Appendix II: An Assignment Model of Knowledge Diffusion and Income Inequality

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Contents

1. Introduction

This appendix proves Proposition 1 of Luttmer [2015]. For completeness, it begins, in Section 2, with an economy with only one ability type. The resulting Kolmogorov forward equation is a linear second-order differential equation that is easy to solve.

The rest of this appendix is about the main scenario treated in the paper: an economy with two types of individuals, indexed by their ability $\lambda \in \{\beta, \gamma\}$ to learn from others, in which the slow learners are indifferent students. Section 3 describes the Kolmogorov forward equations for this case. These are not linear but piecewise linear second-order differential equations. A smooth solution consists of linear combinations of exponential functions that differ across subsets of the domain. The equations that determine the 8 coefficients of these linear combinations are linear and can be solved explicitly. The solution involves a 2×2 matrix that must be shown to be invertible, and the resulting densities $m(\beta, z)$ and $m(\gamma, z)$ have to be positive on the domains where managers choose not to exit.

The explicit solution given in Section 3 is relatively easy to obtain. Showing that the solution is well defined and positive is more challenging. The proof given here amounts to verifying that a determinant is non-zero, and that the functions obtained as candidate stationary densities are positive. Specifically, Section 4 shows the required invertibility and proves that the resulting densities are positive on the part of the domain where type-γ managers choose to be students. Section 5 shows the densities are positive on the part of the domain where type-γ managers choose to be teachers and everyone teaches type-β students. Section 6 shows the densities are positive in the right tail, where everyone teaches type-γ students.

The bottom line is that long-run growth and stationary distributions emerge with only two simple assumptions: the productivities of individual managers are subject to random shocks, and fast learners are expected to live longer than it takes them to learn from others.

2. One Learning Ability Type

Write $f(z) \propto m(\gamma, z)$ and take $f(z)$ to be a probability density. Write b for the exit threshold $b(\gamma)$. The Kolmogorov forward equation for $f(z)$ is

$$0 = -(\mu - \kappa)\mathrm{D}f(z) + \frac{1}{2}\sigma^2\mathrm{D}^2 f(z) + (\gamma - \delta)f(z), \quad z \in (b, \infty),$$

with the boundary condition $f(b) = 0$. This is a homogeneous equation with solutions of the type $e^{-\zeta(z-b)}$, where ζ must solve the characteristic equation

$$0 = -(\kappa - \mu)\zeta + \frac{1}{2}\sigma^2\zeta^2 + \gamma - \delta.$$

The solutions are

$$\zeta_{\pm} = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma - \delta}{\sigma^2/2}} \tag{1}$$

If $\delta \geq \gamma$ then these roots are real. But then at most one root would be strictly positive. The candidate solution would then be an exponential function, and such a function cannot satisfy the boundary condition $f(b) = 0$. Suppose therefore that $\gamma > \delta$. This opens up the possibility of complex roots. This has to be ruled out because complex roots would result in a density that oscillates around zero. The possibility $\kappa \leq \mu$ must be rule out as well because in that case, again, at most one root would be strictly positive. A minimal requirement is therefore

$$\frac{\kappa - \mu}{\sigma^2} \geq \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}. \tag{2}$$

If (2) holds strictly, then $\zeta_+ > \zeta_- > 0$. Imposing $f(b) = 0$ and requiring $f(z)$ to be a probability density then yields

$$f(z) = \zeta_+\zeta_- \times \frac{e^{-\zeta_-(z-b)} - e^{-\zeta_+(z-b)}}{\zeta_+ - \zeta_-}, \quad z \geq b. \tag{3}$$

Define

$$\zeta_\gamma = \sqrt{\frac{\gamma - \delta}{\sigma^2/2}}.$$

Letting κ approach the lower bound (2) from above gives $\zeta_\pm \to \zeta_\gamma$. The density (3) then converges to

$$f(z) = \zeta_\gamma^2(z - b)e^{-\zeta_\gamma(z-b)}, \quad z \geq b. \tag{4}$$

This follows from l'Hôpital's rule. This Gamma density also appears in Luttmer [2007]. Although any κ that satisfies (2) corresponds to a stationary distribution, given in (3), the technical appendix of Luttmer [2007] explains why one can expect (4) to arise when the initial distribution has bounded support.

3. Two Learning Ability Types with Indifferent Type-β Students

The maintained parameter assumptions are

$$0 \le \delta < \gamma, \quad 0 < \beta < \gamma, \quad 0 < \sigma, \tag{5}$$

as well as (2). There are thresholds $b(\beta)$, $b(\gamma)$, $x(\gamma)$, and y that satisfy $b(\beta) < x(\gamma)$ and $b(\gamma) < x(\gamma) < y$. It will be convenient to define

$$\Theta = x(\gamma) - b(\beta), \quad \Omega = x(\gamma) - b(\gamma), \quad \Delta = y - x(\gamma). \tag{6}$$

These parameters are positive and can vary independently. The joint density of managers of type $\lambda \in \{\beta, \gamma\}$ in state $z \in (-\infty, \infty)$ is $m(\lambda, z)$. The marginal density of managerial productivity states is

$$m(z) = m(\beta, z) + m(\gamma, z).$$

The forward equations for $m(\beta, z)$ and $m(\gamma, z)$ are

$$\begin{aligned} \delta m(\beta, z) &= -(\mu - \kappa)\mathrm{D}_z m(\beta, z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} m(\beta, z) \\ &\quad + \begin{cases} \beta m(\beta, z), & z \in (b(\beta), x(\gamma)) \\ \beta[m(\beta, z) + m(\gamma, z)], & z \in (x(\gamma), y) \\ 0, & z \in (y, \infty). \end{cases} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \delta m(\gamma, z) &= -(\mu - \kappa)\mathrm{D}_z m(\gamma, z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} m(\gamma, z) \\ &\quad + \begin{cases} -\gamma m(\gamma, z), & z \in (b(\gamma), x(\gamma)) \\ 0, & z \in (x(\gamma), y) \\ \gamma[m(\beta, z) + m(\gamma, z)], & z \in (y, \infty). \end{cases} \end{aligned} \tag{8}$$

The boundary conditions are

$$m(\beta, b(\beta)) = 0 \tag{9}$$

$$m(\gamma, b(\gamma)) = 0 \tag{10}$$

$$-(\mu - \kappa)m(\lambda, z) + \frac{1}{2}\sigma^2 \mathrm{D}_z m(\lambda, z) \text{ is continuous at } x(\gamma) \text{ and } y, \text{ for } \lambda \in \{\beta, \gamma\}. \tag{11}$$

$$\lim_{z \to \infty} m(\lambda, z) = 0, \quad \text{for } \lambda \in \{\beta, \gamma\}. \tag{12}$$

This is a homogeneous differential equation. The overall equilibrium conditions for a balanced growth path will pin down the scale. An implication of (7)-(12) is

$$\begin{aligned} \delta \int_{b(\beta)}^{\infty} m(\beta, z) \mathrm{d}z + \frac{1}{2}\sigma^2 \mathrm{D}_z m(\beta, b(\beta)) &= \beta \left(\int_{b(\beta)}^{y} m(\beta, z)\mathrm{d}z + \int_{x(\gamma)}^{y} m(\gamma, z)\mathrm{d}z \right), \\ \delta \int_{b(\gamma)}^{\infty} m(\gamma, z)\mathrm{d}z + \frac{1}{2}\sigma^2 \mathrm{D}_z m(\gamma, b(\gamma)) &= \gamma \left(\int_{y}^{\infty} [m(\beta, z) + m(\gamma, z)]\mathrm{d}z - \int_{b(\gamma)}^{x(\gamma)} m(\gamma, z)\mathrm{d}z \right). \end{aligned}$$

These are the flows of exiting managers. The scale of $m(\lambda, z)$ is determined by requiring the flow of exiting type-γ managers to match the flow of entering type-γ managers. In this scenario, the supply of type-β managers is assumed to be large enough to ensure there are enough type-β workers who can become managers to match the flow of exiting type-β managers.

