Lecture Notes on Knowledge Diffusion, Growth, and Income Inequality

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these notes

are based on my

1. "Selection, Growth, and the Size Distribution of Firms"

   Quarterly Journal of Economics, vol. 122, no. 3 (2007), 1103-1144.

2. "An Assignment Model of Knowledge Diffusion and Income Inequality"

   Federal Reserve Bank of Minneapolis working paper 715 (Sept 2014)

► see original papers for references to related literature

two models of social learning

1. individuals randomly select others at rate β and copy if “better”

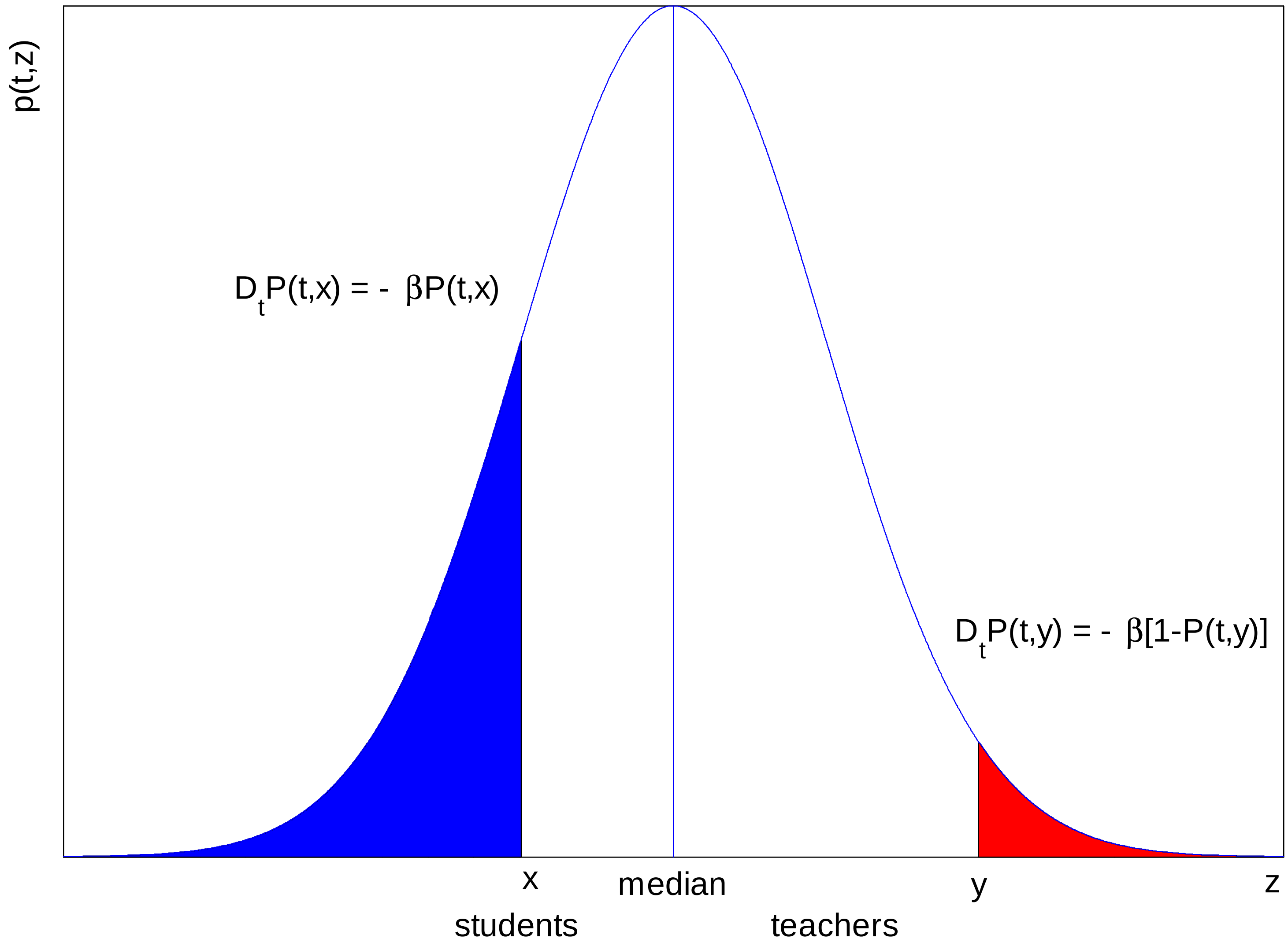
$$\mathrm{D}_t P(t,z) \;=\; -\beta P(t,z)[1-P(t,z)]$$

2. “students” match 1-on-1 with “teachers” and learn at rate β

$$\mathrm{D}_t P(t,z) \;=\; -\beta \min\left\{P(t,z), 1-P(t,z)\right\}$$

▶ a parabola or a tent

the ODE for one-on-one knowledge transfer



the solution

1. random matching delays

$$P(t,z) = \frac{1}{1 + \left(\frac{1}{P(0,z)} - 1\right) e^{\beta t}}$$

2. random learning delays

$$P(t,z) = \begin{cases} e^{-\beta t} P(0,z) & z \in (-\infty, x_0] \\ \frac{1}{2} \frac{1/2}{e^{\beta t}[1 - P(0,z)]} & z \in [x_0, x_t] \\ 1 - e^{\beta t}[1 - P(0,z)] & z \in [x_t, \infty) \end{cases}$$

with a median x_t defined by

$$\frac{1}{2} = P(t, x_t) = e^{\beta t} [1 - P(0, x_t)] \qquad (!)$$

▶ in both cases, stationary solutions of the form

$$P(t,z) = P(0, z - \kappa t) \quad \text{and} \quad P(t,z) = P(0, z e^{-\kappa t})$$

for any κ positive

individual creativity & social learning

- two independent standard Brownian motions $B_{1,t}$, $B_{2,t}$,

$$\mathrm{E}\left[\max\left\{\sigma B_{1,t}, \sigma B_{2,t}\right\}\right] = \sigma\sqrt{t/\pi}$$

- reset to max at random time $\tau_{j+1} > \tau_j$

$$z_{\tau_{j+1}} = z_{\tau_j} + \sigma \max\left\{B_{1,\tau_{j+1}} - B_{1,\tau_j}, B_{2,\tau_{j+1}} - B_{2,\tau_j}\right\}$$

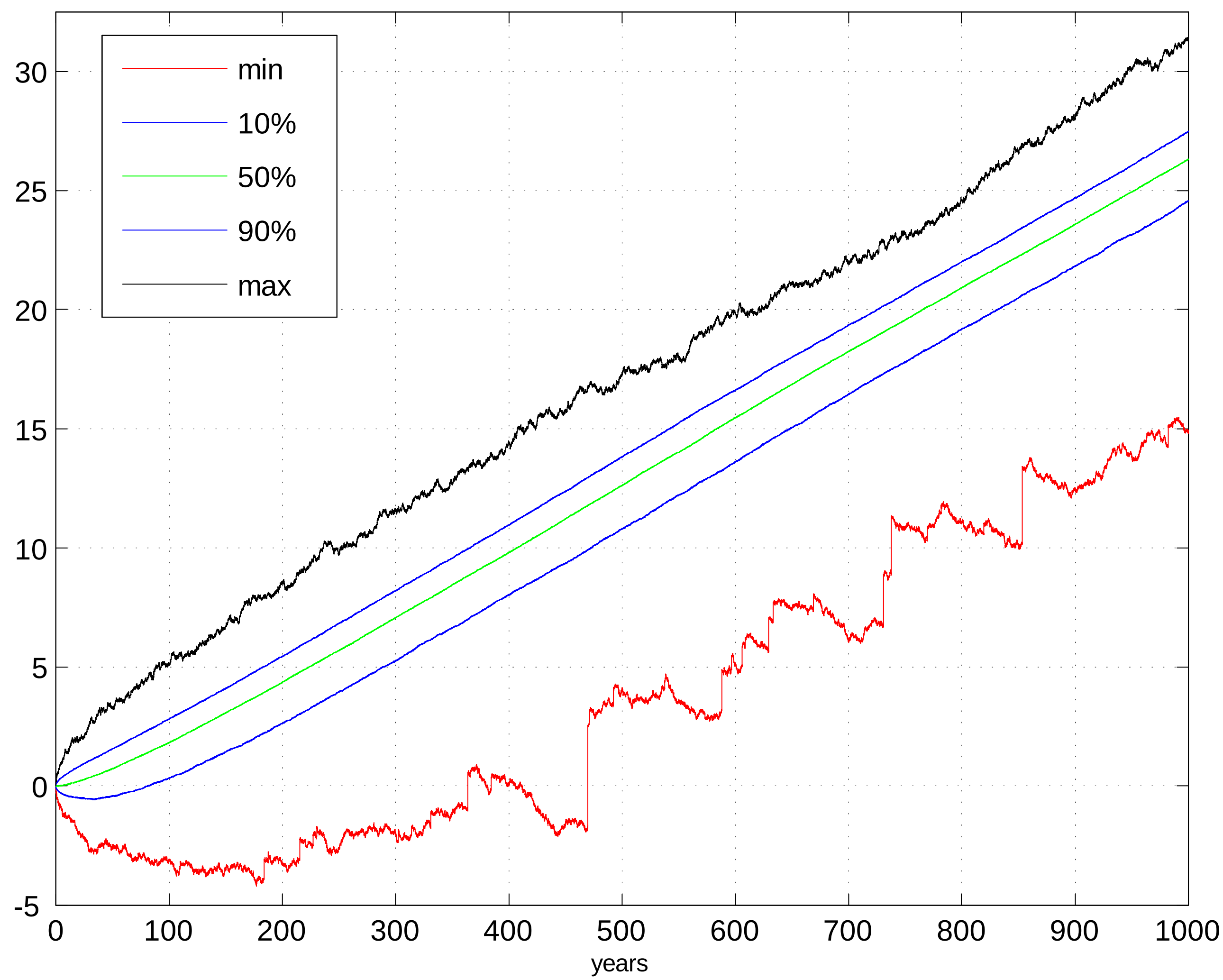
- reset times arrive randomly at rate β

$$\begin{aligned}\frac{\mathrm{E}\left[z_{\tau_{j+1}} - z_{\tau_j} | z_{\tau_j}\right]}{\mathrm{E}\left[\tau_{j+1} - \tau_j | z_{\tau_j}\right]} &= \frac{1}{1/\beta}\int_0^\infty \sigma\left(t/\pi\right)^{1/2}\beta e^{-\beta t}\mathrm{d}t \\ &= \frac{1}{2}\sigma\sqrt{\beta}\int_0^\infty 2\left(u/\pi\right)^{1/2}e^{-u}\mathrm{d}u = \frac{1}{2}\sigma\sqrt{\beta}\end{aligned}$$

