Lecture Notes on Knowledge Diffusion, Growth, and Income Inequality

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these notes

are based on my

1. “Selection, Growth, and the Size Distribution of Firms”
   

2. “An Assignment Model of Knowledge Diffusion and Income Inequality”
   
   Federal Reserve Bank of Minneapolis working paper 715 (Sept 2014)

▶ see original papers for references to related literature
two models of social learning

1. individuals randomly select others at rate $\beta$ and copy if “better”

\[ D_t P(t, z) = -\beta P(t, z)[1 - P(t, z)] \]

2. “students” match 1-on-1 with “teachers” and learn at rate $\beta$

\[ D_t P(t, z) = -\beta \min \{ P(t, z), 1 - P(t, z) \} \]

▸ a parabola or a tent
the ODE for one-on-one knowledge transfer

\[ D_t P(t,x) = - \beta P(t,x) \]

\[ D_t P(t,y) = - \beta [1 - P(t,y)] \]
the solution

1. random matching delays

\[ P(t, z) = \frac{1}{1 + \left( \frac{1}{P(0, z)} - 1 \right) e^{\beta t}} \]

2. random learning delays

\[ P(t, z) = \begin{cases} 
  e^{-\beta t} P(0, z) & z \in (-\infty, x_0] \\
  \frac{1}{2} e^{\beta t} \frac{1/2}{1 - P(0, z)} & z \in [x_0, x_t] \\
  1 - e^{\beta t} [1 - P(0, z)] & z \in [x_t, \infty) 
\end{cases} \]

with a median \( x_t \) defined by

\[ \frac{1}{2} = P(t, x_t) = e^{\beta t} [1 - P(0, x_t)] \]  (!)

▶ in both cases, stationary solutions of the form

\[ P(t, z) = P(0, z - \kappa t) \quad \text{and} \quad P(t, z) = P(0, z e^{-\kappa t}) \]

for any \( \kappa \) positive
individual creativity & social learning

- two independent standard Brownian motions $B_{1,t}, B_{2,t}$,
  
  $$E \left[ \max \{ \sigma B_{1,t}, \sigma B_{2,t} \} \right] = \sigma \sqrt{t/\pi}$$

- reset to max at random time $\tau_{j+1} > \tau_j$
  
  $$z_{\tau_{j+1}} = z_{\tau_j} + \sigma \max \{ B_{1,\tau_{j+1}} - B_{1,\tau_j}, B_{2,\tau_{j+1}} - B_{2,\tau_j} \}$$

- reset times arrive randomly at rate $\beta$

\[
\frac{E \left[ z_{\tau_{j+1}} - z_{\tau_j} \big| z_{\tau_j} \right]}{E \left[ \tau_{j+1} - \tau_j \big| z_{\tau_j} \right]} = \frac{1}{1/\beta} \int_0^\infty \sigma \left( t/\pi \right)^{1/2} \beta e^{-\beta t} dt \\
= \frac{1}{2} \sigma \sqrt{\beta} \int_0^\infty 2 \left( u/\pi \right)^{1/2} e^{-u} du = \frac{1}{2} \sigma \sqrt{\beta}
\]

- large populations

$$\text{trend} = \sigma^2 \sqrt{\frac{\beta}{\sigma^2/2}} > \sigma \sqrt{\beta} = E \left[ \frac{z_{\tau_{j+1}} - z_{\tau_j}}{\tau_{j+1} - \tau_j} \big| z_{\tau_j} \right] > \frac{1}{2} \sigma \sqrt{\beta} \ldots$$
10K agents: every 2.4 days, someone imitates someone else

- $\sigma = 0.12$, $\beta = 0.015$, implies trend $= 0.0147$
the random imitation economy

• demography and preferences

\[ \int_0^\infty e^{-\rho t} \ln(C_t) dt \]

- unit measure of dynasties
- generations die randomly at the rate \( \delta \)
- replaced immediately with next generation
- complete markets, interest rate \( r_t = \rho + DC_t/C_t \)

• (Lucas, 1978) manager in state \( z \) and \( l \) workers can produce consumption,

\[ c = \left( \frac{e^z}{1 - \alpha} \right)^{1-\alpha} \left( \frac{l}{\alpha} \right)^\alpha \]

• economy-wide state at \( t \)

a measure of managers \( M(t, z) \)
the human resource constraint

\[ L_t + E_t + (1 + \phi)N_t = 1 \]

- \( L_t \): production workers, one unit of labor per worker
- \( E_t \): entrants, trying to become managers
- \( N_t \): managers, \( N_t = M(t, \infty) \), overhead of \( \phi \) workers per manager

