \right)^{1/(\varepsilon-1)}$$

▶ conjecture that there will (somehow) be a steady state of the form

$$\left[e^{Z_t}, w_t, c_t\right] = \left[e^{Z}, w, c\right] e^{\kappa t}, \qquad \kappa = \theta + \frac{\eta}{\varepsilon - 1}$$

for some θ to be determined

– this θ will generally differ from θ_z

– the key assumption will be about the productivity of entrants

marginal utility weighted profits

- recall that

$$\frac{v_{\omega,t}}{c_t} = \frac{1}{\varepsilon} \times e^{(\varepsilon-1)(z_{\omega,t}-Z_t)} H_t$$

with, in a steady state

$$\mathrm{d}z_{\omega,t} = \theta_z \mathrm{d}t + \sigma_z \mathrm{d}W_{\omega,t}, \qquad \mathrm{d}Z_t = \kappa \mathrm{d}t = \left(\theta + \frac{\eta}{\varepsilon - 1}\right) \mathrm{d}t$$

▶ Ito's lemma implies

$$\mathrm{d}\ln\left(\frac{v_{\omega,t}}{c_t}\right) \;=\; \mu \mathrm{d}t + \sigma \mathrm{d}W_{\omega,t}$$

where

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = (\varepsilon - 1) \begin{bmatrix} \theta_z - \theta \\ \sigma_z \end{bmatrix}$$

– the calculation is

$$\begin{aligned} (\varepsilon - 1)(\theta_z - \kappa) + \eta \;&=\; (\varepsilon - 1)\left(\theta_z - \left(\theta + \frac{\eta}{\varepsilon - 1}\right)\right) + \eta \\ &=\; (\varepsilon - 1)\left(\theta_z - \theta\right) \end{aligned}$$

the value of a firm

- the marginal utility weighted price of a firm

$$\tilde{V}_t = \frac{1}{c_t} \times \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} \exp\left(- \int_t^s r_u \mathrm{d}u \right) (v_{\omega,s} - \phi w_s) \,\mathrm{d}s \right]$$

- recall, from logarithmic utility

$$\exp\left(- \int_t^s r_u \mathrm{d}u \right) = e^{-\rho t} \times \frac{c_t}{c_s}$$

- therefore

$$\begin{aligned} \tilde{V}_t &= \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} \exp\left(- \int_t^s r_u \mathrm{d}u \right) \frac{c_s}{c_t} \left(\frac{v_{\omega,s}}{c_s} - \frac{\phi w_s}{c_s} \right) \mathrm{d}s \right] \\ &= \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho s} \left(\frac{v_{\omega,s}}{c_s} - \frac{\phi w_s}{c_s} \right) \mathrm{d}s \right] \end{aligned}$$

a convenient state variable

- in units of fixed cost labor

$$V_t = \frac{\tilde{V}_t}{\phi w_t/c_t} = \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho s} \times \frac{w_s/c_s}{w_t/c_t} \times \left(\frac{v_{\omega,s}}{\phi w_s} - 1 \right) \mathrm{d}s \right]$$

- in a steady state

$$\frac{w_t}{c_t} = \left(1 - \frac{1}{\varepsilon}\right) \frac{H_t}{L_t} \text{ will be constant}$$

and then

$$V_t = \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho s} \left(\frac{v_{\omega,s}}{\phi w_s} - 1 \right) \mathrm{d}s \right]$$

- define

$$e^{y_t} = \frac{v_{\omega,t}}{\phi w_t}$$

  – in a steady state,

$$\mathrm{d}y_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

the Bellman equation

- given some exit threshold b, the Bellman equation is then

$$\rho V(y) = e^y - 1 + \mu \mathrm{D}V(y) + \frac{1}{2}\sigma^2 \mathrm{D}^2 V(y), \quad y > b$$

  – and the boundary conditions are

$$0 = V(b) \quad \lim_{y\to\infty} V(y) = \frac{e^y}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)}$$

- the optimal b must be such that

$$0 = \mathrm{D}V(b).$$

- the solution is

$$V(y) = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{y-b} - 1 - \frac{1 - e^{-\xi(y-b)}}{\xi}\right)$$

where

$$e^b = \frac{\xi}{1+\xi}\left(1 - \frac{\mu + \sigma^2/2}{\rho}\right), \quad \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}$$

entry

- let $z_{X,t}$ the log productivity of exiting firms
- suppose entrants can use

$$z_{E,t} = z_{X,t} + \frac{\Delta}{\varepsilon - 1}$$

  – standing on the shoulders of midgets...