▶ large populations

$$\texttt{trend} = \sigma^2\sqrt{\frac{\beta}{\sigma^2/2}} > \sigma\sqrt{\beta} = \mathrm{E}\left[\frac{z_{\tau_{j+1}} - z_{\tau_j}}{\tau_{j+1} - \tau_j} \mid z_{\tau_j}\right] > \frac{1}{2}\sigma\sqrt{\beta}\ldots$$

10K agents: every 2.4 days, someone imitates someone else



- $\sigma = 0.12$, $\beta = 0.015$, implies $\texttt{trend} = 0.0147$

the random imitation economy

- demography and preferences

$$\int_0^\infty e^{-\rho t} \ln(C_t) \mathrm{d}t$$

  – unit measure of dynasties
  – generations die randomly at the rate δ
  – replaced immediately with next generation
  – complete markets, interest rate $r_t = \rho + \mathrm{D}C_t/C_t$

- (Lucas, 1978) manager in state z and l workers can produce consumption,

$$c = \left(\frac{e^z}{1-\alpha}\right)^{1-\alpha} \left(\frac{l}{\alpha}\right)^{\alpha}$$

- economy-wide state at t

a measure of managers $M(t, z)$

the human resource constraint

$$L_t + E_t + (1+\phi)N_t = 1$$

- L_t : production workers, one unit of labor per worker
- E_t : entrants, trying to become managers
- N_t : managers, $N_t = M(t,\infty)$, overhead of ϕ workers per manager

▶ transitions:

- newborn individuals start in $L_t + E_t + \phi N_t$
- back and forth between L_t, E_t and ϕN_t instantaneously
- $N_t \to L_t + E_t + \phi N_t$ instantaneous when manager chooses
- $E_t \to N_t$ after random delay with mean $1/\gamma$

production of consumption, as usual

- managerial profit maximization

$$\max_{l}\left\{\left(\frac{e^{z}}{1-\alpha}\right)^{1-\alpha}\left(\frac{l}{\alpha}\right)^{\alpha}-w_t l\right\}=v_t e^{z}$$

yields

$$\frac{w_t l_t(z)}{v_t e^{z}}=\frac{\alpha}{1-\alpha},\qquad v_t^{1-\alpha}w_t^{\alpha}=1$$

- factor prices and aggregate consumption

$$\begin{bmatrix} w_t L_t \\ v_t K_t \end{bmatrix}=\begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}C_t,\qquad C_t=\left(\frac{K_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{L_t}{\alpha}\right)^{\alpha}$$

given

$$\begin{bmatrix} L_t \\ K_t \end{bmatrix}=\int\begin{bmatrix} l_t(z) \\ e^{z} \end{bmatrix}M(t,\mathrm{d}z)$$

as long as a manager continues in a job

$$\mathrm{d}z_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$$

- idiosyncratic shock B_t is a standard Brownian motion

- add learning jumps later

- must pay flow of $\phi \geq 0$ units of labor to continue
  - if not, lose z_t and become a worker again

workers and entrants

- workers supply one unit of labor at wage w_t
- entrants sample incumbent managers at the rate γ, and imitate perfectly
- time-t present value of dynastic earnings
  - when worker or entrant: W_t
  - when manager in state z: $V_t(z)$

▶ random imitation

$$q_t = \frac{1}{N_t} \int V_t(z) M(t, \mathrm{d}z)$$

▶ because production workers are essential

$$w_t \geq \gamma(q_t - W_t) \qquad \text{w.e. if } E_t > 0$$

Ito, and a piece of convenient notation

$$\mathrm{d}z_t \;=\; \mu \mathrm{d}t + \sigma \mathrm{d}B_t$$

- for a sufficiently nice $f(t, z)$,

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathrm{E}\left[f(t+\Delta, z_{t+\Delta}) - f(t, z_t) | z_t = z \right] \;=\; \mathcal{A} f(t, z)$$

▶ where

$$\mathcal{A} f(t, z) \;=\; \mathrm{D}_t f(t, z) + \mu \mathrm{D}_z f(t, z) + \frac{1}{2} \sigma^2 \mathrm{D}_{zz} f(t, z)$$

– depends on μ and σ^2

Bellman equations

- workers and entrants

$$r_t W_t = w_t + \mathrm{D}_t W_t$$

- managers

$$r_t V_t(z) = v_t e^z - \phi w_t + \mathcal{A} V_t(z) + \delta \left[W_t - V_t(z)\right]$$

for all $z > b_t$,

$$V_t(b_t) = W_t$$

▶ implied managerial surplus

$$(r_t + \delta)\left[V_t(z) - W_t\right] \;=\; v_t e^z - (1+\phi) w_t + \mathcal{A}\left[V_t(z) - W_t\right]$$

– effective fixed cost is $1 + \phi$ units of labor

– managerial opportunity cost

▶ crucial transversality conditions omitted

population dynamics

- density $m(t,z)$ of $M(t,z)$
- Kolmogorov forward equation

$$\mathrm{D}_t m(t,z) \;=\; -\mu \mathrm{D}_z m(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} m(t,z) + \left(\frac{\gamma E_t}{N_t} - \delta\right) m(t,z)$$

density and derivatives vanish as $z \to \infty$, and

$$m(t,b_t) \;=\; 0$$

▶ this implies

$$\mathrm{D}N_t \;=\; \frac{\partial}{\partial t}\int_{b_t}^{\infty} m(t,z)\mathrm{d}z = \int_{b_t}^{\infty} \mathrm{D}_t m(t,z)\mathrm{d}z = -\frac{1}{2}\sigma^2 \mathrm{D}_z m(t,b_t) + \gamma E_t - \delta N_t$$

balanced growth

- *conjecture* growth rate κ so that cross-section of $z_t - \kappa t$ time-invariant
  - ▶ notation: $z_t - \kappa t \to z$
- constant numbers of individuals in various occupations

$$L + E + (1+\phi)N = 1$$

- density of managers

$$m(t, z + \kappa t) = m(z)$$

- consumption and factor prices

$$[C_t, w_t] = [C, w]\, e^{(1-\alpha)\kappa t}, \quad v_t = v e^{-\alpha t}$$

- value functions

$$[W_t, V_t(z + \kappa t)] = [W, V(z)]\, e^{(1-\alpha)\kappa t}$$

- interest rate $r_t = r$,

$$r = \rho + (1-\alpha)\kappa$$

level of the balanced growth path

- Cobb-Douglas consumption sector

$$\frac{L}{N} = \frac{\alpha}{1-\alpha} \times \frac{ve^b}{w} \times \frac{Ke^{-b}}{N}$$

- stock of managerial knowledge capital

$$\frac{Ke^{-b}}{N} = \frac{1}{N}\int_b^\infty e^{z-b} m(z)\mathrm{d}z$$

- entry and exit

$$\frac{\gamma E}{N} = \delta + \frac{1}{2}\sigma^2 \times \frac{\mathrm{D}m(b)}{N}$$

- human resource constraint

$$N = \left(\frac{L}{N} + \frac{E}{N} + 1 + \phi\right)^{-1}$$

▶ just need ve^b/w and $m(b+\bullet)/N$

stationary value functions

- value of workers and entrants is $W = w/\rho$
- the Bellman equation for managers is

$$(\rho+\delta)V(z) = ve^{z} - \phi w + (\mu-\kappa)\mathrm{D}V(z) + \frac{1}{2}\sigma^2\mathrm{D}^2V(z)$$

with boundary conditions

$$0 = V(b) - W = \mathrm{D}V(b)$$

▶ change variables

$$e^{\widehat{z}} = \frac{1}{1+\phi}\frac{ve^{z}}{w}, \qquad e^{\widehat{b}} = \frac{1}{1+\phi}\frac{ve^{b}}{w}$$