→ transitions:
  - newborn individuals start in \( L_t + E_t + \phi N_t \)
  - back and forth between \( L_t \), \( E_t \) and \( \phi N_t \) instantaneously
  - \( N_t \to L_t + E_t + \phi N_t \) instantaneous when manager chooses
  - \( E_t \to N_t \) after random delay with mean \( 1/\gamma \)
production of consumption, as usual

- managerial profit maximization

\[
\max_l \left\{ \left( \frac{e^z}{1 - \alpha} \right)^{1-\alpha} \left( \frac{l}{\alpha} \right)^\alpha - w_tl \right\} = v_te^z
\]

yields

\[
\frac{w_tl_t(z)}{v_te^z} = \frac{\alpha}{1 - \alpha}, \quad v_t^{1-\alpha}w_t^\alpha = 1
\]

- factor prices and aggregate consumption

\[
\begin{bmatrix} w_tL_t \\ v_tK_t \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 - \alpha \end{bmatrix} C_t, \quad C_t = \left( \frac{K_t}{1 - \alpha} \right)^{1-\alpha} \left( \frac{L_t}{\alpha} \right)^\alpha
\]

given

\[
\begin{bmatrix} L_t \\ K_t \end{bmatrix} = \int \begin{bmatrix} l_t(z) \\ e^z \end{bmatrix} M(t, \text{d}z)
\]
as long as a manager continues in a job

\[ dz_t = \mu dt + \sigma dB_t \]

- idiosyncratic shock \( B_t \) is a standard Brownian motion

- add learning jumps later

- must pay flow of \( \phi \geq 0 \) units of labor to continue
  - if not, lose \( z_t \) and become a worker again
workers and entrants

- workers supply one unit of labor at wage $w_t$
- entrants sample incumbent managers at the rate $\gamma$, and imitate perfectly
- time-$t$ present value of dynastic earnings
  - when worker or entrant: $W_t$
  - when manager in state $z$: $V_t(z)$

» random imitation

$$q_t = \frac{1}{N_t} \int V_t(z) M(t, dz)$$

» because production workers are essential

$$w_t \geq \gamma(q_t - W_t) \quad \text{w.e. if } E_t > 0$$
Ito, and a piece of convenient notation

\[ dz_t = \mu dt + \sigma dB_t \]

• for a sufficiently nice \( f(t, z) \),

\[
\lim_{\Delta \downarrow 0} \frac{1}{\Delta} E \left[ f(t + \Delta, z_{t+\Delta}) - f(t, z_{t}) \middle| z_{t} = z \right] = \mathcal{A}f(t, z)
\]

► where

\[
\mathcal{A}f(t, z) = D_t f(t, z) + \mu D_z f(t, z) + \frac{1}{2} \sigma^2 D_{zz} f(t, z)
\]

– depends on \( \mu \) and \( \sigma^2 \)
Bellman equations

- workers and entrants
  \[ r_t W_t = w_t + D_t W_t \]

- managers
  \[ r_t V_t(z) = v_t e^z - \phi w_t + \mathcal{A} V_t(z) + \delta [W_t - V_t(z)] \]
  for all \( z > b_t \),
  \[ V_t(b_t) = W_t \]

- implied managerial surplus
  \[ (r_t + \delta) [V_t(z) - W_t] = v_t e^z - (1 + \phi) w_t + \mathcal{A} [V_t(z) - W_t] \]

  - effective fixed cost is \( 1 + \phi \) units of labor
  - managerial opportunity cost

- crucial transversality conditions omitted
population dynamics

- density $m(t, z)$ of $M(t, z)$
- Kolmogorov forward equation

$$D_t m(t, z) = -\mu D_z m(t, z) + \frac{1}{2} \sigma^2 D_{zz} m(t, z) + \left( \frac{\gamma E_t}{N_t} - \delta \right) m(t, z)$$

density and derivatives vanish as $z \to \infty$, and

$$m(t, b_t) = 0$$

- this implies

$$DN_t = \frac{\partial}{\partial t} \int_{b_t}^{\infty} m(t, z) dz = \int_{b_t}^{\infty} D_t m(t, z) dz = -\frac{1}{2} \sigma^2 D_z m(t, b_t) + \gamma E_t - \delta N_t$$
balanced growth

• *conjecture* growth rate $\kappa$ so that cross-section of $z_t - \kappa t$ time-invariant
  
  ➤ notation: $z_t - \kappa t \rightarrow z$

• constant numbers of individuals in various occupations

\[ L + E + (1 + \phi)N = 1 \]

• density of managers

\[ m(t, z + \kappa t) = m(z) \]