- translates into entry state

$$e^{x_t} = e^{\Delta} \times \left(\frac{v_{\omega,t}}{\phi w_t} \right)_{\text{exiting firms}}$$

  in a steady state

$$x = b + \Delta$$

- price of a new firm in units of fixed cost labor

$$s = V(x)$$

- as before, a Roy model delivers
  – a flow of entrants $E(s)$
  – labor supply $L(s)$

variable labor as a function of the state

- the derived firm state variables are

$$e^{y_{\omega,t}} = \frac{v_{\omega,t}}{\phi w_t}$$

  – depends on the individual productivities $z_{\omega,t}$

  – and on the aggregate state

- recall

$$\frac{v_{\omega,t}}{w_t l_{\omega,t}} = \frac{1/\varepsilon}{1-1/\varepsilon}$$

▶ so variable labor is

$$l_{\omega,t} = (\varepsilon - 1)\phi \times e^{y_{\omega,t}}$$

the balanced growth path

- the number of firms is $M_t = Me^{\eta t}$

▶ the steady state market clearing conditions are

1. the market for labor

$$L(s)H \; = \; \left(1 + (\varepsilon - 1)\int_b^{\infty} e^y f(y)\mathrm{d}y\right)\phi M$$

2. the market for entrepreneurial services

$$E(s)H \; = \; \left(\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(b)\right) M$$

▶ b is the optimal exit threshold

▶ $f(\cdot)$ is the stationary density on (b, ∞)

– both functions only of the firm growth rate $\mu = (\varepsilon - 1)(\theta_z - \theta)$

summary of balanced growth conditions

- demand for firms

$$\frac{M}{H} = \frac{1}{\phi}\frac{L(s)}{1+(\varepsilon-1)\int_b^\infty e^y f(y)\mathrm{d}y}$$

- supply of firms

$$\frac{M}{H} = \frac{E(s)}{\eta+\frac{1}{2}\sigma^2 \mathrm{D}f(b)}$$

where

$$s = V(b+\Delta),$$

and we have a mapping

$$\mu \mapsto (b, V(\cdot), f(\cdot))$$

- the growth rate is

$$\kappa = \theta + \frac{\eta}{\varepsilon-1},$$

where

$$\theta = \theta_z - \frac{\mu}{\varepsilon-1}$$

is the growth rate of entrant productivities

the mapping $\mu \mapsto (b, V(\cdot), f(\cdot))$

- the value function is

$$V(y) = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{y-b} - 1 - \frac{1-e^{-\xi(y-b)}}{\xi}\right)$$

and

$$e^b = \frac{\xi}{1+\xi}\left(1 - \frac{\mu+\sigma^2/2}{\rho}\right), \quad \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}$$

- the stationary density is

$$f(y) = \frac{\alpha e^{-\alpha(y-b)}}{(e^{\alpha_*\Delta}-1)/\alpha_*} \times \min\left\{\frac{e^{(\alpha+\alpha_*)(y-b)}-1}{\alpha+\alpha_*}, \frac{e^{(\alpha+\alpha_*)\Delta}-1}{\alpha+\alpha_*}\right\}$$

and

$$\alpha = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \quad \alpha_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

key properties of the mapping $\mu \mapsto (b, V(\cdot), f(\cdot))$

- recall that $\mu = (\varepsilon - 1)(\theta_z - \theta)$

▶ the mean

$$\frac{\partial}{\partial \mu} \int_b^\infty e^y f(y) \mathrm{d}y > 0$$

– importantly,

$$\mu + \frac{1}{2}\sigma^2 \uparrow \eta \text{ implies } \int_b^\infty e^y f(y) \mathrm{d}y \to \infty$$

▶ the exit rate

$$\frac{\partial}{\partial \mu} \left(\frac{1}{2}\sigma^2 \mathrm{D} f(b) \right) < 0$$

▶ the value of an entrant

$$\frac{\partial s}{\partial \mu} = \frac{\partial V(b + \Delta)}{\partial \mu} > 0$$

▶ the tail index

$$\frac{\partial \alpha}{\partial \mu} = \frac{\partial}{\partial \mu} \left(-\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} \right) < 0$$

demand and supply curves have the usual slopes

- rapid firm growth increases the value of entrants

$$\frac{\partial s}{\partial \mu} = \frac{\partial V(b+\Delta)}{\partial \mu} > 0$$