▶ the normalized value function

$$\widehat{V}(\widehat{z}) = \frac{V(\widehat{z}+\ln(1+\phi)-\ln(v/w)) - W}{(1+\phi)w}$$

satisfies

$$(\rho+\delta)\widehat{V}(\widehat{z}) = e^{\widehat{z}} - 1 + (\mu-\kappa)\mathrm{D}\widehat{V}(\widehat{z}) + \frac{1}{2}\sigma^2\mathrm{D}^2\widehat{V}(\widehat{z})$$

the stationary value function

- ▶ $\widehat{V}(\cdot)$ and $\widehat{b}$ only depend on growth rate κ, and *nothing* else
- • solution for $V(\cdot)$

$$\frac{V(z) - W}{(1+\phi)w} = \frac{1}{\rho+\delta}\frac{\xi}{1+\xi}\left(e^{z-b} - 1 - \frac{1 - e^{-\xi(z-b)}}{\xi}\right)$$

for all $z \geq b$, where

$$e^{\widehat{b}} = \frac{1}{1+\phi}\frac{ve^b}{w}$$

and

$$e^{\widehat{b}} = \frac{\xi}{1+\xi}\left(1 - \frac{\mu - \kappa + \sigma^2/2}{\rho+\delta}\right), \quad \xi = \frac{\mu-\kappa}{\sigma^2} + \sqrt{\left(\frac{\mu-\kappa}{\sigma^2}\right)^2 + \frac{\rho+\delta}{\sigma^2/2}}$$

- ▶ key implication

$$\frac{ve^b}{w} \text{ is a function } \textit{only} \text{ of the growth rate } \kappa$$

- • $\partial\widehat{b}/\partial\kappa > 0$, so incumbent managers quit more easily when κ high

stationary densities

- from the KFE

$$0 = -(\mu - \kappa)\mathrm{D}m(z) + \frac{1}{2}\sigma^2\mathrm{D}^2 m(z) + \left(\frac{\gamma E}{N} - \delta\right) m(z)$$

with $m(b) = 0$, and density and derivatives vanish as $z \to \infty$

▶ solution must be

$$m(z) \propto e^{-\zeta_+(z-b)} - e^{-\zeta_-(z-b)}$$

where

$$\zeta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{(\gamma E/N) - \delta}{\sigma^2/2}}$$

▶ need $\zeta_\pm$ real and positive,

$$\kappa \geq \mu + \sigma^2\sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}} \qquad (!)$$

growth at lower bound

▶ if initial distribution has bounded support then long-run κ at lower bound

• this yields $\zeta_{\pm} \to \zeta$ and

$$\frac{m(z)}{N} = \zeta^2 (z-b) e^{-\zeta(z-b)}$$

where

$$\zeta = \frac{\kappa - \mu}{\sigma^2} = \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}$$

▶ hence

$$\kappa \ = \ \mu + \sigma^2 \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}$$

– yet to determine the entry rate $\gamma E/N$

– anything that raises $\gamma E/N$ increases growth

determining the entry rate $\gamma E/N$

- workers and entrants indifferent

$$w = \gamma(q - W), \quad q - W = \frac{1}{N}\int_b^\infty (V(z) - W)m(z)\mathrm{d}z$$

▶ yields

$$\frac{1}{\gamma} = \int_b^\infty \left(\frac{V(z) - W}{w}\right)\zeta^2(z-b)e^{-\zeta(z-b)}\mathrm{d}z$$

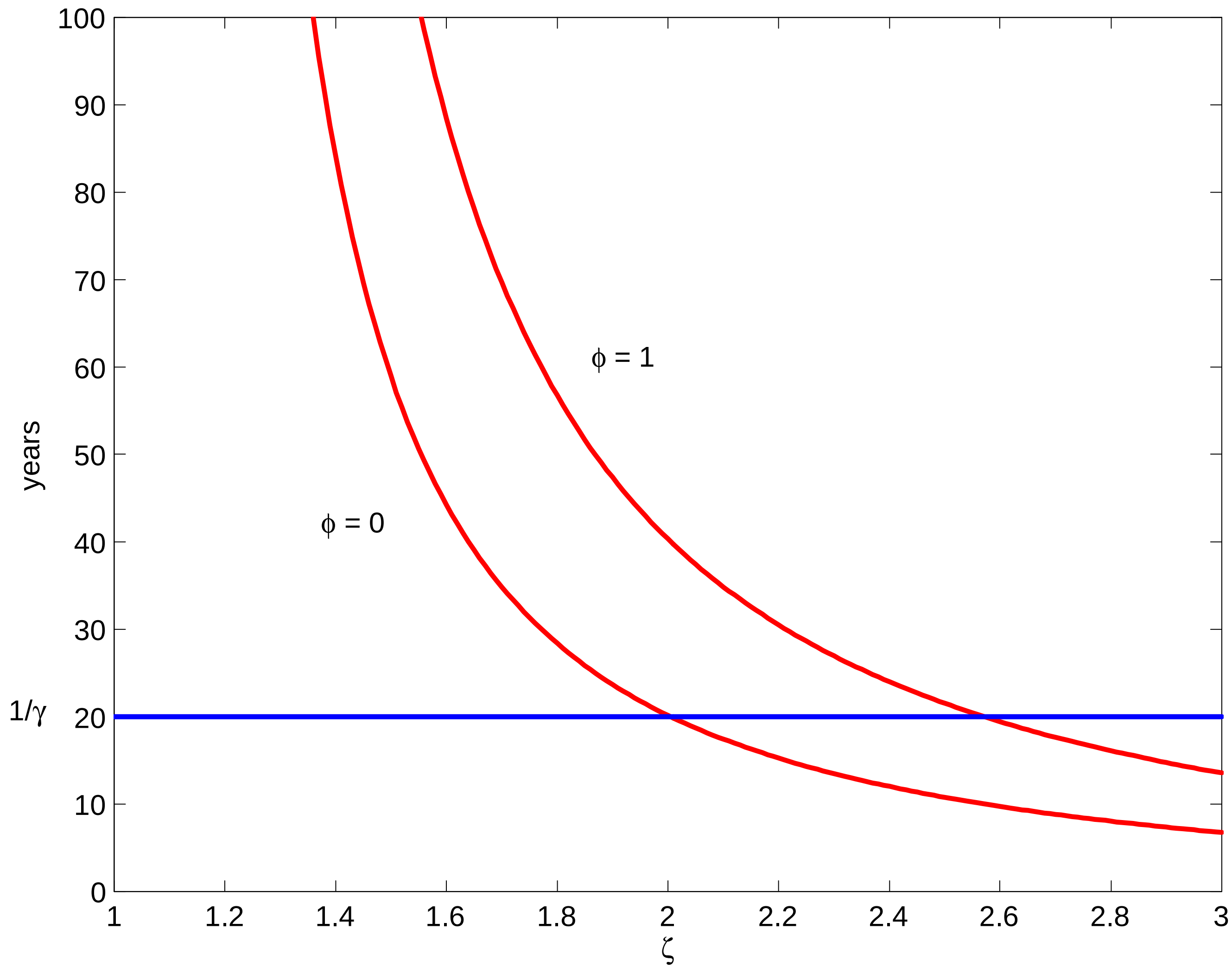
where

$$\frac{V(z) - W}{w} = \frac{1+\phi}{\rho+\delta}\frac{\xi}{1+\xi}\left(e^{z-b} - 1 - \frac{1 - e^{-\xi(z-b)}}{\xi}\right)$$

and

$$\xi = -\zeta + \sqrt{\zeta^2 + \frac{\rho+\delta}{\sigma^2/2}}$$

▶ equilibrium condition in ζ


100
90
80
70
60
50
40
30
20
10
0
1/γ
years
φ = 0
φ = 1
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
ζ

the competitive assignment economy

- one-on-one assignment of "students" to "teachers"
  - learn to be like teacher, randomly at rate γ
  - teacher-manager in state z charges flow tuition $T_t(z)$

▶ *new* definition of q_t

$$\gamma q_t = \sup_{\widetilde{z}} \{\gamma V_t(\widetilde{z}) - T_t(\widetilde{z})\}$$

- net gain for student-manager in state z

$$\gamma \left(q_t - V_t(z)\right) = \sup_{\widetilde{z}} \{\gamma \left[V_t(\widetilde{z}) - V_t(z)\right] - T_t(\widetilde{z})\}$$

- net gain for entrant same as manager at $z = b_t$

$$\gamma \left(q_t - W_t\right) = \gamma(q_t - V_t(b_t))$$

▶ *same* equilibrium condition for entry

$$w_t \geq \gamma(q_t - W_t), \quad \text{w.e. if } E_t > 0$$

equilibrium tuition

- a positive density of managers on (b_t, ∞)

▶ by definition of q_t

$$T_t(z) \geq \gamma\left(V_t(z) - q_t\right) \qquad (*)$$

– with equality if students select teachers in state z

▶ if $q_t - V_t(z) < 0$ then manager in state z prefers to teach at any $T_t(z) \geq 0$