3.1 A Particular Solution

Although the differential equation for $\{m(\lambda, z) : \lambda \in \{\beta, \gamma\}\}$ is homogenous, fixing $m(\gamma, z)$ makes (7) an inhomogeneous equation for $m(\beta, z)$, and vice versa for (8). The resulting inhomogeneous equations have particular solutions that are easy to guess. If $m(\gamma, z)$ satisfies (8) on $(x(\gamma), y)$, then $-m(\gamma, z)$ is a particular solution for (7) on $(x(\gamma), y)$. Similarly, if $m(\beta, z)$ satisfies (7) on (y, ∞), then $-m(\beta, z)$ is a particular solution for (8) on (y, ∞).

3.2 The Homogeneous Solutions

3.2.1 The Interval (y, ∞)

On this interval, (7) has solutions of the form $e^{-\xi_{\pm} z}$ and the homogeneous part of (8) has solutions of the form $e^{-\zeta_{\pm} z}$. The exponents $\zeta_{\pm}$ were defined in (1), and

$$\xi_{\pm} = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 + \frac{\delta}{\sigma^2/2}} \tag{13}$$

The assumption that $\delta \geq 0$ implies that $\xi_{+} > 0 \geq \xi_{-}$. The integrability condition (12) then implies that $e^{-\xi_{-} z}$ cannot be part of the solution for $m(\beta, z)$ on (y, ∞). As before, complex roots $\zeta_{\pm}$ have to be ruled out. The assumption that $\gamma > \delta$ implies that $\kappa \leq \mu$ would imply $\zeta_{-} \leq \zeta_{+} \leq 0$, and (12) would then rule out both $e^{-\zeta_{\pm} z}$ as part of the solution. This would only leave the particular solution $-m(\beta, z)$ for (8) on (y, ∞). Since $m(\beta, z)$ will be positive, this would imply a negative solution for $m(\gamma, z)$ on (y, ∞). So it has to be the case that $\kappa \geq \mu$, and then the requirement that the $\zeta_\pm$ be real implies that κ has to satisfy the lower bound (2), as before.

Note that if (2) holds with equality, then $\zeta_\pm = \zeta_\gamma$ and $\xi_+ = \zeta_\beta$, where

$$\zeta_\beta = \zeta_\gamma + \sqrt{\frac{\gamma}{\sigma^2/2}}. \tag{14}$$

This explains why the right tail of $m(\gamma, z)$ will be thicker than that of $m(\beta, z)$.

3.2.2 The Interval $(x(\gamma), y)$

The homogeneous equation for $m(\beta, z)$ on this interval is of the same form as the homogeneous equation for $m(\gamma, z)$ on (y, ∞), with γ replaced by β. The homogeneous solutions for $m(\beta, z)$ are thus $e^{-\theta_\pm z}$, where

$$\theta_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\beta - \delta}{\sigma^2/2}}. \tag{15}$$

Since $\gamma > \beta$, it follows from (2) that $\theta_\pm$ is real and that $\theta_+ > 0$. The equation for $m(\gamma, z)$ on $(x(\gamma), y)$ is homogeneous and identical to the equation for $m(\beta, z)$ on (y, ∞). The homogenous solutions for $m(\gamma, z)$ are thus $e^{-\xi_\pm z}$.

3.2.3 The Interval $(b(\beta), x(\gamma))$

The homogeneous equation for $m(\beta, z)$ on this interval is the same as it is on $(x(\gamma), y)$, and so the solutions are again $e^{-\theta_\pm z}$.

3.2.4 The Interval $(b(\gamma), x(\gamma))$

The equation for $m(\gamma, z)$ on this interval is homogeneous, with solutions of the form $e^{-\omega_\pm z}$, where

$$\omega_\pm = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 + \frac{\gamma + \delta}{\sigma^2/2}}. \tag{16}$$

These roots satisfy $\omega_+ > 0 > \omega_-$.

3.2.5 Comparing Characteristic Roots

The various characteristic roots are defined in (1), (13), (15) and (16). The parameter restrictions (2) and (5) imply that these roots are real and can be ranked as follows

$$\omega_+ > \xi_+ > \theta_+ > \zeta_+ \geq \zeta_- > \theta_- > \xi_- > \omega_-. \tag{17}$$

Furthermore,

$$\zeta_- > 0 \geq \xi_- \tag{18}$$

while θ_- can be of either sign. The two roots ζ_+ and ζ_- merge if and only if (2) holds with equality. Most of what follows relies only on (17)-(18), but at key points in the argument it will be essential to use the fact that all $\pm$ pairs add up to $(\kappa - \mu)/\sigma^2$.

3.3 The General Solution

Combining the particular and homogeneous solutions and imposing the boundary conditions (9)-(10) at the exit thresholds gives general solutions of the form

$$m(\beta, z) = \begin{cases} A(\beta)\left[e^{-\theta_-(z-b(\beta))} - e^{-\theta_+(z-b(\beta))}\right], & z \in (b(\beta), x(\gamma)) \\ B_-(\beta)e^{-\theta_-(z-x(\gamma))} + B_+(\beta)e^{-\theta_+(z-x(\gamma))} - m(\gamma, z), & z \in (x(\gamma), y) \\ C_+(\beta)e^{-\xi_+(z-y)}, & z \in (y, \infty). \end{cases} \tag{19}$$

and

$$m(\gamma, z) = \begin{cases} A(\gamma)\left[e^{-\omega_-(z-b(\gamma))} - e^{-\omega_+(z-b(\gamma))}\right], & z \in (b(\gamma), x(\gamma)) \\ B_-(\gamma)e^{-\xi_-(z-x(\gamma))} + B_+(\gamma)e^{-\xi_+(z-x(\gamma))}, & z \in (x(\gamma), y) \\ C_-(\gamma)e^{-\zeta_-(z-y)} + C_+(\gamma)e^{-\zeta_+(z-y)} - m(\beta, z), & z \in (y, \infty). \end{cases} \tag{20}$$

Note that $\theta_+ > \theta_-$ and $\omega_+ > \omega_-$ imply that $A(\beta)$ and $A(\gamma)$ will have to be positive. Clearly, $C_+(\beta)$ will have to be positive as well.

3.3.1 Continuously Differentiable Solutions

Recall that $\theta_+ > \theta_-$, $\omega_+ > \omega_-$ and $\xi_+ > \xi_-$. Suppose (2) holds strictly, so that ζ_+ and ζ_- are distinct as well. There are then 9 undetermined coefficients, $[A(\beta), B_-(\beta), B_+(\beta), C_+(\beta)]$ and $[A(\gamma), B_-(\gamma), B_+(\gamma), C_-(\gamma), C_+(\gamma)]$. The overall scale is determined by a market clearing condition. This leaves 8 coefficients. The continuous flow restrictions (11) yield 4 boundary conditions. This means there are 4 more degrees of freedom. In the following, consider densities $m(\beta, z)$ and $m(\gamma, z)$ that are in fact continuous at $x(\gamma)$ and y. This yields 4 more boundary conditions. It then follows from (11) that these densities must also continuously differentiable. The resulting boundary conditions therefore amount to requiring that $[m(\beta, z), m(\gamma, z), \mathrm{D}m(\beta, z), \mathrm{D}m(\gamma, z)]$ is continuous at $x(\gamma)$ and y.