▶ demand

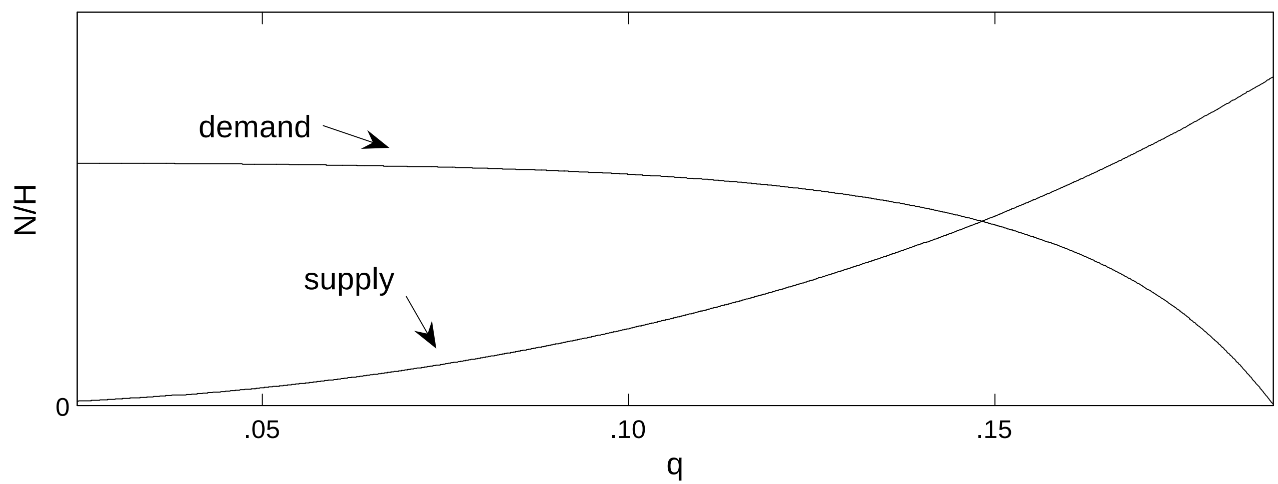
$$\frac{M}{H} = \frac{1}{\phi}\frac{L(s)}{1+(\varepsilon-1)\int_b^\infty e^y f(y)\mathrm{d}y} \Rightarrow \frac{\partial}{\partial s}\left(\frac{M}{H}\right) < 0$$

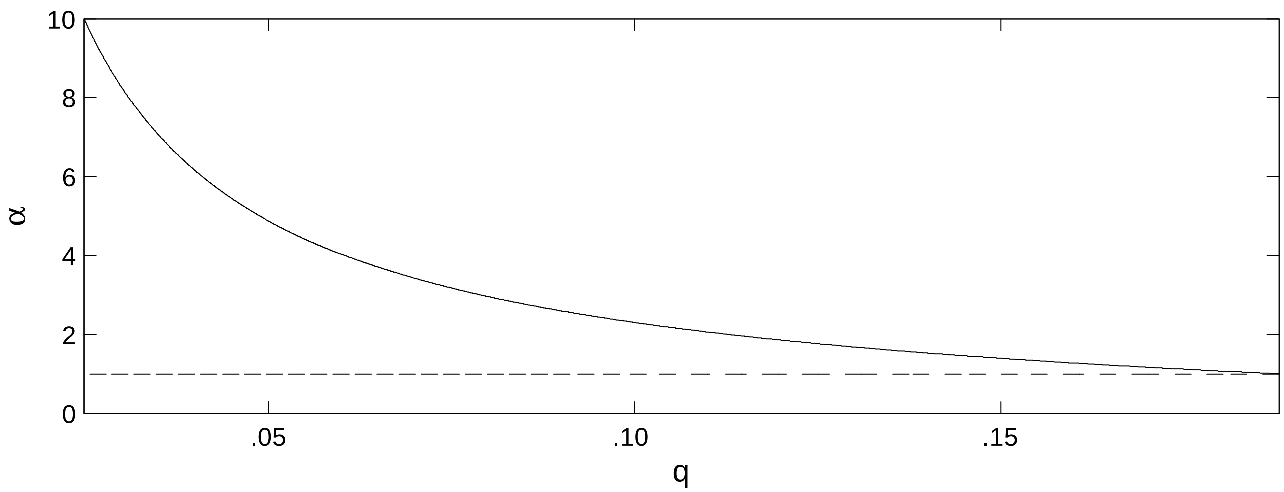
– converges to zero as $\mu + \frac{1}{2}\sigma^2 \uparrow \eta$