– market clearing: must have students; hence (*) holds with equality

$$T_t(z) = \gamma\left(V_t(z) - q_t\right)$$

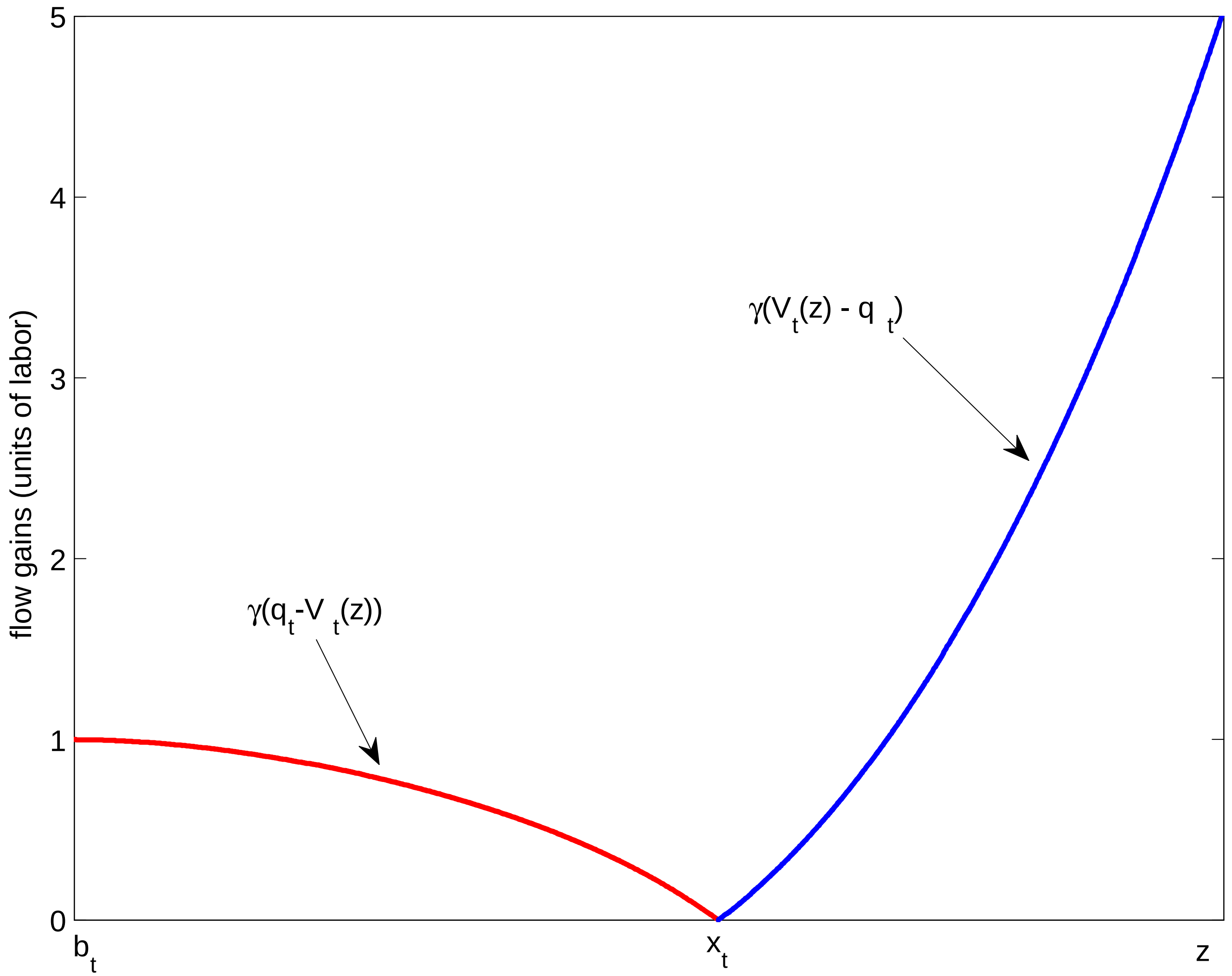
▶ if $q_t - V_t(z) > 0 = T_t(z)/\gamma$ then manager in state z prefers to study

$$T_t(z) = \gamma\left[V_t(z) - q_t\right]^+$$

– could raise to $\gamma\left|V_t(z) - q_t\right|$

- marginal teacher $x_t > b_t$

$$\gamma\left(q_t - V_t(x_t)\right) = 0 < w_t = \gamma(q_t - V_t(b_t))$$



flow gains (units of labor)
5
4
3
2
1
0
b_t
x_t
z
γ(q_t-V_t(z))
γ(V_t(z) - q_t)

Bellman equations

- ▶ workers and entrants $r_t W_t = w_t + \mathrm{D}W_t$
- • flow gains for teacher/student managers

$$\max\{\gamma(q_t - V_t(z)), T_t(z)\} = \gamma\,|V_t(z) - q_t|$$

- ▶ surplus of managers

$$(r_t + \delta)\,[V_t(z) - W_t] \;=\; v_t e^z - (1+\phi)w_t + \gamma\,|V_t(z) - q_t| + \mathcal{A}\,[V_t(z) - W_t]$$

– exit and teaching thresholds

$$0 = V_t(b_t) - W_t, \quad q_t - W_t = V_t(x_t) - W_t$$

- • as long as $E_t > 0$

$$w_t = \gamma(q_t - W_t) \qquad (!)$$

and hence

$$\gamma\,|V_t(z) - q_t| = |\gamma\,(V_t(z) - W_t) - w_t| \qquad (!!)$$

• again

$$(r_t + \delta)\left[V_t(z) - W_t\right] \;=\; v_t e^z - (1+\phi) w_t + \gamma\left|V_t(z) - q_t\right| + \mathcal{A}\left[V_t(z) - W_t\right]$$

– as long as $E_t > 0$, $w_t = \gamma(q_t - W_t)$ and hence

$$\gamma\left|V_t(z) - q_t\right| \;=\; \left|\gamma\left(V_t(z) - W_t\right) - w_t\right|$$

► therefore, on (b_t, x_t) and (x_t, ∞) respectively,

$$\left.\begin{array}{l} (r_t + \delta + \gamma)\left[V_t(z) - W_t\right] - (v_t e^z - \phi\, w_t) \\ (r_t + \delta - \gamma)\left[V_t(z) - W_t\right] - (v_t e^z - (2+\phi) w_t) \end{array}\right\} \;=\; \mathcal{A}\left[V_t(z) - W_t\right]$$

– ability to learn on the job lowers apparent fixed cost on (b_t, x_t)

– therefore assume $\phi > 0$

population dynamics

- Kolmogorov forward equation

$$\mathrm{D}_t m(t,z) \;=\; -\mu \mathrm{D}_z m(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} m(t,z) + \begin{cases} (-\gamma-\delta)m(t,z), & z \in (b_t, x_t) \\ (\ \ \gamma-\delta)m(t,z), & z \in (x_t, \infty) \end{cases}$$

$$m(t,b_t) = 0 \text{ and } \quad -\mu m(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_z m(t,z) \text{ continuous}$$

- market clearing

$$E_t + \int_{b_t}^{x_t} m(t,z)\mathrm{d}z = \int_{x_t}^{\infty} m(t,z)\mathrm{d}z$$

– state x_t of marginal teacher and E_t can adjust instantaneously

► *same* implication as before

$$\mathrm{D}N_t \;=\; \frac{\partial}{\partial t}\int_{b_t}^{\infty} m(t,z)\mathrm{d}z = \int_{b_t}^{\infty} \mathrm{D}_t m(t,z)\mathrm{d}z = -\frac{1}{2}\sigma^2 \mathrm{D}_z m(t,b_t) + \gamma E_t - \delta N_t$$

balanced growth (*same*)

- *conjecture* growth rate κ so that cross-section of $z_t - \kappa t$ time-invariant
  - ▶ notation: $z_t - \kappa t \to z$
- constant numbers of individuals in various occupations

$$L + E + (1+\phi)N = 1$$

- density of managers

$$m(t, z + \kappa t) = m(z)$$

- consumption and factor prices

$$[C_t, w_t] = [C, w]\, e^{(1-\alpha)\kappa t}, \;\; v_t = v e^{-\alpha t}$$

- value functions

$$[W_t, V_t(z + \kappa t)] = [W, V(z)]\, e^{(1-\alpha)\kappa t}$$

- interest rate $r_t = r$,

$$r = \rho + (1-\alpha)\kappa$$

level of the balanced growth path (*same*)

- Cobb-Douglas consumption sector

$$\frac{L}{N} = \frac{\alpha}{1-\alpha} \times \frac{ve^b}{w} \times \frac{Ke^{-b}}{N}$$

- stock of managerial knowledge capital

$$\frac{Ke^{-b}}{N} = \frac{1}{N}\int_b^\infty e^{z-b} m(z)\mathrm{d}z$$

- entry and exit

$$\frac{\gamma E}{N} = \delta + \frac{1}{2}\sigma^2 \times \frac{\mathrm{D}m(b)}{N}$$

- human resource constraint

$$N = \left(\frac{L}{N} + \frac{E}{N} + 1 + \phi\right)^{-1}$$

▶ just need ve^b/w and $m(b+\bullet)/N$

stationary value functions

- the value of workers and entrants is $W/w = 1/\rho$, and $(q - W)/w = 1/\gamma$

▶ the Bellman equation for managers is

$$
\begin{aligned}
&(\mu - \kappa)\mathrm{D}\left[V(z) - W\right] + \frac{1}{2}\sigma^2\mathrm{D}^2\left[V(z) - W\right] \\
&\quad = \begin{cases} (\rho + \delta + \gamma)\left[V(z) - W\right] - (ve^z - \phi\, w), & z \in (b, x) \\ (\rho + \delta - \gamma)\left[V(z) - W\right] - (ve^z - (2 + \phi)w), & z \in (x, \infty) \end{cases}
\end{aligned}
$$

at the exit threshold

$$
\begin{aligned}
0 &= V(b) - W \\
0 &= \mathrm{D}V(b)
\end{aligned}
$$

at the teaching threshold

$$
\begin{aligned}
\gamma\left(V(x_-) - W\right) &= \gamma\left(V(x_+) - W\right) = w \\
\mathrm{D}V(x_-) &= \mathrm{D}V(x_+)
\end{aligned}
$$

a familiar change of variables

- define

$$\left[e^{\widehat{z}}, e^{\widehat{b}}, e^{\widehat{x}}\right] = \frac{v}{w} \times \left[e^{z}, e^{b}, e^{x}\right]$$