The assumed continuity immediately implies $C_+(\beta) = m(\beta, y)$. The boundary conditions can then be summarized as

$$\begin{aligned}
m(\beta, x(\gamma)) &= A(\beta)\left[e^{-\theta_-(x(\gamma)-b(\beta))} - e^{-\theta_+(x(\gamma)-b(\beta))}\right] \\
m(\beta, x(\gamma)) + m(\gamma, x(\gamma)) &= B_-(\beta) + B_+(\beta) \\
m(\beta, y) + m(\gamma, y) &= B_-(\beta)e^{-\theta_-(y-x(\gamma))} + B_+(\beta)e^{-\theta_+(y-x(\gamma))}
\end{aligned}$$

and

$$\begin{aligned}
m(\gamma, x(\gamma)) &= A(\gamma)\left[e^{-\omega_-(x(\gamma)-b(\gamma))} - e^{-\omega_+(x(\gamma)-b(\gamma))}\right] \\
m(\gamma, x(\gamma)) &= B_-(\gamma) + B_+(\gamma) \\
m(\gamma, y) &= B_-(\gamma)e^{-\xi_-(y-x(\gamma))} + B_+(\gamma)e^{-\xi_+(y-x(\gamma))} \\
m(\beta, y) + m(\gamma, y) &= C_-(\gamma) + C_+(\gamma)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{D}m(\beta, x(\gamma)) &= -A(\beta)\left[\theta_- e^{-\theta_-(x(\gamma)-b(\beta))} - \theta_+ e^{-\theta_+(x(\gamma)-b(\beta))}\right] \\
\mathrm{D}m(\beta, x(\gamma)) + \mathrm{D}m(\gamma, x(\gamma)) &= -B_-(\beta)\theta_- - B_+(\beta)\theta_+ \\
\mathrm{D}m(\beta, y) + \mathrm{D}m(\gamma, y) &= -B_-(\beta)\theta_- e^{-\theta_-(y-x(\gamma))} - B_+(\beta)\theta_+ e^{-\theta_+(y-x(\gamma))} \\
\mathrm{D}m(\beta, y) &= -m(\beta, y)\xi_+
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{D}m(\gamma, x(\gamma)) &= -A(\gamma)\left[\omega_- e^{-\omega_-(x(\gamma)-b(\gamma))} - \omega_+ e^{-\omega_+(x(\gamma)-b(\gamma))}\right] \\
\mathrm{D}m(\gamma, x(\gamma)) &= -B_-(\gamma)\xi_- - B_+(\gamma)\xi_+ \\
\mathrm{D}m(\gamma, y) &= -B_-(\gamma)\xi_- e^{-\xi_-(y-x(\gamma))} - B_+(\gamma)\xi_+ e^{-\xi_+(y-x(\gamma))} \\
\mathrm{D}m(\beta, y) + \mathrm{D}m(\gamma, y) &= -C_-(\gamma)\zeta_- - C_+(\gamma)\zeta_+
\end{aligned}$$

This can be viewed as a linear system of 15 equations in the 16 variables

$$\begin{aligned}
&[m(\beta, x(\gamma)), m(\gamma, x(\gamma)), m(\beta, y), m(\gamma, y)]\,, \\
&[\mathrm{D}m(\beta, x(\gamma)), \mathrm{D}m(\gamma, x(\gamma)), \mathrm{D}m(\beta, y), \mathrm{D}m(\gamma, y)]\,, \\
&[A(\beta), A(\gamma), B_-(\beta), B_+(\beta), B_-(\gamma), B_+(\gamma), C_-(\gamma), C_+(\gamma)]\,.
\end{aligned}$$

The scale is determined by a market-clearing condition.

3.3.2 Block Recursivity

Note that the system of boundary conditions is block recursive in the sense that the coefficients $[C_-(\gamma), C_+(\gamma)]$ only appear in the two equations

$$\begin{bmatrix} m(y) \\ -\mathrm{D}m(y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \zeta_- & \zeta_+ \end{bmatrix} \begin{bmatrix} C_-(\gamma) \\ C_+(\gamma) \end{bmatrix}. \tag{21}$$

The other boundary conditions must determine the left-hand side, and then $[C_-(\gamma), C_+(\gamma)]$ follows from this pair of equations as long as $\zeta_+ > \zeta_-$. Thus one can reduce the problem to solving 13 equations and a scale condition for 14 coefficients.

3.4 The Right Tail

Throughout this section, consider $m(z)$, $m(\beta, z)$ and $m(\gamma, z)$ on the domain $[y, \infty)$. The z-marginal is $m(z) = C_-(\gamma)e^{-\zeta_-(z-y)} + C_+(\gamma)e^{-\zeta_+(z-y)}$. Suppose the roots ζ_+ and ζ_- are distinct. Solving (21) gives

$$m(z) = \begin{bmatrix} e^{-\zeta_-(z-y)} & e^{-\zeta_+(z-y)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \zeta_- & \zeta_+ \end{bmatrix}^{-1} \begin{bmatrix} m(y) \\ -\mathrm{D}m(y) \end{bmatrix}.$$

Calculating the inverse gives

$$\begin{aligned} \begin{bmatrix} e^{-\zeta_-(z-y)} & e^{-\zeta_+(z-y)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \zeta_- & \zeta_+ \end{bmatrix}^{-1} &= \frac{\begin{bmatrix} e^{-\zeta_-(z-y)} & e^{-\zeta_+(z-y)} \end{bmatrix}}{\zeta_+ - \zeta_-} \begin{bmatrix} \zeta_+ & -1 \\ -\zeta_- & 1 \end{bmatrix} \\ &= \begin{bmatrix} \zeta_+\zeta_- \left(\frac{\frac{e^{-\zeta_-(z-y)}}{\zeta_-} - \frac{e^{-\zeta_+(z-y)}}{\zeta_+}}{\zeta_+ - \zeta_-} \right) & -\frac{e^{-\zeta_-(z-y)} - e^{-\zeta_+(z-y)}}{\zeta_+ - \zeta_-} \end{bmatrix} \end{aligned}$$

and hence

$$\frac{m(z)}{m(y)} = \zeta_+\zeta_- \left(\frac{\frac{e^{-\zeta_-(z-y)}}{\zeta_-} - \frac{e^{-\zeta_+(z-y)}}{\zeta_+}}{\zeta_+ - \zeta_-} \right) - \frac{e^{-\zeta_-(z-y)} - e^{-\zeta_+(z-y)}}{\zeta_+ - \zeta_-} \left(-\frac{\mathrm{D}m(y)}{m(y)} \right)$$

for all $z \geq y$. The (λ, z)-density follows from $m(\beta, z) = m(\beta, y)e^{-\xi_+(z-y)}$ and $m(\gamma, z) = m(z) - m(\beta, z)$ on this domain.

3.4.1 The $(\kappa - \mu)/\sigma^2 \downarrow \zeta_\gamma$ Limit

The inequality (2) becomes an equality as $(\kappa - \mu)/\sigma^2 \downarrow \zeta_\gamma$. This implies $\zeta_\pm \to \zeta_\gamma$ and so the matrix on the right-hand side of (21) becomes singular. But l'Hôpital's rule gives

$$\begin{bmatrix} \zeta_+\zeta_- \left(\frac{\frac{e^{-\zeta_-(z-y)}}{\zeta_-} - \frac{e^{-\zeta_+(z-y)}}{\zeta_+}}{\zeta_+ - \zeta_-} \right) & -\frac{e^{-\zeta_-(z-y)} - e^{-\zeta_+(z-y)}}{\zeta_+ - \zeta_-} \end{bmatrix} \to \begin{bmatrix} 1 + \zeta_\gamma(z-y) & -(z-y) \end{bmatrix} e^{-\zeta_\gamma(z-y)}$$

as $\zeta_\pm \to \zeta_\gamma$. The limiting density is therefore

$$\frac{m(z)}{m(y)} = \left\{1 + \left(\zeta_\gamma + \frac{\mathrm{D}m(y)}{m(y)}\right)(z-y)\right\} e^{-\zeta_\gamma(z-y)},$$

for all $z \geq y$. All other coefficients are continuous functions of $(\kappa - \mu)/\sigma^2 \geq \zeta_\gamma$, and by implication so is $\mathrm{D}m(y)/m(y)$. Thus $m(z)$ converges to a well-defined limit as $\kappa \downarrow \mu + \sigma^2\zeta_\gamma$. The resulting density is a weighted sum of an exponential density and the Gamma density encountered above in (4).