• consumption and factor prices

\[ [C_t, w_t] = [C, w] e^{(1-\alpha)\kappa t}, \quad v_t = v e^{-\alpha t} \]

• value functions

\[ [W_t, V_t(z + \kappa t)] = [W, V(z)] e^{(1-\alpha)\kappa t} \]

• interest rate $r_t = r,$

\[ r = \rho + (1 - \alpha)\kappa \]
level of the balanced growth path

- Cobb-Douglas consumption sector

\[
\frac{L}{N} = \frac{\alpha}{1 - \alpha} \times \frac{ve^b}{w} \times \frac{Ke^{-b}}{N}
\]

- stock of managerial knowledge capital

\[
\frac{Ke^{-b}}{N} = \frac{1}{N} \int_b^\infty e^{z-b}m(z)dz
\]

- entry and exit

\[
\frac{\gamma E}{N} = \delta + \frac{1}{2} \sigma^2 \times \frac{Dm(b)}{N}
\]

- human resource constraint

\[
N = \left( \frac{L}{N} + \frac{E}{N} + 1 + \phi \right)^{-1}
\]

\* just need \(ve^b/w\) and \(m(b + \bullet)/N\)
stationary value functions

- value of workers and entrants is \( W = w / \rho \)
- the Bellman equation for managers is
  \[
  (\rho + \delta)V(z) = ve^z - \phi w + (\mu - \kappa)DV(z) + \frac{1}{2}\sigma^2D^2V(z)
  \]
  with boundary conditions
  \[
  0 = V(b) - W = DV(b)
  \]

- change variables
  \[
  e^\hat{z} = \frac{1}{1 + \phi} \frac{ve^z}{w}, \quad e^b = \frac{1}{1 + \phi} \frac{ve^b}{w}
  \]

- the normalized value function
  \[
  \hat{V}(\hat{z}) = \frac{V(\hat{z} + \ln(1 + \phi) - \ln(w/v)) - W}{(1 + \phi)w}
  \]
  satisfies
  \[
  (\rho + \delta)\hat{V}(\hat{z}) = e^\hat{z} - 1 + (\mu - \kappa)D\hat{V}(\hat{z}) + \frac{1}{2}\sigma^2D^2\hat{V}(\hat{z})
  \]
the stationary value function

- \( \widehat{V}(\cdot) \) and \( \widehat{b} \) only depend on growth rate \( \kappa \), and nothing else

- solution for \( V(\cdot) \)

\[
\frac{V(z) - W}{(1 + \phi)w} = \frac{1}{\rho + \delta} \frac{\xi}{1 + \xi} \left( e^{z-b} - 1 - \frac{1 - e^{-\xi(z-b)}}{\xi} \right)
\]

for all \( z \geq b \), where

\[
\widehat{e}^b = \frac{1}{1 + \phi} \frac{\nu e^b}{w}
\]

and

\[
\widehat{e}^b = \frac{\xi}{1 + \xi} \left( 1 - \frac{\mu - \kappa + \sigma^2/2}{\rho + \delta} \right), \quad \xi = \frac{\mu - \kappa}{\sigma^2} + \sqrt{\left( \frac{\mu - \kappa}{\sigma^2} \right)^2 + \frac{\rho + \delta}{\sigma^2/2}}
\]

- key implication

\[
\frac{\nu e^b}{w} \text{ is a function only of the growth rate } \kappa
\]

- \( \partial b / \partial \kappa > 0 \), so incumbent managers quit more easily when \( \kappa \) high
stationary densities

- from the KFE

\[ 0 = -(\mu - \kappa)Dm(z) + \frac{1}{2}\sigma^2 D^2 m(z) + \left(\frac{\gamma E}{N} - \delta\right)m(z) \]

with \( m(b) = 0 \), and density and derivatives vanish as \( z \to \infty \)

solution must be

\[ m(z) \propto e^{-\zeta_+(z-b)} - e^{-\zeta_-(z-b)} \]

where

\[
\zeta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{(\gamma E/N) - \delta}{\sigma^2/2}}
\]

need \( \zeta_\pm \) real and positive,

\[
\kappa \geq \mu + \sigma^2 \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}} \quad (!)
\]
growth at lower bound

- if initial distribution has bounded support then long-run $\kappa$ at lower bound
  - this yields $\zeta_+ \to \zeta$ and
    \[
    \frac{m(z)}{N} = \zeta^2(z - b)e^{-\zeta(z-b)}
    \]
    where
    \[
    \zeta = \frac{\kappa - \mu}{\sigma^2} = \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}
    \]
  - hence
    \[
    \kappa = \mu + \sigma^2 \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}
    \]
    - yet to determine the entry rate $\gamma E/N$
    - anything that raises $\gamma E/N$ increases growth
determining the entry rate $\gamma E/N$