- the normalized value function

$$\widehat{V}(\widehat{z}) = \left(V(\widehat{z} - \ln(v/w)) - W\right)/w$$

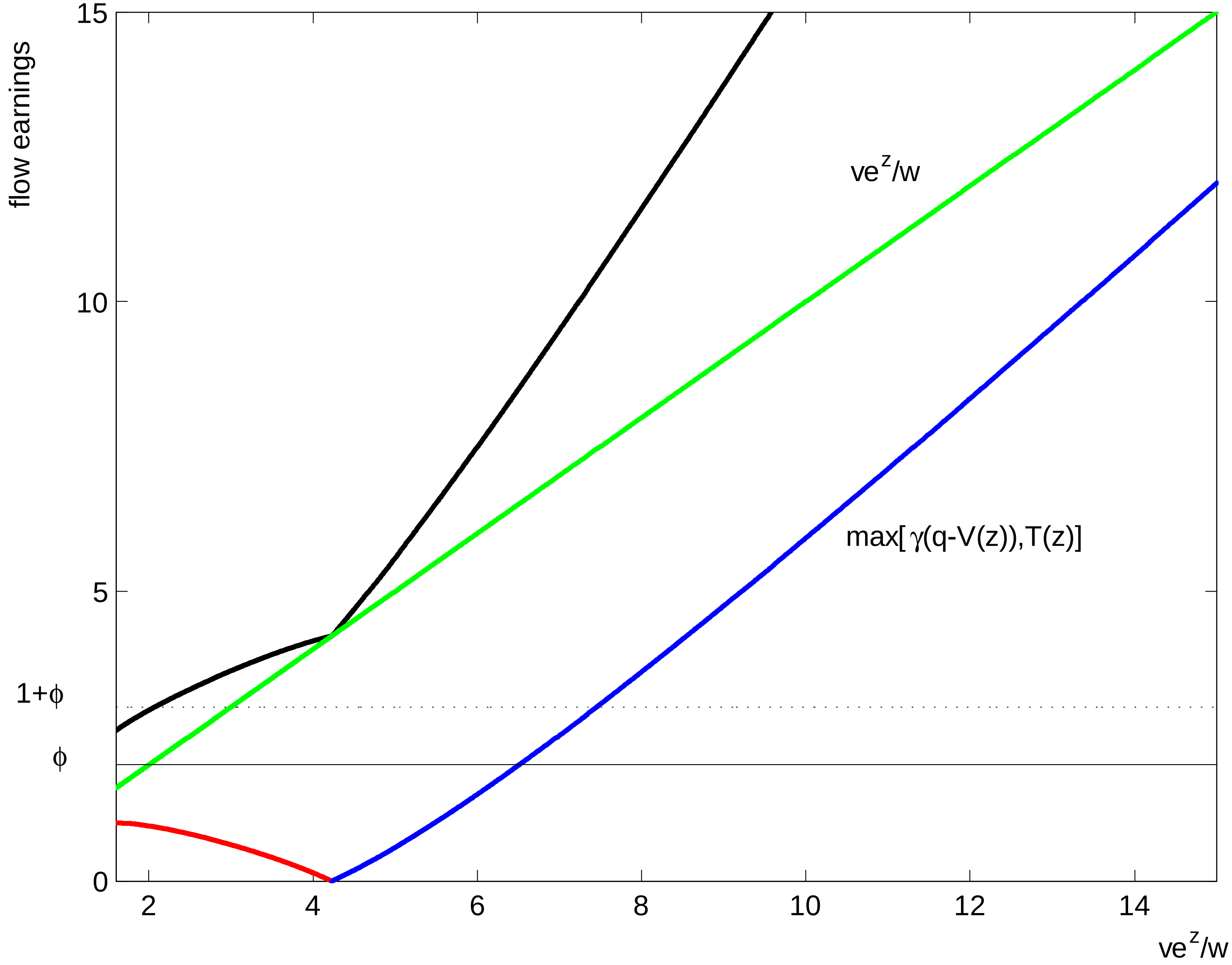
satisfies

$$(\mu - \kappa)\mathrm{D}\widehat{V}(\widehat{z}) + \frac{1}{2}\sigma^2\mathrm{D}^2\widehat{V}(\widehat{z})$$

$$= \begin{cases} (\rho + \delta + \gamma)\widehat{V}(\widehat{z}) - \left(e^{\widehat{z}} - \phi\, w\right), & \widehat{z} \in (\widehat{b}, \widehat{x}) \\ (\rho + \delta - \gamma)\widehat{V}(\widehat{z}) - \left(e^{\widehat{z}} - (2 + \phi)w\right), & \widehat{z} \in (\widehat{x}, \infty) \end{cases}$$

- key implication

$ve^{b}/w = e^{\widehat{b}}$ and $x - b = \widehat{x} - \widehat{b}$ depend *only* on conjectured κ


flow earnings
15
10
5
1+ϕ
ϕ
0
2
4
6
8
10
12
14
ve^z/w
ve^z/w
max[γ(q-V(z)),T(z)]

average versus marginal $q \ldots$

- in both economies $W = w/\rho$ and $w = \gamma(q - W)$ gives

$$\frac{W}{w} = \frac{1}{\rho}, \qquad \frac{q}{w} = \frac{1}{\rho} + \frac{1}{\gamma}$$

1. *random imitation*

$$\frac{q - W}{w} = \frac{1}{N}\int_b^\infty \left(\frac{V(z) - W}{w}\right) m(z)\mathrm{d}z = \frac{1}{N}\int_0^\infty \widehat{V}(\widehat{b} + u)m(b + u)\mathrm{d}u$$

   – and $\widehat{V}(\widehat{b} + \bullet)$ and $m(b + \bullet)$ only depend on κ

   – this condition determines κ

2. *competitive assignment*

$$\frac{q - W}{w} = \frac{V(x) - W}{w} = \widehat{V}(\widehat{x})$$

   – used already in the construction of the normalized value function

   – this condition holds identically in κ

stationary densities

- from the KFE: $m(b) = 0$ and

$$0 = -(\mu - \kappa)\mathrm{D}m(z) + \frac{1}{2}\sigma^2\mathrm{D}^2 m(z) + \begin{cases} (-\gamma - \delta)m(z), & z \in (b, x) \\ (\gamma - \delta)m(z), & z \in (x, \infty) \end{cases}$$

▶ on (b, x)

$$m(z) \propto e^{-\theta_+(z-b)} - e^{-\theta_-(z-b)}, \qquad \theta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 + \frac{\gamma + \delta}{\sigma^2/2}}$$

▶ on (x, ∞)

$$m(z) \propto A_+ e^{-\zeta_+(z-x)} + A_- e^{-\zeta_-(z-x)}, \quad \zeta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma - \delta}{\sigma^2/2}}$$

must have

$$\kappa \geq \mu + \sigma^2 \sqrt{\frac{\gamma - \delta}{\sigma^2/2}} \qquad (!)$$

growth at lower bound

- Kolmogorov-Petrovsky-Piskounov suggests: lower bound, so $\zeta_{\pm} \to \zeta$ and

$$m(z) \propto \ell(x-b, z-x)e^{-\zeta(z-x)}, \qquad z \in (x, \infty)$$

where

$$\zeta = \frac{\kappa - \mu}{\sigma^2} = \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}$$

▶ hence

$$\kappa = \mu + \sigma^2 \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}$$