▶ supply

$$\frac{M}{H} = \frac{E(s)}{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(b)} \Rightarrow \frac{\partial}{\partial s}\left(\frac{M}{H}\right) > 0$$

market clearing and the tail index α





the Zipf asymptote

- demand and supply for firms

$$\frac{M}{H} = \frac{1}{\phi}\frac{L(s)}{1+(\varepsilon-1)\int_b^\infty e^y f(y)\mathrm{d}y}$$

$$\frac{M}{H} = \frac{E(s)}{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(b)}$$

  where $s = V(b+\Delta)$

  – average firm size explodes as s increases and $\mu + \frac{1}{2}\sigma^2 \uparrow \eta$

  – hence, the demand for firms goes to zero

- to approach Zipf

  – shift the supply curve in along the downward sloping demand curve

  – can use shifts in $E(\cdot)$ and $L(\cdot)$ (from the Roy model)

so what determines growth?

- recall

$$e^{Z_t} = \left(\frac{\int e^{(\varepsilon-1)z_{\omega,t}} \mathrm{d}M_t(\omega)}{\int \mathrm{d}M_t(\omega)} \right)^{1/(\varepsilon-1)} \times \left(\int \mathrm{d}M_t(\omega) \right)^{1/(\varepsilon-1)}$$

- two components

1. improvements in some "average" of the individual productivities

2. gains from variety

growth with a constant population

- this implies $\alpha = -\mu/(\sigma^2/2)$ and $\alpha_* = 0$
  - the density near b is then

$$f(y) = \frac{1 - e^{-\alpha(y-b)}}{\Delta}, \quad y \in [b, b+\Delta]$$

- implied entry and exit rates

$$\epsilon = \frac{1}{2}\sigma^2 \mathrm{D} f(b) = \frac{1}{2}\sigma^2 \times \frac{\alpha}{\Delta} = -\frac{\mu}{\Delta} = -\frac{(\varepsilon - 1)(\theta_z - \theta)}{\Delta}$$

  - this can be written as

$$\theta = \theta_z + \epsilon \times \frac{\Delta}{\varepsilon - 1}.$$

▶ so growth follows from

1. incumbent firms improving their own productivities at the rate θ_z
2. replacing firms, selectively, with firms that are better

$$z_t[entry] = z_t[exit] + \frac{\Delta}{\varepsilon - 1}$$

▶ the entry rate ϵ is endogenous

  - could enrich the model by making θ_z and Δ endogenous as well

randomly copying incumbents

- suppose entrants draw random incumbent and copy productivity
- stationary density must satisfy

$$\eta f(y) = -\mu \mathrm{D} f(y) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(y) + \epsilon f(y)$$

together with the boundary conditions

$$f(b) = 0 = \lim_{y\to\infty} f(y)$$

- solutions of form $e^{-\alpha y}$ imply

$$\alpha_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 - \frac{\epsilon - \eta}{\sigma^2/2}}$$

  – need $\epsilon > \eta$ to replace exit at b, and need real roots

$$\left(\frac{\mu}{\sigma^2}\right)^2 \geq \frac{\epsilon - \eta}{\sigma^2/2}$$

  – if $\mu < 0$ then both $\alpha_+ > \alpha_- > 0$

  – not "enough" boundary conditions

► continuum of stationary densities

initial conditions matter

- recall

$$\alpha_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 - \frac{\epsilon - \eta}{\sigma^2/2}}$$

  – need real roots

$$\left(\frac{\mu}{\sigma^2}\right)^2 \geq \frac{\epsilon - \eta}{\sigma^2/2}$$

- when this holds with equality, $\alpha_+ = \alpha_- = \alpha > 0$, and

$$f(y) = \alpha^2 (y - b) e^{-\alpha(y-b)}$$

  – take limit as $\alpha_+ - \alpha_- \downarrow 0$

  – log firm size follows a Gamma density

▶ Luttmer [2007] argues this is what will happen when the economy starts with an initial productivity distribution that has bounded support