- workers and entrants indifferent

$$w = \gamma(q - W), \quad q - W = \frac{1}{N} \int_{b}^{\infty} (V(z) - W)m(z)dz$$

- yields

$$\frac{1}{\gamma} = \int_{b}^{\infty} \left( \frac{V(z) - W}{w} \right) \zeta^2(z - b)e^{-\zeta(z-b)}dz$$

where

$$\frac{V(z) - W}{w} = \frac{1 + \phi}{\rho + \delta} \frac{\xi}{1 + \xi} \left( e^{z-b} - 1 - \frac{1 - e^{-\xi(z-b)}}{\xi} \right)$$

and

$$\xi = -\zeta + \sqrt{\zeta^2 + \frac{\rho + \delta}{\sigma^2/2}}$$

- equilibrium condition in $\zeta$
the competitive assignment economy

- one-on-one assignment of “students” to “teachers”
  - learn to be like teacher, randomly at rate $\gamma$
  - teacher-manager in state $z$ charges flow tuition $T_t(z)$

- new definition of $q_t$
  $$\gamma q_t = \sup_{\tilde{z}} \{ \gamma V_t(\tilde{z}) - T_t(\tilde{z}) \}$$

- net gain for student-manager in state $z$
  $$\gamma (q_t - V_t(z)) = \sup_{\tilde{z}} \{ \gamma [V_t(\tilde{z}) - V_t(z)] - T_t(\tilde{z}) \}$$

- net gain for entrant same as manager at $z = b_t$
  $$\gamma (q_t - W_t) = \gamma (q_t - V_t(b_t))$$

- same equilibrium condition for entry
  $$w_t \geq \gamma (q_t - W_t), \text{ w.e. if } E_t > 0$$
equilibrium tuition

• a positive density of managers on \((b_t, \infty)\)

\[ T_t(z) \geq \gamma (V_t(z) - q_t) \]  

\(^(*)\)

– with equality if students select teachers in state \(z\)

\(\downarrow\) if \(q_t - V_t(z) < 0\) then manager in state \(z\) prefers to teach at any \(T_t(z) \geq 0\)

– market clearing: must have students; hence \(^(*)\) holds with equality

\[ T_t(z) = \gamma (V_t(z) - q_t) \]

\(\downarrow\) if \(q_t - V_t(z) > 0 = T_t(z) / \gamma\) then manager in state \(z\) prefers to study

\[ T_t(z) = \gamma [V_t(z) - q_t]^+ \]

– could raise to \(\gamma |V_t(z) - q_t|\)

• marginal teacher \(x_t > b_t\)

\[ \gamma (q_t - V_t(x_t)) = 0 < w_t = \gamma (q_t - V_t(b_t)) \]
flow gains (units of labor)

\[ \gamma (V_t(z) - q_t) \]

\[ \gamma (q_t - V_t(z)) \]
Bellman equations

- workers and entrants $r_t W_t = w_t + D W_t$

- flow gains for teacher/student managers

$$\max \{\gamma (q_t - V_t(z)), T_t(z)\} = \gamma |V_t(z) - q_t|$$

- surplus of managers

$$(r_t + \delta) [V_t(z) - W_t] = v_t e^{z} - (1 + \phi) w_t + \gamma |V_t(z) - q_t| + A [V_t(z) - W_t]$$

- exit and teaching thresholds

$$0 = V_t(b_t) - W_t, \quad q_t - W_t = V_t(x_t) - W_t$$

- as long as $E_t > 0$

$$w_t = \gamma (q_t - W_t) \quad (!)$$

and hence

$$\gamma |V_t(z) - q_t| = |\gamma (V_t(z) - W_t) - w_t| \quad (!!)$$
\begin{itemize}
  \item again

  \[(r_t + \delta) [V_t(z) - W_t] = v_t e^z - (1 + \phi)w_t + \gamma |V_t(z) - q_t| + A [V_t(z) - W_t]\]

  \[= \text{as long as } E_t > 0, w_t = \gamma (q_t - W_t) \text{ and hence}\]

  \[\gamma |V_t(z) - q_t| = |\gamma (V_t(z) - W_t) - w_t|\]

  \[\Rightarrow \text{therefore, on } (b_t, x_t) \text{ and } (x_t, \infty) \text{ respectively,}\]

  \[\left\{ (r_t + \delta + \gamma) [V_t(z) - W_t] - (v_t e^z - \phi w_t) \right\} = A [V_t(z) - W_t] \]

