# THE EXISTENCE OF SUBGAME-PERFECT EQUILIBRIUM IN CONTINUOUS GAMES WITH ALMOST PERFECT INFORMATION: A COMMENT

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Harris (1985) has shown that subgame-perfect equilibria exist in deterministic continuous games with perfect information.[1] A recent influential paper by Harris, Reny and Robson (1995) shows that public randomization ensures the existence of subgame-perfect equilibria in continuous games with almost perfect information. The authors exhibit an example of a game with almost perfect information in which no subgame-perfect equilibrium exists without public randomization. In addition, their Proposition 36 argues that public randomization is not required in games with perfect information. Contrary to this proposition, we give here an example of a continuous game with perfect information in which no subgame-perfect equilibrium exists if public randomization is not allowed. The result of Harris (1985) implies that our example must be a game in which Nature is an active player. Intrapersonal games for consumers with changing preferences are usually games of this type, as in, for example, stochastic versions of Peleg and Yaari (1973) and Goldman (1980).

The example has five stages. In stage 1, player 1 chooses $a_1 \in [0,1]$. In stage 2, player 2 chooses $a_2 \in [0,1]$. In stage 3, Nature chooses $x$ by randomizing uniformly over the interval $[-2+a_1+a_2, 2-a_1-a_2]$. After this, players 3 and 4 move sequentially. The subgame following a history $(a_1, a_2, x)$ and the associated payoffs for all four players are shown in Figure 1. In the following, let $\alpha$ and $\beta$ denote the probabilities with which players 3 and 4 choose $U$ and $u$, respectively.

—Insert Figure 1 Here—

Consider first the subgame defined by $a_1 = a_2 = 1$. Nature's move is degenerate in this subgame: $x = 0$. The set of subgame-perfect equilibrium paths in the resulting subgame is characterized by three segments of mixing probabilities: $\alpha = 0$ and $\beta \in [0, .5]$; $\alpha \in [0, 1]$ and $\beta = .5$; and $\alpha = 1$ and $\beta \in [.5, 1]$. The set of equilibrium expected payoffs for players 1 and 2 implied by these three segments is given by:

$$(1) \qquad \{(1, .5\alpha + 2(1 - \alpha)) \mid \alpha \in [0, 1]\} \cup \{(2\beta, \beta) \mid \beta \in [.5, 1]\}.$$

Note that this is not convex.

The implied set of subgame-perfect equilibrium payoffs for players 3 and 4 is given by the union of $\{(1, 1 - \alpha) \mid \alpha \in [0, 1]\}$ and $\{(2\beta, 0) \mid \beta \in [.5, 1]\}$. This is also not convex. Contrary to the second part of Proposition 36 of Harris, Reny and Robson (1995), this shows that the set of subgame-perfect equilibrium payoffs in a game with perfect information need not be convex, even if mixed strategies are allowed. Public randomization would restore convexity.

In any subgame in which $a_1 + a_2 < 2$, Nature's move $x$ is uniformly distributed on a non-degenerate interval that is symmetric around zero. If $x < 0$, then the continuation path is $(U, u)$. If $x > 0$, then the continuation path is $D$. The expected continuation payoffs for players 1 and 2 are therefore $1.5a_1$ and $1.5a_2$, respectively.

Note that by choosing $a_1$ and $a_2$ strictly smaller than 1, players 1 and 2 can both guarantee an expected payoff arbitrarily close to 1.5, irrespective of the actions of the other player. In contrast, any selection from (1) yields a payoff of no more than 1 for at least one of the players 1 and 2. This means that for any continuation, there is always one of the two players 1 and 2 who wants to prevent a selection from (1) to occur by choosing an action slightly less than 1. More precisely, player 2 has a best response ($a_2 = 1$) in the subgame defined by $a_1 = 1$ if and only if he or she receives at least 1.5.

This requires $\alpha \in [0, 1/3]$ and $\beta = .5$ or $\alpha = 0$ and $\beta \leq .5$, and the resulting payoff for player 1 is 1. Since this is less than 1.5, player 1 will want to choose $a_1 < 1$, but arbitrarily close to 1. Thus player 1 has no best response.

Technically, the problem is the fact that the set $C_4(a_1, a_2, x)$ of equilibrium probabilities over continuation paths is not convex for every subgame $(a_1, a_2, x)$. Convexity fails for $(a_1, a_2, x) = (1, 1, 0)$. The set $C_3(a_1, a_2)$ of equilibrium probabilities over continuation paths following $(a_1, a_2)$ is constructed from the marginal of Nature's move and conditionals in $C_4(a_1, a_2, x)$. Although $C_4(a_1, a_2, x)$ is upper hemicontinuous, the non-convexity of $C_4(a_1, a_2, x)$ causes $C_3(a_1, a_2)$ to fail to be upper hemicontinuous. In turn, this generates an openness problem in the game played by players 1 and 2.

Intuitively, the crucial feature of this example is that Nature acts as a public randomization device, but only as long as $a_1 + a_2 < 2$. However, this strict inequality is not compatible with players 1 and 2's incentives, and the endogenous public signal vanishes in the limit. This is analogous to Harris' (1990) original counterexample to the existence of subgame-perfect equilibrium in continuous games with simultaneous moves (see Fudenberg and Tirole (1991, Exercise 13.4)).

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## FOOTNOTES

[1] Hellwig and Leininger (1987) show the measurability of equilibrium strategies, and Hellwig et al. (1990) provide an elementary proof of existence using finite approximations.

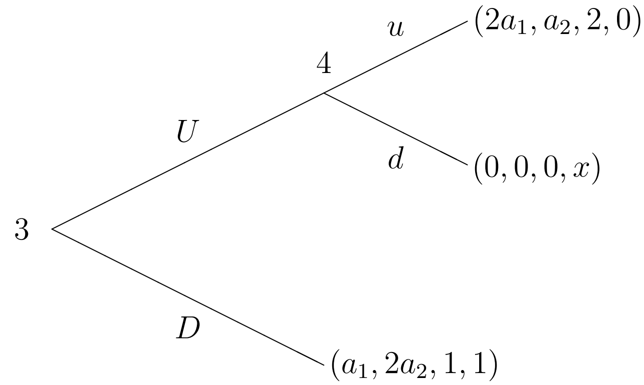


FIGURE 1