3.5 The Left Tail

Block recursivity means that the boundary conditions for the left tail are determined separately. Using the notation (6), the system can be written as

$$\begin{bmatrix} m(\beta, x(\gamma)) \\ -\mathrm{D}m(\beta, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta)$$

$$\begin{bmatrix} m(\gamma, x(\gamma)) \\ -\mathrm{D}m(\gamma, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

and

$$\begin{bmatrix} m(\beta, x(\gamma)) + m(\gamma, x(\gamma)) \\ m(\beta, y) + m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-\theta_-\Delta} & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix}$$

$$-\begin{bmatrix} \mathrm{D}m(\beta, x(\gamma)) + \mathrm{D}m(\gamma, x(\gamma)) \\ \mathrm{D}m(\beta, y) + \mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-\theta_-\Delta} & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} \theta_- & 0 \\ 0 & \theta_+ \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix}$$

and

$$\begin{bmatrix} m(\gamma, x(\gamma)) \\ m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-\xi_-\Delta} & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}$$

$$-\begin{bmatrix} \mathrm{D}m(\gamma, x(\gamma)) \\ \mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-\xi_-\Delta} & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} \xi_- & 0 \\ 0 & \xi_+ \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}.$$

and

$$\mathrm{D}m(\beta, y) = -\xi_+ m(\beta, y).$$

This is now a system of 13 equations for the 14 variables

$$\begin{aligned} &[A(\beta), A(\gamma), B_-(\beta), B_+(\beta), B_-(\gamma), B_+(\gamma)], \\ &[m(\beta, x(\gamma)), m(\gamma, x(\gamma)), m(\beta, y), m(\gamma, y)], \\ &[\mathrm{D}m(\beta, x(\gamma)), \mathrm{D}m(\gamma, x(\gamma)), \mathrm{D}m(\beta, y), \mathrm{D}m(\gamma, y)]. \end{aligned}$$

From this system, it is easy to eliminate $[A(\beta), A(\gamma), B_-(\beta), B_+(\beta), B_-(\gamma), B_+(\gamma)]$. What remains then is a system of 7 equations in the 4 levels and 4 slopes of $m(\beta, z)$ and $m(\gamma, z)$ at $x(\gamma)$ and y. Again, the scale is determined elsewhere.

3.5.1 Boundary Conditions by Threshold

Sorting these equations by thresholds gives

$$\begin{bmatrix} m(\beta, x(\gamma)) \\ -\mathrm{D}m(\beta, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta)$$

$$\begin{bmatrix} m(\gamma, x(\gamma)) \\ -\mathrm{D}m(\gamma, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

and

$$\begin{bmatrix} m(\beta, x(\gamma)) + m(\gamma, x(\gamma)) \\ -\mathrm{D}m(\beta, x(\gamma)) - \mathrm{D}m(\gamma, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix}$$

$$\begin{bmatrix} m(\beta, y) + m(\gamma, y) \\ -\mathrm{D}m(\beta, y) - \mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix}$$

and

$$\begin{bmatrix} m(\gamma, x(\gamma)) \\ -\mathrm{D}m(\gamma, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}$$

$$\begin{bmatrix} m(\gamma, y) \\ -\mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}.$$

and $\mathrm{D}m(\beta, y) = -\xi_+ m(\beta, y)$.

3.5.2 A Six-Dimensional System

Eliminating the $m(\lambda, z)$ and $\mathrm{D}m(\lambda, z)$ gives

$$\begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta) + \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

$$\begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

and

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} m(\beta,y) = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}.$$

This system must be solved for $[A(\beta), A(\gamma), B_-(\beta), B_+(\beta), B_-(\gamma), B_+(\gamma)]/m(\beta, y)$.

3.5.3 The Solution

Eliminating $[B_-(\beta), B_+(\beta), B_-(\gamma), B_+(\gamma)]$ gives

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-(\Theta+\Delta)} & 0 \\ 0 & e^{-\theta_+(\Theta+\Delta)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{A(\beta)}{m(\beta,y)} + \left\{ \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} - \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \right\} \times \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{A(\gamma)}{m(\beta,y)} \tag{22}$$

This leaves two equations in the two unknowns $[A(\beta), A(\gamma)]/m(\beta, y)$.

The Two-Dimensional Problem Equation (22) can be written more explicitly as

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} = \begin{bmatrix} e^{-\theta_-(\Theta+\Delta)} - e^{-\theta_+(\Theta+\Delta)} \\ \theta_- e^{-\theta_-(\Theta+\Delta)} - \theta_+ e^{-\theta_+(\Theta+\Delta)} \end{bmatrix} \frac{A(\beta)}{m(\beta,y)} + \left\{ \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} - \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \right\} \begin{bmatrix} e^{-\omega_-\Omega} - e^{-\omega_+\Omega} \\ \omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega} \end{bmatrix} \frac{A(\gamma)}{m(\beta,y)}.$$

With an obvious change of notation, this becomes

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\theta_- e^{-\theta_-(\Theta+\Delta)} - \theta_+ e^{-\theta_+(\Theta+\Delta)}}{e^{-\theta_-(\Theta+\Delta)} - e^{-\theta_+(\Theta+\Delta)}} \end{bmatrix} n(\beta) + \\ \left\{ \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} - \right. \\ \left. \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \right\} \begin{bmatrix} 1 \\ \frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} \end{bmatrix} n(\gamma) \tag{23}$$

We need to solve (23) for $[n(\beta), n(\gamma)]$ and show that the solution is positive.

The Rest of the Solution Given a solution for $[A(\beta), A(\gamma)]$, the other coefficients are determined by

$$\begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} = \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta) \\ + \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

and

$$\begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma).$$

At the thresholds, the densities and their slopes are

$$\begin{bmatrix} m(\beta, x(\gamma)) \\ -\mathrm{D}m(\beta, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta)$$

$$\begin{bmatrix} m(\gamma, x(\gamma)) \\ -\mathrm{D}m(\gamma, x(\gamma)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma)$$

$$\begin{bmatrix} m(\gamma, y) \\ -\mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix}.$$

and $\mathrm{D}m(\beta, y) = -\xi_+ m(\beta, y)$.