  \[= \left\{ (r_t + \delta - \gamma) [V_t(z) - W_t] - (v_t e^z - (2 + \phi)w_t) \right\} \]

  \[= \text{ability to learn on the job lowers apparent fixed cost on } (b_t, x_t)\]

  \[= \text{therefore assume } \phi > 0\]
\end{itemize}
population dynamics

- Kolmogorov forward equation

\[ D_t m(t, z) = -\mu D_z m(t, z) + \frac{1}{2}\sigma^2 D_{zz} m(t, z) + \begin{cases} (-\gamma - \delta) m(t, z), & z \in (b_t, x_t) \\ (\gamma - \delta) m(t, z), & z \in (x_t, \infty) \end{cases} \]

\[ m(t, b_t) = 0 \text{ and } -\mu m(t, z) + \frac{1}{2}\sigma^2 D_z m(t, z) \text{ continuous} \]

- market clearing

\[ E_t + \int_{b_t}^{x_t} m(t, z) dz = \int_{x_t}^{\infty} m(t, z) dz \]

- state \( x_t \) of marginal teacher and \( E_t \) can adjust instantaneously

- same implication as before

\[ D N_t = \frac{\partial}{\partial t} \int_{b_t}^{\infty} m(t, z) dz = \int_{b_t}^{\infty} D_t m(t, z) dz = -\frac{1}{2}\sigma^2 D_z m(t, b_t) + \gamma E_t - \delta N_t \]
balanced growth (same)

- conjecture growth rate $\kappa$ so that cross-section of $z_t - \kappa t$ time-invariant
  - notation: $z_t - \kappa t \rightarrow z$
- constant numbers of individuals in various occupations
  $$L + E + (1 + \phi)N = 1$$
- density of managers
  $$m(t, z + \kappa t) = m(z)$$
- consumption and factor prices
  $$[C_t, w_t] = [C, w] e^{(1-\alpha)\kappa t}, \quad v_t = v e^{-\alpha t}$$
- value functions
  $$[W_t, V_t(z + \kappa t)] = [W, V(z)] e^{(1-\alpha)\kappa t}$$
- interest rate $r_t = r$
  $$r = \rho + (1 - \alpha)\kappa$$
level of the balanced growth path (same)

- Cobb-Douglas consumption sector
  \[
  \frac{L}{N} = \frac{\alpha}{1 - \alpha} \times \frac{ve^b}{w} \times \frac{Ke^{-b}}{N}
  \]

- Stock of managerial knowledge capital
  \[
  \frac{Ke^{-b}}{N} = \frac{1}{N} \int_{b}^{\infty} e^{z-b}m(z)dz
  \]

- Entry and exit
  \[
  \frac{\gamma E}{N} = \delta + \frac{1}{2} \sigma^2 \times \frac{Dm(b)}{N}
  \]

- Human resource constraint
  \[
  N = \left( \frac{L}{N} + \frac{E}{N} + 1 + \phi \right)^{-1}
  \]

  \(\blacktriangleright\) Just need \(ve^b/w\) and \(m(b + \bullet)/N\)
stationary value functions

- the value of workers and entrants is $W/w = 1/\rho$, and $(q - W)/w = 1/\gamma$

- the Bellman equation for managers is

\[
(\mu - \kappa)D[V(z) - W] + \frac{1}{2}\sigma^2D^2[V(z) - W] = \begin{cases} 
(\rho + \delta + \gamma) [V(z) - W] - (ve^z - \phi w), & z \in (b, x) \\
(\rho + \delta - \gamma) [V(z) - W] - (ve^z - (2 + \phi)w), & z \in (x, \infty) 
\end{cases}
\]

at the exit threshold

\[
0 = V(b) - W \\
0 = DV(b)
\]

at the teaching threshold

\[
\gamma (V(x_-) - W) = \gamma (V(x_+) - W) = w \\
DV(x_-) = DV(x_+)
\]
a familiar change of variables

- define

\[
\begin{bmatrix} e^{\hat{z}}, e^{\hat{b}}, e^{\hat{x}} \end{bmatrix} = \frac{v}{w} \times [e^{z}, e^{b}, e^{x}]
\]

- the normalized value function

\[
\hat{V}(\hat{z}) = (V(\hat{z} - \ln(v/w)) - W) / w
\]

satisfies

\[
(\mu - \kappa)D\hat{V}(\hat{z}) + \frac{1}{2}\sigma^2D^2\hat{V}(\hat{z})
\]

\[
= \begin{cases}
(\rho + \delta + \gamma)\hat{V}(\hat{z}) - (e^{\hat{z}} - \phi w), & \hat{z} \in (\hat{b}, \hat{x}) \\
(\rho + \delta - \gamma)\hat{V}(\hat{z}) - (e^{\hat{z}} - (2 + \phi)w), & \hat{z} \in (\hat{x}, \infty)
\end{cases}
\]