  – this determines the growth rate κ

- could make endogenous by making γ depend on effort

▶ preferences do affect $m(z)$ and level of the balanced growth path

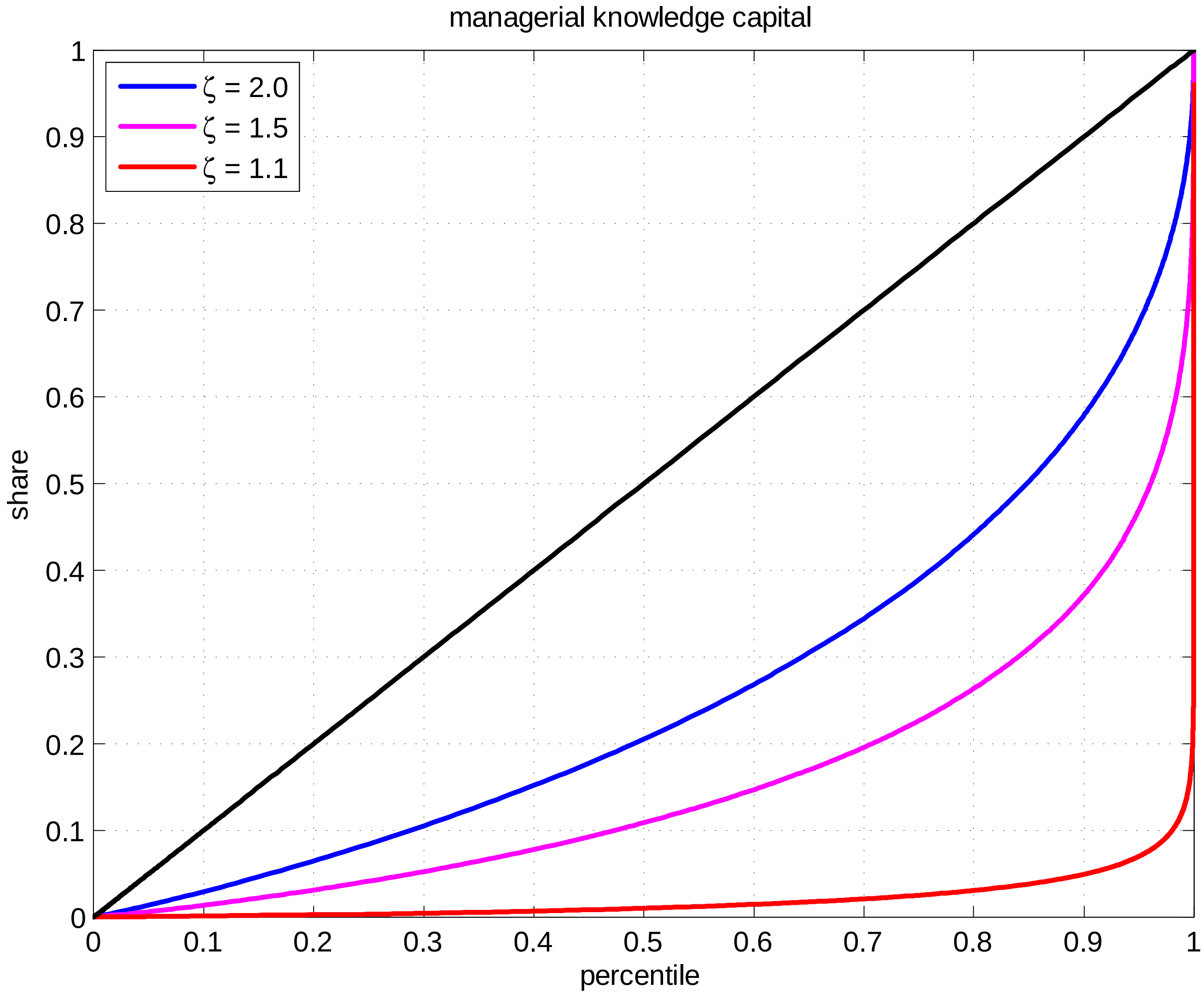
an empirical difficulty

- employment size distribution of firms: $\zeta = 1.1$

- income distribution: $\zeta = 2$ in the 1960s, $\zeta = 1.5$ now

(US data)

▶ these are very different distributions



Lorenz Curves

heterogeneous ability

- individuals can learn at rates $\lambda \in \Lambda$
  - a finite number of learning types, measure $M(\lambda)$ of type λ
  - learning ability an attribute of the dynasty
  - will specialize to $\Lambda = \{\beta, \gamma\}$, with $\gamma > \beta > 0$

▶ notation of *w.p. 715* (Luttmer, 2014)

$$S_t(\lambda) \;=\; \lambda q_t(\lambda) = \sup_z \{\lambda V_t(z|\lambda) - T_t(z)\}$$

▶ a change in assumptions

workers can learn and supply labor at the same time

- this assumption will be replaced by costly worker learning at a later date

Bellman equations

- workers sort

$$r_t W_t(\lambda) \; = \; w_t + \max\left\{0, S_t(\lambda) - \lambda V_t(z|\lambda)\right\} + \mathrm{D}W_t(\lambda)$$

- managers study or teach

$$r_t V_t(z) \; = \; v_t e^z - \phi w_t + \max\left\{T_t(z), S_t(\lambda) - \lambda V_t(z|\lambda)\right\}$$
$$+\mathcal{A}V_t(z|\lambda) + \delta\left(W_t(\lambda) - V_t(z|\lambda)\right)$$

for $z > b_t(\lambda)$, $V_t(b_t(\lambda)) = W_t(\lambda)$

– where

$$S_t(\lambda) = \sup_z \left\{\lambda V_t(z|\lambda) - T_t(z)\right\}$$

need to guess and verify

- conjecture shape of $V_t(z)$

$$V_t(b_t(\lambda)|\lambda) \;=\; W_t(\lambda) \text{ for some } b_t(\lambda) > -\infty$$

$$V_t(z|\lambda) \text{ increasing in } z \;>\; b_t(\lambda), \quad \lim_{z\to\infty} V_t(z|\lambda) = \infty$$

$$V_t(z|\lambda) \text{ increasing in } \lambda$$

▶ then equilibrium of the form

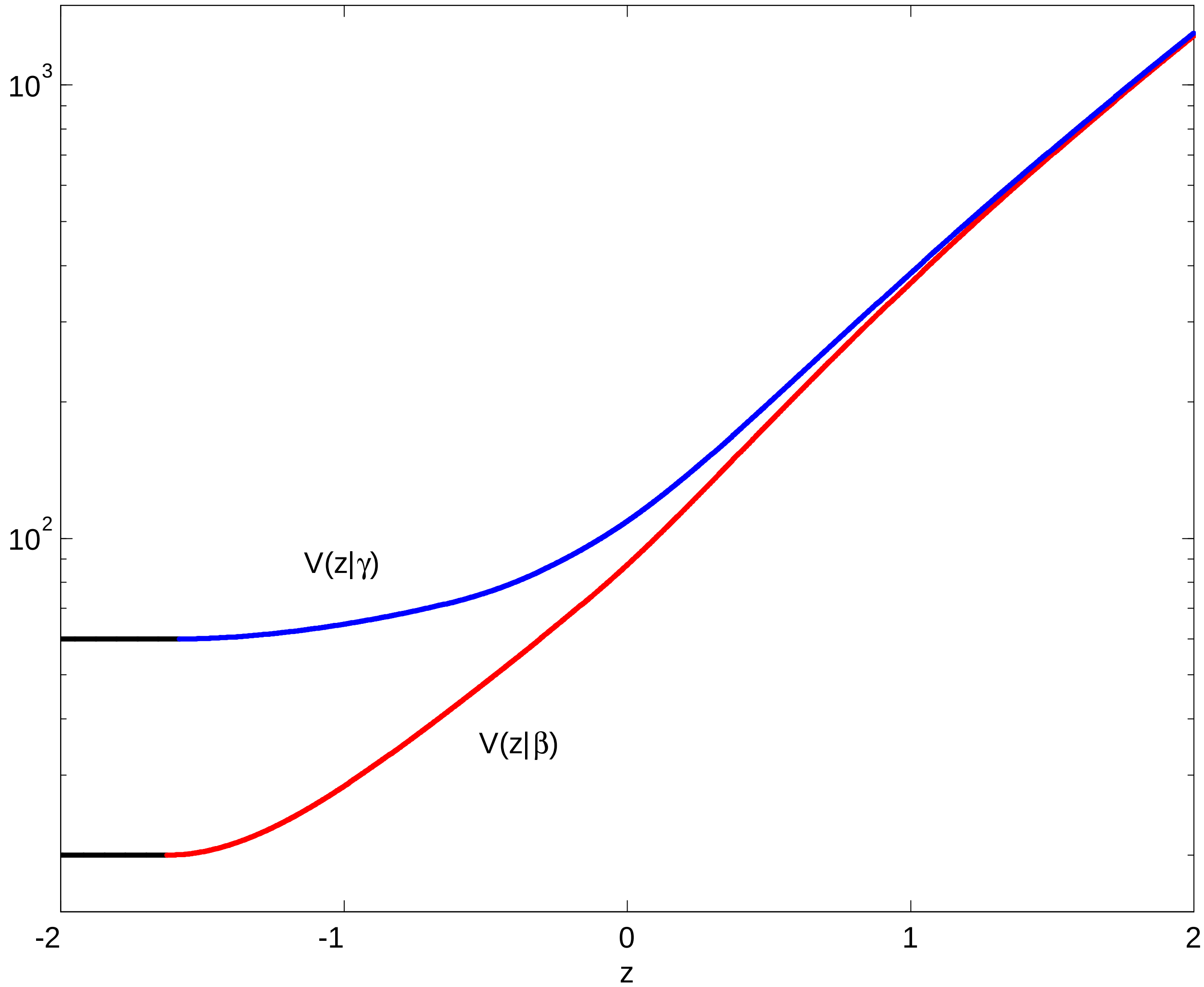
$$S_t(\lambda) \;=\; \sup_z \{\lambda V_t(z|\lambda) - T_t(z)\}$$

$$T_t(z) \;=\; \max_{\lambda\in\Lambda} \left\{[\lambda V_t(z|\lambda) - S_t(\lambda)]^+\right\}$$