4. The Density $m(\lambda, z)$ is Positive on $(b(\lambda), x(\gamma)]$

Define

$$\theta(\Delta) = \theta_- - \left(\frac{e^{(\theta_+ - \theta_-)(\Theta+\Delta)} - 1}{\theta_+ - \theta_-}\right)^{-1}, \quad \omega_* = \omega_- - \left(\frac{e^{(\xi_+ - \xi_-)\Omega} - 1}{\xi_+ - \xi_-}\right)^{-1}$$

and

$$\begin{aligned} F(\Delta) \;&=\; \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} - \\ &\quad \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix} \begin{bmatrix} e^{-\xi_-\Delta} & 0 \\ 0 & e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \end{aligned}$$

With this notation, equation (23) can be written as

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} = \begin{bmatrix} 1 \\ \theta(\Delta) \end{bmatrix} n(\beta) + F(\Delta) \begin{bmatrix} 1 \\ \omega_* \end{bmatrix} n(\gamma).$$

We need to show that the solution for $[n(\beta), n(\gamma)]$ is positive. Note that

$$\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \omega_* \end{bmatrix} = \frac{1}{b-a} \begin{bmatrix} b & -1 \\ -a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega_* \end{bmatrix} = \frac{1}{b-a} \begin{bmatrix} b - \omega_* \\ -(a - \omega_*) \end{bmatrix}$$

and thus

$$\begin{aligned} F(\Delta) \begin{bmatrix} 1 \\ \omega_* \end{bmatrix} \;&=\; \frac{1}{\theta_+ - \theta_-} \begin{bmatrix} e^{-\theta_-\Delta} & -e^{-\theta_+\Delta} \\ \theta_- e^{-\theta_-\Delta} & -\theta_+ e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} \theta_+ - \omega_* \\ \theta_- - \omega_* \end{bmatrix} - \\ &\quad \frac{1}{\xi_+ - \xi_-} \begin{bmatrix} e^{-\xi_-\Delta} & -e^{-\xi_+\Delta} \\ \xi_- e^{-\xi_-\Delta} & -\xi_+ e^{-\xi_+\Delta} \end{bmatrix} \begin{bmatrix} \xi_+ - \omega_* \\ \xi_- - \omega_* \end{bmatrix} = \begin{bmatrix} f(\Delta) - g(\Delta) \\ -[\mathrm{D}f(\Delta) - \mathrm{D}g(\Delta)] \end{bmatrix}. \end{aligned}$$

The weights $\theta_\pm - \omega_*$ and $\xi_\pm - \omega_*$ are all positive, as are $\theta_+ - \theta_-$ and $\xi_+ - \xi_-$. Equation (23) becomes

$$\begin{bmatrix} 1 \\ \xi_+ \end{bmatrix} = \begin{bmatrix} 1 & f(\Delta) - g(\Delta) \\ \theta(\Delta) & -[\mathrm{D}f(\Delta) - \mathrm{D}g(\Delta)] \end{bmatrix} \begin{bmatrix} n(\beta) \\ n(\gamma) \end{bmatrix}.$$

If the matrix on the right-hand side is non-singular, then the solution is

$$\begin{bmatrix} n(\beta) \\ n(\gamma) \end{bmatrix} = \frac{1}{\det(\Delta)} \begin{bmatrix} \mathrm{D}g(\Delta) - \mathrm{D}f(\Delta) + \xi_+ [g(\Delta) - f(\Delta)] \\ \xi_+ - \theta(\Delta) \end{bmatrix},$$

where $\det(\Delta)$ is the determinant

$$\det(\Delta) = \mathrm{D}g(\Delta) - \mathrm{D}f(\Delta) + \theta(\Delta)\left[g(\Delta) - f(\Delta)\right].$$

Observe that $n(\gamma)$ equals $1/\det(\Delta)$ times

$$\xi_+ - \theta(\Delta) = \xi_+ - \theta_- + \left(\frac{e^{(\theta_+ - \theta_-)(\Theta + \Delta)} - 1}{\theta_+ - \theta_-}\right)^{-1} > \xi_+ - \theta_- > 0.$$

We need the solution for $[n(\beta), n(\gamma)]$ to be positive, and so a minimal requirement is that $\det(\Delta)$ is positive.

4.1 A Convenient Scaling

Define $[p(\Delta), q(\Delta)] = e^{\omega_* \Delta}[f(\Delta), g(\Delta)]$, or

$$\begin{aligned} p(\Delta) &= \frac{(\theta_+ - \omega_*)e^{-(\theta_- - \omega_*)\Delta} - (\theta_- - \omega_*)e^{-(\theta_+ - \omega_*)\Delta}}{(\theta_+ - \omega_*) - (\theta_- - \omega_*)}, \\ q(\Delta) &= \frac{(\xi_+ - \omega_*)e^{-(\xi_- - \omega_*)\Delta} - (\xi_- - \omega_*)e^{-(\xi_+ - \omega_*)\Delta}}{(\xi_+ - \omega_*) - (\xi_- - \omega_*)}. \end{aligned}$$

These functions now only depend on the positive parameters $\theta_\pm - \omega_*$ and $\xi_\pm - \omega_*$. Differentiating $[f(\Delta), g(\Delta)] = [p(\Delta), q(\Delta)]e^{-\omega_* \Delta}$ with respect to Δ gives

$$\begin{bmatrix} \mathrm{D}f(\Delta) \\ \mathrm{D}g(\Delta) \end{bmatrix} = \left(-\omega_* \begin{bmatrix} p(\Delta) \\ q(\Delta) \end{bmatrix} + \begin{bmatrix} \mathrm{D}p(\Delta) \\ \mathrm{D}q(\Delta) \end{bmatrix}\right) e^{-\omega_* \Delta}.$$

The solution for $[n(\beta), n(\gamma)]$ can then be written as

$$\begin{bmatrix} n(\beta) \\ n(\gamma) \end{bmatrix} = \frac{1}{\det(\Delta)} \begin{bmatrix} \left(\mathrm{D}q(\Delta) - \mathrm{D}p(\Delta) + (\xi_+ - \omega_*)\left[q(\Delta) - p(\Delta)\right]\right) e^{-\omega_* \Delta} \\ \xi_+ - \omega_* - (\theta(\Delta) - \omega_*) \end{bmatrix}, \qquad (24)$$

with

$$\det(\Delta) = \left(\mathrm{D}q(\Delta) - \mathrm{D}p(\Delta) + (\theta(\Delta) - \omega_*)\left[q(\Delta) - p(\Delta)\right]\right) e^{-\omega_* \Delta}. \qquad (25)$$

We are going to show that $q(\Delta) - p(\Delta) > 0$ and $\det(\Delta) > 0$ for all $\Delta > 0$. As already argued, $\xi_+ - \omega_* > \theta(\Delta) - \omega_*$, and hence it will follow that $n(\beta)$ and $n(\gamma)$ are both positive.

4.2 Proof that $n(\beta)$ and $n(\gamma)$ are Positive

Because $\xi_+ + \xi_- = \theta_+ - \theta_-$ one can define $u = (\xi_+ - \theta_+)\Delta = -(\xi_- - \theta_-)\Delta$ and write

$$\begin{bmatrix} \theta_- - \omega_* \\ \xi_- - \omega_* \end{bmatrix} \Delta = \begin{bmatrix} a \\ a - u \end{bmatrix}, \quad \begin{bmatrix} \theta_+ - \omega_* \\ \xi_+ - \omega_* \end{bmatrix} \Delta = \begin{bmatrix} b \\ b + u \end{bmatrix}.$$

The functions $p(\Delta)$ and $q(\Delta)$ are then

$$p(\Delta) = \frac{be^{-a} - ae^{-b}}{b-a}, \quad q(\Delta) = \frac{(b+u)e^{-(a-u)} - (a-u)e^{-(b+u)}}{b+u-(a-u)},$$

and the parameters satisfy $u \in (0, a)$ and $b > a > 0$. The derivatives of $p(\Delta)$ and $q(\Delta)$ can be written as

$$\Delta \mathrm{D}p(\Delta) = -\frac{ba\left(e^{-a} - e^{-b}\right)}{b-a}, \quad \Delta \mathrm{D}q(\Delta) = -\frac{(b+u)(a-u)\left(e^{-(a-u)} - e^{-(b+u)}\right)}{b+u-(a-u)}.$$

Furthermore, the fact that Θ is positive implies that

$$\begin{aligned}(\theta(\Delta) - \omega_*)\,\Delta &= (\theta_- - \omega_*)\Delta - \left(\frac{e^{(\theta_+ - \theta_-)(\Theta+\Delta)} - 1}{(\theta_+ - \theta_-)\Delta}\right)^{-1} \\ &> (\theta_- - \omega_*)\Delta - \left(\frac{e^{(\theta_+ - \theta_-)\Delta} - 1}{(\theta_+ - \theta_-)\Delta}\right)^{-1} = a - \left(\frac{e^{b-a} - 1}{b-a}\right)^{-1} = \frac{ae^{-a} - be^{-b}}{e^{-a} - e^{-b}}.\end{aligned}$$

Because $q(\Delta) - p(\Delta)$ will be shown to be positive, this means that it suffices to show that $\Delta \det(\Delta)$ is positive for $\Theta = 0$.