- key implication

\[
v e^{b} / w = e^{\hat{b}} \text{ and } x - b = \hat{x} - \hat{b} \text{ depend only on conjectured } \kappa
\]
average versus marginal $q \ldots$

• in both economies $W = w/\rho$ and $w = \gamma(q - W)$ gives

$$\frac{W}{w} = \frac{1}{\rho}, \quad \frac{q}{w} = \frac{1}{\rho} + \frac{1}{\gamma}$$

1. random imitation

$$\frac{q - W}{w} = \frac{1}{N} \int_b^\infty \left( \frac{V(z) - W}{w} \right) m(z)dz = \frac{1}{N} \int_0^\infty \hat{V}(\hat{b} + u)m(\hat{b} + u)du$$

– and $\hat{V}(\hat{b} + \bullet)$ and $m(\hat{b} + \bullet)$ only depend on $\kappa$

– this condition determines $\kappa$

2. competitive assignment

$$\frac{q - W}{w} = \frac{V(x) - W}{w} = \hat{V}(\hat{x})$$

– used already in the construction of the normalized value function

– this condition holds identically in $\kappa$
stationary densities

• from the KFE: $m(b) = 0$ and

$$0 = -(\mu - \kappa)Dm(z) + \frac{1}{2}\sigma^2D^2m(z) + \begin{cases} (-\gamma - \delta)m(z), & z \in (b, x) \\ (\gamma - \delta)m(z), & z \in (x, \infty) \end{cases}$$

► on $(b, x)$

$$m(z) \propto e^{-\theta_+(z-b)} - e^{-\theta_-(z-b)}, \quad \theta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 + \frac{\gamma + \delta}{\sigma^2/2}}$$

► on $(x, \infty)$

$$m(z) \propto A_+e^{-\zeta_+(z-x)} + A_-e^{-\zeta_-(z-x)}, \quad \zeta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma - \delta}{\sigma^2/2}}$$

must have

$$\kappa \geq \mu + \sigma^2\sqrt{\frac{\gamma - \delta}{\sigma^2/2}}$$

(!)
growth at lower bound

- Kolmogorov-Petrovsky-Piskounov suggests: lower bound, so $\zeta_{\pm} \to \zeta$ and

$$m(z) \propto \ell(x - b, z - x) e^{-\zeta(z-x)}, \quad z \in (x, \infty)$$

where

$$\zeta = \frac{\kappa - \mu}{\sigma^2} = \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}$$

- hence

$$\kappa = \mu + \sigma^2 \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}$$

- this determines the growth rate $\kappa$

- could make endogenous by making $\gamma$ depend on effort

- preferences do affect $m(z)$ and level of the balanced growth path
an empirical difficulty

- employment size distribution of firms: $\zeta = 1.1$

- income distribution: $\zeta = 2$ in the 1960s, $\zeta = 1.5$ now

(US data)

▶ these are very different distributions
Lorenz Curves
heterogeneous ability

- individuals can learn at rates $\lambda \in \Lambda$
  - a finite number of learning types, measure $M(\lambda)$ of type $\lambda$
  - learning ability an attribute of the dynasty
  - will specialize to $\Lambda = \{\beta, \gamma\}$, with $\gamma > \beta > 0$

$\uparrow$ notation of w.p. 715 (Luttmer, 2014)

$$S_t(\lambda) = \lambda q_t(\lambda) = \sup_z \{\lambda V_t(z|\lambda) - T_t(z)\}$$

$\uparrow$ a change in assumptions

_workers can learn and supply labor at the same time_

- this assumption will be replaced by costly worker learning at a later date
Bellman equations

- workers sort

\[ r_t W_t(\lambda) = w_t + \max \{0, S_t(\lambda) - \lambda V_t(z|\lambda)\} + D W_t(\lambda) \]

- managers study or teach

\[ r_t V_t(z) = v_t e^z - \phi w_t + \max \{T_t(z), S_t(\lambda) - \lambda V_t(z|\lambda)\} \]

\[ + A V_t(z|\lambda) + \delta (W_t(\lambda) - V_t(z|\lambda)) \]

for \( z > b_t(\lambda) \), \( V_t(b_t(\lambda)) = W_t(\lambda) \)

- where

\[ S_t(\lambda) = \sup_z \{\lambda V_t(z|\lambda) - T_t(z)\} \]
need to guess and verify