- will have

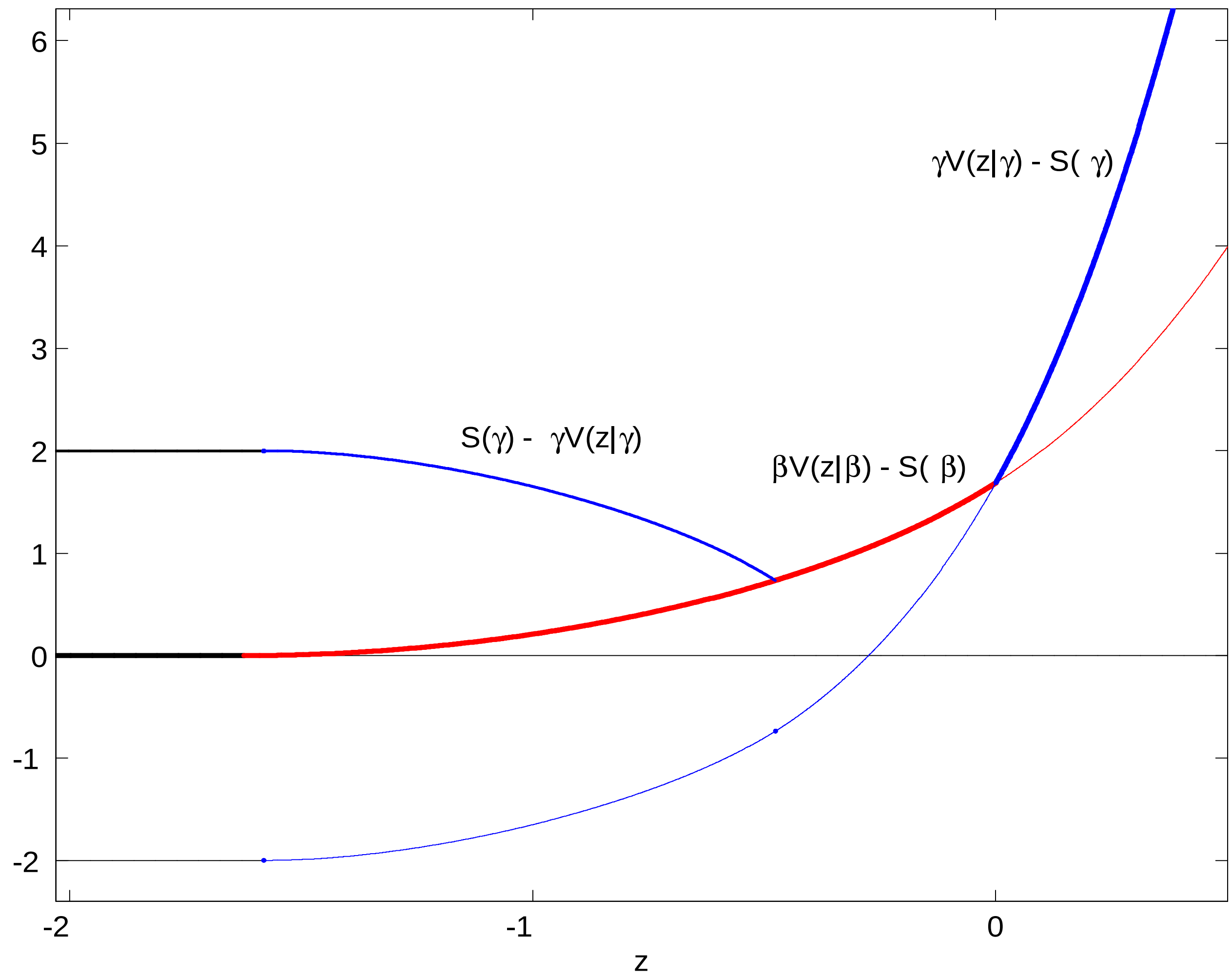
$$S_t(\lambda) - \lambda W_t(\lambda) \geq 0, \quad \lambda \in \Lambda$$

conjecture value functions



- now consider $\lambda V(z|\lambda) - S(\lambda)$

scenario: $S_t(\gamma) - \gamma W_t > S_t(\beta) - \beta W_t = 0$



- learning gains $S(\lambda) - \lambda V(z|\lambda)$ satisfy a single-crossing property

thresholds in this diagram

- exit thresholds $b(\lambda)$

$$V(b(\lambda)|\lambda) \;=\; W(\lambda), \quad \lambda \in \{\beta, \gamma\}$$

- type-γ managers switch into teaching at $x(\gamma)$ (type-β students)

$$S(\gamma) - \gamma V(x(\gamma)|\beta) \;=\; \beta V(x(\gamma)|\beta) - S(\beta),$$

- teaching managers switch into teaching type-γ students at $y > x(\gamma)$

$$\gamma V(y|\gamma) - S(\gamma) \;=\; \beta V(y|\beta) - S(\beta)$$

a familiar change of variables

- write

$$\rho V(z)/w = e^{z+\ln(v/w)} - \phi + \max\{T(z), S(\lambda) - \lambda V(z|\lambda)\}/w$$

$$+\mathcal{A}[V(z|\lambda)/w] + \delta(W(\lambda) - V(z|\lambda))/w$$

where

$$T(z) = \max_{\lambda\in\{\beta,\gamma\}} \left\{[\lambda V(z|\lambda) - S(\lambda)]^+\right\}$$

▶ normalized Bellman equation in $\widehat{z} = z + \ln(v/w)$

▶ this determines

$$\left[e^{\widehat{b}(\beta)}, e^{\widehat{b}(\gamma)}, e^{\widehat{x}(\gamma)}, e^{\widehat{y}}\right] = \frac{v}{w} \times \left[e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y}\right]$$

as a function of $[S(\beta), S(\gamma)]/w$

the key implication of the Bellman equation

- tuition schedules parameterized by $[S(\beta), S(\gamma)]/w$
- scenario of indifferent slow learners pins down

$$S(\beta) = \beta W(\beta) = \frac{\beta w}{\rho}$$

- the normalized Bellman equation determines a curve

$$\frac{S(\gamma)}{w} \mapsto \frac{v}{w} \times \left[e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y}\right]$$

  – can invert and take ve^y/w as the independent variable

▶ will use

$$ve^y/w \mapsto [y - b(\beta), y - b(\gamma), y - x(\gamma)]$$

stationary densities

- forward equations ($\theta = \mu - \kappa$)

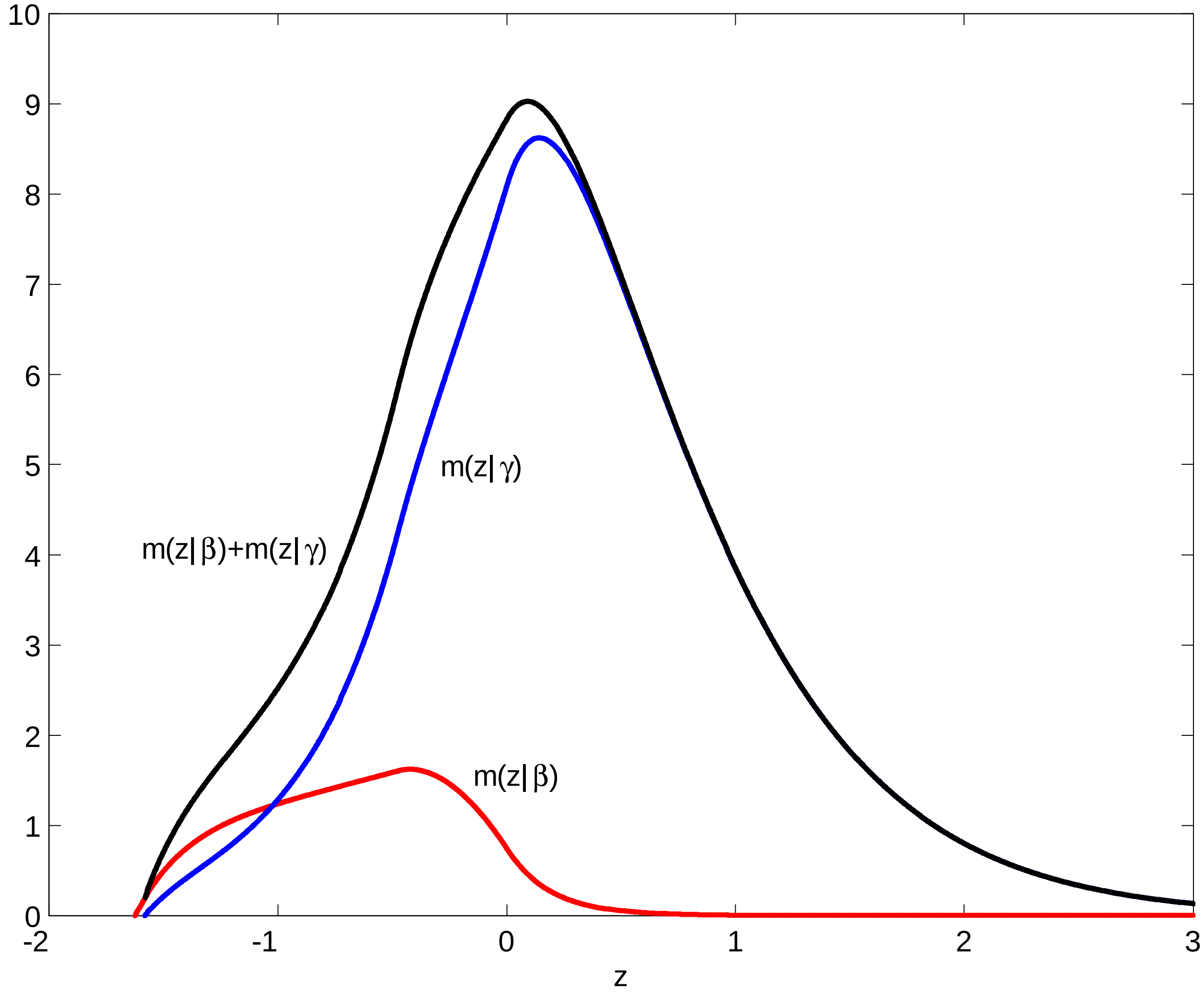
$$\delta m(z|\beta) = -\theta \mathrm{D} m(z|\beta) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(z|\beta) + \begin{cases} \beta m(z|\beta) & , \ z \in (b(\beta), x(\gamma)) \\ \beta[m(z|\beta) + m(z|\gamma)] & , \ z \in (x(\gamma), y) \\ 0 & , \ z \in (y, \infty) \end{cases}$$

and

$$\delta m(z|\gamma) = -\theta \mathrm{D} m(z|\gamma) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(z|\gamma) + \begin{cases} -\gamma m(z|\gamma) & , \ z \in (b(\gamma), x(\gamma)) \\ 0 & , \ z \in (x(\gamma), y) \\ \gamma[m(z|\beta) + m(z|\gamma)] & , \ z \in (y, \infty) \end{cases}$$