To summarize, the requirement that $q(\Delta) - p(\Delta)$ be positive can be written as

$$\frac{(b+u)e^{-(a-u)} - (a-u)e^{-(b+u)}}{b+u-(a-u)} - \frac{be^{-a} - ae^{-b}}{b-a} > 0, \tag{26}$$

and the requirement that $\det(\Delta)$ be positive follows from

$$\begin{aligned}&\left(-\frac{(b+u)(a-u)\left(e^{-(a-u)} - e^{-(b+u)}\right)}{b+u-(a-u)}\right) - \left(-\frac{ba\left(e^{-a} - e^{-b}\right)}{b-a}\right) \\ &+ \left(\frac{ae^{-a} - be^{-b}}{e^{-a} - e^{-b}}\right)\left(\frac{(b+u)e^{-(a-u)} - (a-u)e^{-(b+u)}}{b+u-(a-u)} - \frac{be^{-a} - ae^{-b}}{b-a}\right) > 0.\end{aligned} \tag{27}$$

These conditions have to hold for $u \in (0, a)$ and $b > a > 0$.

4.2.1 The Function $q(\Delta) - p(\Delta)$ is Positive

Fix $G = a + b > 0$ and define

$$g(b) = \frac{be^{b} - (G-b)e^{G-b}}{b-(G-b)}$$

for any $b \in [0, G]$. Multiply (26) by e^{a+b} and observe that this inequality holds for any $u \in (0, a) \subset (0, b)$ if and only if $g(b+u) > g(b)$ for any $u \in (0, G-b)$ and $b \in (G/2, G)$.

That is, $g(b)$ must be strictly increasing on $(G/2, G)$. To prove this, let $\psi(x) = (e^x - 1)/x$ and note that one can write

$$g(b) = (1 + b\psi(b - (G - b)))\, e^{G-b}.$$

The derivative of $g(b)$ can therefore be written as

$$\mathrm{D}g(b) = [\psi(b - (G - b)) + 2b\mathrm{D}\psi(b - (G - b)) - b\psi(b - (G - b)) - 1]\, e^{G-b}.$$

Thus $g(b)$ is strictly increasing on $(G/2, G)$ if and only if

$$\mathrm{D}\psi(b - (G - b)) - \frac{1}{2}\psi(b - (G - b)) > \frac{1 - \psi(b - (G - b))}{2b}$$

for all $b \in (G/2, G)$. The right-hand side is non-positive because $\psi(x) \geq 1$. So we simply have to show that $\mathrm{D}\psi(x) - \psi(x)/2 > 0$ for all $x > 0$. Using $\mathrm{D}\psi(x) = (1 + (x - 1)e^x)/x^2$, this condition can written as

$$1 + \frac{1}{2}x + \left(\frac{1}{2}x - 1\right) e^x > 0.$$

The left-hand side is zero at $x = 0$. Its first derivative is $(1 + (x - 1)\, e^x)/2$ and its second derivative is $xe^x/2$. From this the result follows.

4.2.2 The Determinant is Positive

The expression for the determinant in (27) can be written as the sum of a term that depends on u,

$$-\frac{(b + u)(a - u)\left(e^{-(a-u)} - e^{-(b+u)}\right)}{b + u - (a - u)} + \frac{ae^{-a} - be^{-b}}{e^{-a} - e^{-b}} \frac{(b + u)e^{-(a-u)} - (a - u)e^{-(b+u)}}{b + u - (a - u)},$$

and a term that does not depend on u, and that simplifies as follows

$$-\left(-\frac{ba\left(e^{-a} - e^{-b}\right)}{b - a}\right) + \left(\frac{ae^{-a} - be^{-b}}{e^{-a} - e^{-b}}\right)\left(-\frac{be^{-a} - ae^{-b}}{b - a}\right) = -\frac{a - b}{e^{-a} - e^{-b}} \times e^{-(a+b)}.$$

We therefore have to show that

$$\frac{ae^{-a} - be^{-b}}{e^{-a} - e^{-b}} \frac{(b + u)e^{-(a-u)} - (a - u)e^{-(b+u)}}{b + u - (a - u)} > \frac{(b + u)(a - u)\left(e^{-(a-u)} - e^{-(b+u)}\right)}{b + u - (a - u)} + \frac{a - b}{e^{-a} - e^{-b}} \times e^{-(a+b)}$$

for all $u \in (0, a)$ and $b > a > 0$. Multiplying by $(b + u - (a - u))(e^{-a} - e^{-b})e^{a+b} > 0$ gives

$$\begin{aligned} 0 \ < \ & -(b+u)(a-u)(e^b - e^a)\left(e^{-(a-u)} - e^{-(b+u)}\right) \\ & + \left(ae^b - be^a\right)\left((b+u)e^{-(a-u)} - (a-u)e^{-(b+u)}\right) + (b-a)\left(b+u-(a-u)\right) \end{aligned}$$

for all $u \in (0, a)$ and $b > a > 0$. To reorganize this, observe that

$$\begin{aligned} \left(e^b - e^a\right)\left(e^{-(a-u)} - e^{-(b+u)}\right) &= \left(e^{b-a} - 1\right)e^u - (1 - e^{-(b-a)})e^{-u}, \\ \left(ae^b - be^a\right)\left(be^{-(a-u)} - ae^{-(b+u)}\right) &= \left(ae^{b-a} - b\right)be^u - \left(a - be^{-(b-a)}\right)ae^{-u}, \\ \left(ae^b - be^a\right)\left(ue^{-(a-u)} + ue^{-(b+u)}\right) &= \left(ae^{b-a} - b\right)ue^u + \left(a - be^{-(b-a)}\right)ue^{-u}. \end{aligned}$$

Collecting terms with the factors e^u and e^{-u} gives

$$\begin{aligned} 0 \ < \ & \left(-(a-u)(e^{b-a} - 1) + \left(ae^{b-a} - b\right)\right)(b+u)e^u \\ & + \left((b+u)(1 - e^{-(b-a)}) - \left(a - be^{-(b-a)}\right)\right)(a-u)e^{-u} + (b-a)\left(b+u-(a-u)\right), \end{aligned}$$

which is the same as

$$\begin{aligned} 0 \ < \ & \left(\left(e^{b-a} - 1\right)u - (b-a)\right)(b+u)e^u \\ & + \left(\left(1 - e^{-(b-a)}\right)u + b - a\right)(a-u)e^{-u} + (b-a)\left(b+u-(a-u)\right), \end{aligned}$$

for all $u \in (0, a)$ and $b > a > 0$. Now divide by $b - a > 0$ and rewrite to obtain the condition

$$(b+u)\left(1 + \left(\left(\frac{e^{b-a} - 1}{b-a}\right)u - 1\right)e^u\right) > (a-u)\left(1 - \left(\left(\frac{1 - e^{-(b-a)}}{b-a}\right)u + 1\right)e^{-u}\right),$$

for all $u \in (0, a)$ and $b > a > 0$. On this domain, $(a - u)/(b + u)$ is bounded above by 1. Note that $(e^x - 1)/x$ is increasing and $(1 - e^{-x})/x$ is decreasing. Furthermore, $1 + (u - 1)e^u$ is equal to 0 at $u = 0$ and has a derivative ue^u. This implies the left-hand side is positive. The function $1 - (1 + u)e^{-u}$ is positive as well, and this immediately implies that the right-hand side is positive for all $u \in (0, a)$ and $b > a > 0$. The desired inequality is therefore implied if it can be shown that

$$\frac{1 + (u-1)\,e^u}{1 - (u+1)\,e^{-u}} > 1$$

for positive u. This is the same as $(u - 1)\,e^u + (u + 1)\,e^{-u} > 0$. The left-hand side is zero at $u = 0$ and has a derivative equal to $u(e^u - e^{-u})$. This implies the desired result.