- conjecture shape of $V_t(z)$

$$V_t(b_t(\lambda) | \lambda) = W_t(\lambda) \text{ for some } b_t(\lambda) > -\infty$$

$$V_t(z | \lambda) \text{ increasing in } z > b_t(\lambda), \quad \lim_{z \to \infty} V_t(z | \lambda) = \infty$$

$$V_t(z | \lambda) \text{ increasing in } \lambda$$

- then equilibrium of the form

$$S_t(\lambda) = \sup_z \left\{ \lambda V_t(z | \lambda) - T_t(z) \right\}$$

$$T_t(z) = \max_{\lambda \in \Lambda} \left\{ [\lambda V_t(z | \lambda) - S_t(\lambda)]^+ \right\}$$

- will have

$$S_t(\lambda) - \lambda W_t(\lambda) \geq 0, \quad \lambda \in \Lambda$$
now consider $\lambda V(z|\lambda) - S(\lambda)$
scenario: \( S_t(\gamma) - \gamma W_t > S_t(\beta) - \beta W_t = 0 \)

- learning gains \( S(\lambda) - \lambda V(z|\lambda) \) satisfy a single-crossing property
thresholds in this diagram

- exit thresholds \( b(\lambda) \)

\[
V(b(\lambda)|\lambda) = W(\lambda), \quad \lambda \in \{\beta, \gamma\}
\]

- type-\( \gamma \) managers switch into teaching at \( x(\gamma) \) (type-\( \beta \) students)

\[
S(\gamma) - \gamma V(x(\gamma)|\beta) = \beta V(x(\gamma)|\beta) - S(\beta),
\]

- teaching managers switch into teaching type-\( \gamma \) students at \( y > x(\gamma) \)

\[
\gamma V(y|\gamma) - S(\gamma) = \beta V(y|\beta) - S(\beta)
\]
a familiar change of variables

• write

\[ \rho V(z)/w = e^{z + \ln(v/w)} - \phi + \max \{ T(z), S(\lambda) - \lambda V(z|\lambda) \} /w \]

\[ + \mathcal{A}[V(z|\lambda)/w] + \delta (W(\lambda) - V(z|\lambda)) /w \]

where

\[ T(z) = \max_{\lambda \in \{\beta, \gamma\}} \{ [\lambda V(z|\lambda) - S(\lambda)]^+ \} \]

► normalized Bellman equation in \( \hat{z} = z + \ln(v/w) \)

► this determines

\[ \left[ e^{\hat{b}(\beta)}, e^{\hat{b}(\gamma)}, e^{\hat{x}(\gamma)}, e^{\hat{y}} \right] = \frac{v}{w} \times \left[ e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y} \right] \]

as a function of \( [S(\beta), S(\gamma)]/w \)
the key implication of the Bellman equation

- tuition schedules parameterized by \([S(\beta), S(\gamma)]/w\)
- scenario of indifferent slow learners pins down
  \[S(\beta) = \beta W(\beta) = \frac{\beta w}{\rho}\]
- the normalized Bellman equation determines a curve
  \[
  \frac{S(\gamma)}{w} \rightarrow \frac{v}{w} \times \left[ e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y} \right]
  \]
  – can invert and take \(ve^{y}/w\) as the independent variable

- will use
  \[ve^{y}/w \rightarrow [y - b(\beta), y - b(\gamma), y - x(\gamma)]\]
stationary densities

- forward equations ($\theta = \mu - \kappa$)

\[
\delta m(z|\beta) = -\theta Dm(z|\beta) + \frac{1}{2}\sigma^2 D^2 m(z|\beta) + \begin{cases} 
\beta m(z|\beta), & z \in (b(\beta), x(\gamma)) \\
\beta[m(z|\beta) + m(z|\gamma)], & z \in (x(\gamma), y) \\
0, & z \in (y, \infty)
\end{cases}
\]

and

\[
\delta m(z|\gamma) = -\theta Dm(z|\gamma) + \frac{1}{2}\sigma^2 D^2 m(z|\gamma) + \begin{cases} 
-\gamma m(z|\gamma), & z \in (b(\gamma), x(\gamma)) \\
0, & z \in (x(\gamma), y) \\
\gamma[m(z|\beta) + m(z|\gamma)], & z \in (y, \infty)
\end{cases}
\]

- homogeneous system of two piecewise linear ODE
  - solve for smooth $[m(z|\beta), m(z|\gamma)]$ up to scale
  - the densities $m(y + \bullet|\lambda)$ only depend on $[y - b(\beta), y - b(\gamma), y - x(\gamma)]$