- homogeneous system of two piecewise linear ODE
  - solve for smooth $[m(z|\beta), m(z|\gamma)]$ up to scale
  - the densities $m(y + \bullet|\lambda)$ only depend on $[y - b(\beta), y - b(\gamma), y - x(\gamma)]$
- students assigned to teachers by construction

▶ but implied type distribution may not match supply



10
9
8
7
6
5
4
3
2
1
0
-2
-1
0
1
2
3
z
m(z|γ)
m(z|β)+m(z|γ)
m(z|β)

market clearing conditions

- supplies $M(\lambda)$ of type-λ individuals are given

- equating supplies of students and teachers

$$
\begin{aligned}
M(\beta) - \int_{b(\beta)}^{\infty} m(z|\beta)\mathrm{d}z \;&\geq\; \int_{b(\beta)}^{y} m(z|\beta)\mathrm{d}z + \int_{x(\gamma)}^{y} m(z|\gamma)\mathrm{d}z \\
M(\gamma) - \int_{x(\gamma)}^{\infty} m(z|\gamma)\mathrm{d}z \;&=\; \int_{y}^{\infty} [m(z|\beta) + m(z|\gamma)]\mathrm{d}z
\end{aligned}
$$

▶ not all type-β workers choose to be students

▶ the type-γ condition determines the scale of

$$m(y + \bullet|\lambda),\ \lambda \in \{\beta, \gamma\} \qquad (!)$$

- these conditions depend only on κ and $[y - b(\beta), y - b(\gamma), y - x(\gamma)]$

the fixed point

- Bellman equation, KFE, type-γ workers at corner

$$ve^y/w \;\mapsto\; [y - b(\beta), y - b(\gamma), y - x(\gamma)] \mapsto m(y + \bullet|\lambda),\; \lambda \in \{\beta, \gamma\}$$

- this pins down the number of managers

$$N = \int_{b(\beta)}^{\infty} m(z|\beta)\mathrm{d}z + \int_{b(\gamma)}^{\infty} m(z|\gamma)\mathrm{d}z$$

- implied factor supplies

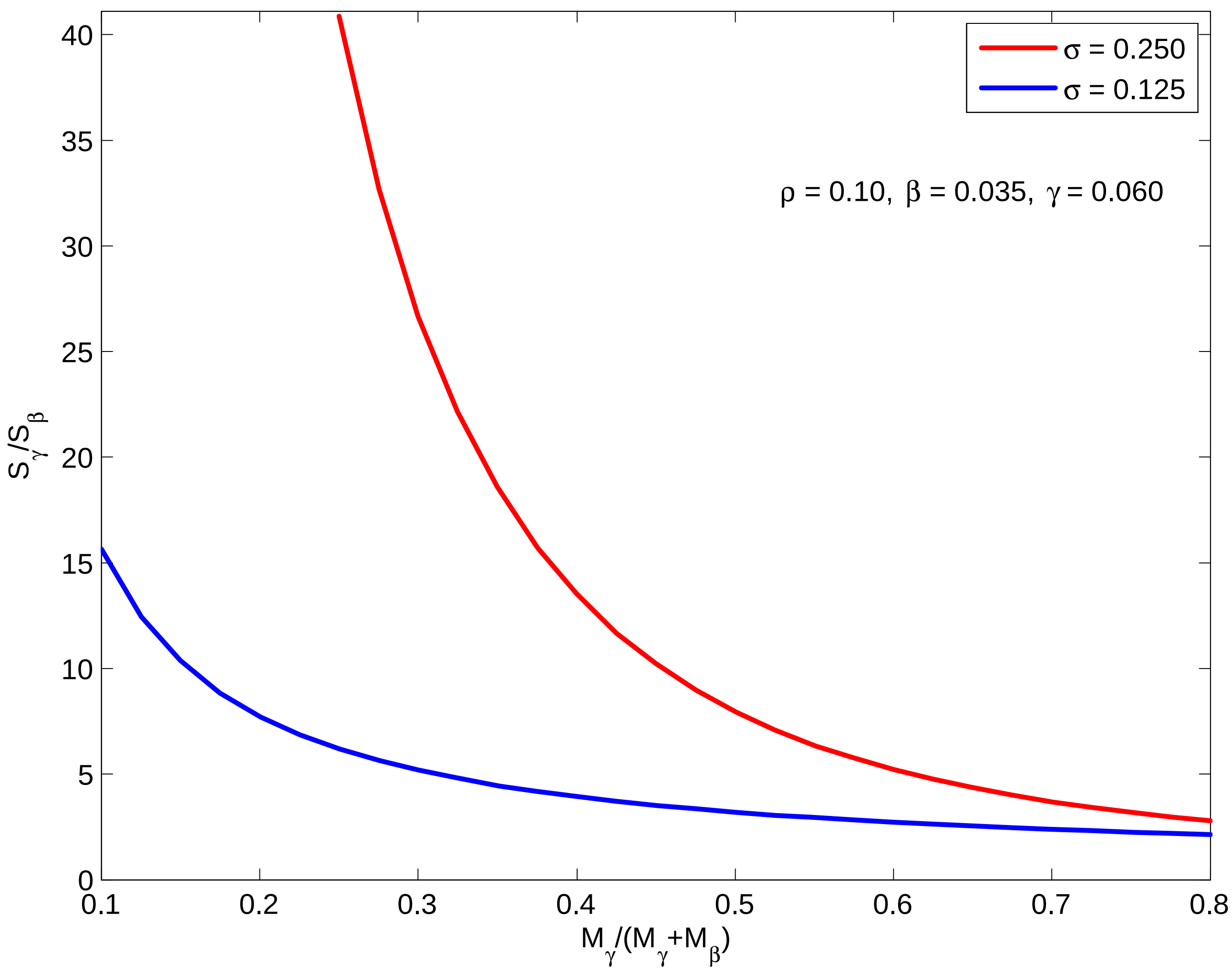
$$L \;=\; M(\beta) + M(\gamma) - (1 + \phi)N$$

$$Ke^{-y} \;=\; \int_{b(\beta)}^{\infty} e^{z-y} m(z|\beta)\mathrm{d}z + \int_{b(\gamma)}^{\infty} e^{z-y} m(z|\gamma)\mathrm{d}z$$

- Cobb-Douglas

$$\frac{ve^y}{w} = \frac{1-\alpha}{\alpha}\frac{L}{Ke^{-y}}$$

ability rents



so why κ at lower bound?

- ignore entry and exit, integrate the forward equation

$$\mathrm{D}_t p(t,z) = -\mu \mathrm{D}_z p(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} p(t,z) + \begin{cases} -\gamma p(t,z) & z < x_t \\ +\gamma p(t,z) & z > x_t \end{cases}$$

  – where x_t is the median

- the right tail $R(t,z) = 1 - P(t,z)$ satisfies

$$\mathrm{D}_t R(t,z) \;=\; -\mu \mathrm{D}_z R(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} R(t,z) + \gamma \min\left\{1 - R(t,z), R(t,z)\right\}$$

▶ a reaction-diffusion equation

▶ in the case of random imitation

$$\text{replace } \min\{1-R, R\} \text{ by } (1-R)R$$

  – parabola instead of a tent

  – no explicit solution, but can use phase diagram

initial conditions with bounded support

- can construct stationary distribution $P(z - \kappa t)$ for any

$$\kappa \geq \mu + \sigma\sqrt{2\gamma}$$

▶ Kolmogorov, Petrovsky, and Piskounov 1937

– and McKean 1975, Bramson 1981, many others

if support $P(0, z)$ *bounded then* $P(t, z - \kappa t)$ *converges for* $\kappa = \mu + \sigma\sqrt{2\gamma}$

- right tail $R(t, z + \kappa t) \sim e^{-\zeta z}$, where

$$\zeta = \frac{\kappa - \mu}{\sigma^2} - \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma}{\sigma^2/2}} = \sqrt{\frac{\gamma}{\sigma^2/2}}$$

this is a new interpretation of an old equation

$$\mathrm{D}_t f(t,z) = \frac{1}{2}\sigma^2 \mathrm{D}_{zz} f(t,z) + \gamma f(t,z)[1 - f(t,z)]$$

- R.A. Fisher "The Wave of Advance of Advantageous Genes" (1937)
  - $f(t,z)$ is a population density at the location z
  - $\gamma f(t,z)[1 - f(t,z)]$ logistic growth of the population at z
  - random migration gives rise to a "diffusion" term $\frac{1}{2}\sigma^2 \mathrm{D}_{zz} f(t,z)$
- Cavalli-Sforza and Feldman (1981)
  - *Cultural Transmission and Evolution: A Quantitative Approach*
  - Section 1.9 applies Fisher's interpretation to memes (Dawkins)
- these interpretations differ from random imitation
  - Staley (*Journal of Mathematical Economics*, 2011) also has the random imitation interpretation