5. The Density is Positive on $[x(\gamma), y]$

Recall that

$$\begin{aligned} m(\beta, z) &= B_-(\beta) e^{-\theta_-(z-x(\gamma))} + B_+(\beta) e^{-\theta_+(z-x(\gamma))} - m(\gamma, z) \\ m(\gamma, z) &= B_-(\gamma) e^{-\xi_-(z-x(\gamma))} + B_+(\gamma) e^{-\xi_+(z-x(\gamma))} \end{aligned}$$

for all $z \in [x(\gamma), y]$. We have already shown that $A(\beta)$ and $A(\gamma)$ are positive, and therefore that $m(\beta, x(\gamma))$ and $m(\gamma, x(\gamma))$ are both positive for any positive choice of the scale parameter $m(\beta, y)$. It will be shown below that $m(\gamma, y)$ is positive as well. This will then be used to show that $m(\beta, z)$ and $m(\gamma, z)$ are also positive on the open interval $(x(\gamma), y)$.

5.1 The Coefficients $B_\pm(\lambda)$

The solutions for $[B_-(\beta), B_+(\beta)]$ and $[B_-(\gamma), B_+(\gamma)]$ are

$$\begin{aligned} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma) \\ &\quad + \begin{bmatrix} e^{-\theta_-\Theta} & 0 \\ 0 & e^{-\theta_+\Theta} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\beta) \\ \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \xi_- & \xi_+ \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} & 0 \\ 0 & e^{-\omega_+\Omega} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(\gamma) \end{aligned}$$

This can be reduced to

$$\begin{aligned} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} &= \frac{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}}{\theta_+ - \theta_-} \begin{bmatrix} \theta_+ - \frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} \\ -\theta_- + \frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} \end{bmatrix} A(\gamma) + \begin{bmatrix} e^{-\theta_-\Xi} \\ -e^{-\theta_+\Xi} \end{bmatrix} A(\beta) \\ \begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix} &= \frac{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}}{\xi_+ - \xi_-} \begin{bmatrix} \xi_+ - \frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} \\ -\xi_- + \frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} \end{bmatrix} A(\gamma) \end{aligned}$$

Note further that

$$\frac{\omega_- e^{-\omega_-\Omega} - \omega_+ e^{-\omega_+\Omega}}{e^{-\omega_-\Omega} - e^{-\omega_+\Omega}} = \omega_- - \left(\frac{e^{(\omega_+ - \omega_-)\Omega} - 1}{\omega_+ - \omega_-} \right)^{-1}$$

and thence

$$\begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} = \frac{e^{(\omega_+-\omega_-)\Omega}-1}{\theta_+-\theta_-}\begin{bmatrix} \theta_+-\omega_-+\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} \\ -(\theta_--\omega_-)-\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} \end{bmatrix} e^{-\omega_+\Omega}A(\gamma)+\begin{bmatrix} e^{-\theta_-\Theta} \\ -e^{-\theta_+\Theta} \end{bmatrix} A(\beta)$$

$$\begin{bmatrix} B_-(\gamma) \\ B_+(\gamma) \end{bmatrix} = \frac{e^{(\omega_+-\omega_-)\Omega}-1}{\xi_+-\xi_-}\begin{bmatrix} \xi_+-\omega_-+\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} \\ -(\xi_--\omega_-)-\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} \end{bmatrix} e^{-\omega_+\Omega}A(\gamma)$$

It follows that

$$B_-(\lambda) > 0 > B_+(\lambda), \quad \lambda \in \{\beta, \gamma\}.$$

A further useful observation is that

$$\begin{aligned} B_-(\beta)+B_+(\beta) &= \left(e^{(\omega_+-\omega_-)\Omega}-1\right)e^{-\omega_+\Omega}A(\gamma)+\left(e^{(\theta_+-\theta_-)\Theta}-1\right)e^{-\theta_+\Theta}A(\beta) \\ B_-(\gamma)+B_+(\gamma) &= \left(e^{(\omega_+-\omega_-)\Omega}-1\right)e^{-\omega_+\Omega}A(\gamma) \end{aligned}$$

This immediately implies that these two sums are strictly positive.

5.2 The Density $m(\gamma, z)$

Note that $e^{-\xi_-(z-x(\gamma))} > 1 > e^{-\xi_+(z-x(\gamma))}$ for all $z > x(\gamma)$. It therefore follows from $B_-(\gamma) > 0 > B_+(\gamma)$ and $B_-(\gamma)+B_+(\gamma) > 0$ that $m(\gamma, z) > 0$ for all $z \in [x(\gamma), y]$. So we have $m(\gamma, z)$ positive on $[x(\gamma), y]$, and $m(\beta, z)$ positive at the endpoints of this interval. It remains to show that $m(\beta, z)$ is positive on the interior of this interval as well.

5.3 The Density $m(\beta, z)$

Define

$$f(z) = B_-(\beta)e^{-\theta_- z}+B_+(\beta)e^{-\theta_+ z}-\left[B_-(\gamma)e^{-\xi_- z}+B_+(\gamma)e^{-\xi_+ z}\right]$$

for all $z \geq 0$. So $m(\beta, x(\gamma)+z) = f(z)$ for $z \in [0, \Delta]$. We know already that $f(0)$ and $f(\Delta)$ are positive. We are going to use information about the shape of $f(z)$ on $(0, \Delta)$ to conclude that it is positive on this interval as well.

From the solutions for $B_\pm(\lambda)$,

$$\begin{aligned}
f(z) &= \left(e^{(\theta_+-\theta_-)(\Theta+z)}-1\right)e^{-\theta_+(\Theta+z)}A(\beta)\\
&\quad+\left[(\theta_+-\omega_-)\,e^{-\theta_- z}-(\theta_--\omega_-)e^{-\theta_+ z}+\left(e^{-\theta_- z}-e^{-\theta_+ z}\right)\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1}\right]\\
&\quad\times\frac{\left(e^{(\omega_+-\omega_-)\Omega}-1\right)e^{-\omega_+\Omega}A(\gamma)}{\theta_+-\theta_-}\\
&\quad-\left[(\xi_+-\omega_-)e^{-\xi_- z}-(\xi_--\omega_-)e^{-\xi_+ z}+\left(e^{-\xi_- z}-e^{-\xi_+ z}\right)\left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1}\right]\\
&\quad\times\frac{\left(e^{(\omega_+-\omega_-)\Omega}-1\right)e^{-\omega_+\Omega}A(\gamma)}{\xi_+-\xi_-}
\end{aligned}$$