- students assigned to teachers by construction

  - but implied type distribution may not match supply
market clearing conditions

- supplies $M(\lambda)$ of type-$\lambda$ individuals are given

- equating supplies of students and teachers

$$M(\beta) - \int_{b(\beta)}^{\infty} m(z|\beta)dz \geq \int_{b(\beta)}^{y} m(z|\beta)dz + \int_{x(\gamma)}^{y} m(z|\gamma)dz$$

$$M(\gamma) - \int_{x(\gamma)}^{\infty} m(z|\gamma)dz = \int_{y}^{\infty} [m(z|\beta) + m(z|\gamma)]dz$$

- not all type-$\beta$ workers choose to be students

- the type-$\gamma$ condition determines the scale of

$$m(y + \bullet|\lambda), \lambda \in \{\beta, \gamma\}$$

- these conditions depend only on $\kappa$ and $[y - b(\beta), y - b(\gamma), y - x(\gamma)]$
the fixed point

● Bellman equation, KFE, type-γ workers at corner

\[ \frac{ve^y}{w} \leftrightarrow [y - b(\beta), y - b(\gamma), y - x(\gamma)] \leftrightarrow m(y + \bullet|\lambda), \lambda \in \{\beta, \gamma\} \]

● this pins down the number of managers

\[ N = \int_{b(\beta)}^{\infty} m(z|\beta)dz + \int_{b(\gamma)}^{\infty} m(z|\gamma)dz \]

● implied factor supplies

\[ L = M(\beta) + M(\gamma) - (1 + \phi)N \]

\[ Ke^{-y} = \int_{b(\beta)}^{\infty} e^{z-y}m(z|\beta)dz + \int_{b(\gamma)}^{\infty} e^{z-y}m(z|\gamma)dz \]

● Cobb-Douglas

\[ \frac{ve^y}{w} = \frac{1 - \alpha}{\alpha} \frac{L}{Ke^{-y}} \]
ability rents

\[ \frac{S_\gamma}{S_\beta} \]

\[ \frac{M_\gamma}{M_\gamma + M_\beta} \]

\[ \sigma = 0.250 \]

\[ \sigma = 0.125 \]

\[ \rho = 0.10, \beta = 0.035, \gamma = 0.060 \]
so why $\kappa$ at lower bound?

- ignore entry and exit, integrate the forward equation

$$D_t p(t, z) = -\mu D_z p(t, z) + \frac{1}{2} \sigma^2 D_{zz} p(t, z) + \begin{cases} -\gamma p(t, z) & z < x_t \\ +\gamma p(t, z) & z > x_t \end{cases}$$

- where $x_t$ is the median

- the right tail $R(t, z) = 1 - P(t, z)$ satisfies

$$D_t R(t, z) = -\mu D_z R(t, z) + \frac{1}{2} \sigma^2 D_{zz} R(t, z) + \gamma \min \{1 - R(t, z), R(t, z)\}$$

- a reaction-diffusion equation

- in the case of random imitation

  replace $\min\{1 - R, R\}$ by $(1 - R)R$

  - parabola instead of a tent
  - no explicit solution, but can use phase diagram
initial conditions with bounded support

- can construct stationary distribution $P(z - \kappa t)$ for any
  $\kappa \geq \mu + \sigma \sqrt{2\gamma}$

- Kolmogorov, Petrovsky, and Piskounov 1937
  – and McKean 1975, Bramson 1981, many others

if support $P(0, z)$ bounded then $P(t, z - \kappa t)$ converges for $\kappa = \mu + \sigma \sqrt{2\gamma}$

- right tail $R(t, z + \kappa t) \sim e^{-\zeta z}$, where

$$
\zeta = \frac{\kappa - \mu}{\sigma^2} - \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma}{\sigma^2/2}} = \sqrt{\frac{\gamma}{\sigma^2/2}}
$$
this is a new interpretation of an old equation

\[ D_t f(t, z) = \frac{1}{2} \sigma^2 D_{zz} f(t, z) + \gamma f(t, z)[1 - f(t, z)] \]

• R.A. Fisher “The Wave of Advance of Advantageous Genes” (1937)
  - \( f(t, z) \) is a population density at the location \( z \)
  - \( \gamma f(t, z)[1 - f(t, z)] \) logistic growth of the population at \( z \)
  - random migration gives rise to a “diffusion” term \( \frac{1}{2} \sigma^2 D_{zz} f(t, z) \)

• Cavalli-Sforza and Feldman (1981)
  - *Cultural Transmission and Evolution: A Quantitative Approach*
  - Section 1.9 applies Fisher’s interpretation to memes (Dawkins)

• these interpretations differ from random imitation
  - Staley (*Journal of Mathematical Economics*, 2011) also has the random imitation interpretation

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