Another way to write this is

$$\begin{aligned}
f(z) &= \left(\frac{e^{(\theta_+-\theta_-)(\Theta+z)}-1}{\theta_+-\theta_-}\right)e^{-\theta_+ z}G+\left(\frac{e^{-\theta_- z}-e^{-\theta_+ z}}{\theta_+-\theta_-}-\frac{e^{-\xi_- z}-e^{-\xi_+ z}}{\xi_+-\xi_-}\right)H\\
&\quad+\left(\frac{(\theta_+-\omega_-)\,e^{-\theta_- z}-(\theta_--\omega_-)e^{-\theta_+ z}}{\theta_+-\omega_--(\theta_--\omega_-)}-\frac{(\xi_+-\omega_-)e^{-\xi_- z}-(\xi_--\omega_-)e^{-\xi_+ z}}{\xi_+-\omega_--(\xi_--\omega_-)}\right)K
\end{aligned}$$

where G, H and K are positive coefficients given by

$$G=(\theta_+-\theta_-)e^{-\theta_+\Theta}A(\beta),\quad H=(\omega_+-\omega_-)e^{-\omega_+\Omega}A(\gamma),\quad K=\left(e^{(\omega_+-\omega_-)\Omega}-1\right)e^{-\omega_+\Omega}A(\gamma).$$

It is useful to factor $f(z)$ as

$$f(z)=e^{-\theta_+ z}\left(\frac{e^{(\theta_+-\theta_-)z}-1}{\theta_+-\theta_-}\right)\left\{\left(\frac{e^{(\theta_+-\theta_-)(\Theta+z)}-1}{e^{(\theta_+-\theta_-)z}-1}\right)G-h(z)H-k(z)K\right\}\qquad(28)$$

where

$$\begin{aligned}
h(z) &= \frac{\frac{e^{-\xi_- z}-e^{-\xi_+ z}}{\xi_+-\xi_-}}{\frac{e^{-\theta_- z}-e^{-\theta_+ z}}{\theta_+-\theta_-}}-1\\
k(z) &= \frac{(\theta_+-\omega_-)\,e^{-\theta_- z}-(\theta_--\omega_-)e^{-\theta_+ z}}{e^{-\theta_- z}-e^{-\theta_+ z}}\left(\frac{\frac{(\xi_+-\omega_-)e^{-\xi_- z}-(\xi_--\omega_-)e^{-\xi_+ z}}{\xi_+-\omega_--(\xi_--\omega_-)}}{\frac{(\theta_+-\omega_-)e^{-\theta_- z}-(\theta_--\omega_-)e^{-\theta_+ z}}{\theta_+-\omega_--(\theta_--\omega_-)}}-1\right)
\end{aligned}$$

The sign of $f(z)$ is determined by the sign of the last factor in (28). To determine this sign, observe first that

$$\frac{e^{(\theta_+-\theta_-)(\Theta+z)}-1}{e^{(\theta_+-\theta_-)z}-1}=1+\frac{e^{(\theta_+-\theta_-)\Theta}-1}{1-e^{-(\theta_+-\theta_-)z}}.$$

This is positive and strictly decreasing for all positive z. Moreover, this function diverges to ∞ at $z = 0$. Note that $k(z)$ can be written as

$$k(z) = \left(\frac{e^{-(\theta_- - \omega_-)z} - e^{-(\theta_+ - \omega_-)z}}{\theta_+ - \omega_- - (\theta_- - \omega_-)}\right)^{-1} \times$$
$$\left\{\frac{(\xi_+ - \omega_-)e^{-(\xi_- - \omega_-)z} - (\xi_- - \omega_-)e^{-(\xi_+ - \omega_-)z}}{\xi_+ - \omega_- - (\xi_- - \omega_-)} - \frac{(\theta_+ - \omega_-)\, e^{-(\theta_- - \omega_-)z} - (\theta_- - \omega_-)e^{-(\theta_+ - \omega_-)z}}{\theta_+ - \omega_- - (\theta_- - \omega_-)}\right\}$$

and observe that $h(0) = k(0) = 0$, by l'Hôpital. We are going to show that $h(z)$ and $k(z)$ are strictly increasing and thus positive for all positive z. The last factor in (28) is therefore strictly positive near $z = 0$ and strictly decreasing in z for all positive z. This means this factor can cross zero at most once. Since we know that $f(\Delta) = m(\beta, y) > 0$, this guarantees that $f(z)$ is strictly positive on $(0, \Delta)$.

5.3.1 The Function $h(z)$

The key observation is that $\xi_+ = \theta_+ + u$ and $\xi_- = \theta_- - u$, where $u > 0$. Write $t = \theta_+ - \theta_- > 0$. Then

$$\frac{e^{-\xi_- z} - e^{-\xi_+ z}}{e^{-\theta_- z} - e^{-\theta_+ z}} = \frac{e^{(\xi_+ - \xi_-)z} - 1}{e^{(\theta_+ - \theta_-)z} - 1} \times e^{-(\xi_+ - \theta_+)z} = \frac{e^{(t+u)z} - e^{-uz}}{e^{tz} - 1}$$

The derivative of this function with respect to z is given by

$$\begin{aligned} \frac{\partial}{\partial z}\frac{e^{(t+u)z} - e^{-uz}}{e^{tz} - 1} &= \frac{(t+u)e^{(t+u)z} + ue^{-uz}}{e^{tz} - 1} - \frac{\left(e^{(t+u)z} - e^{-uz}\right) te^{tz}}{\left(e^{tz} - 1\right)^2} \\ &= \left(\frac{e^{(t+u)z} - e^{-(t+u)z}}{t+u} - \frac{e^{uz} - e^{-uz}}{u}\right)\frac{(t+u)ue^{tz}}{\left(e^{tz} - 1\right)^2} \end{aligned}$$

This is positive because t is positive, and the function $(e^x - e^{-x})/x$ is positive and increasing when x is positive. Its derivative is $[(x-1)e^x + (x+1)e^{-x}]/x^2$, which has a numerator that is 0 at $x = 0$, and its derivative in turn is $x(e^x - e^{-x})$.

5.3.2 The Function $k(z)$

The key fact is again that $\xi_+ - \omega_- = \theta_+ - \omega_- + u$ and $\xi_- - \omega_- = \theta_- - \omega_- - u$ for some u. Write $\beta = \theta_+ - \omega_-$ and $\alpha = \theta_- - \omega_-$, so that $\beta > \alpha > 0$ and $u \in (0, \alpha)$. Then $k(z)$ can be written as

$$k(z) = \left(\frac{e^{\beta z} - e^{\alpha z}}{\beta - \alpha}\right)^{-1}\left\{\frac{(\beta + u)e^{(\beta+u)z} - (\alpha - u)e^{(\alpha-u)z}}{\beta + u - (\alpha - u)} - \frac{\beta e^{\beta z} - \alpha e^{\alpha z}}{\beta - \alpha}\right\},$$

where $u \in (0,\alpha) \subset (0,\beta)$ and $z > 0$. Taking a derivative with respect to z gives

$$\begin{aligned}\frac{\mathrm{D}k(z)}{\beta-\alpha} &= \frac{\frac{(\beta+u)^2 e^{(\beta+u)z}-(\alpha-u)^2 e^{(\alpha-u)z}}{\beta+u-(\alpha-u)} - \frac{\beta^2 e^{\beta z}-\alpha^2 e^{\alpha z}}{\beta-\alpha}}{e^{\beta z}-e^{\alpha z}} \\ &\quad - \frac{\frac{(\beta+u) e^{(\beta+u)z}-(\alpha-u) e^{(\alpha-u)z}}{\beta+u-(\alpha-u)} - \frac{\beta e^{\beta z}-\alpha e^{\alpha z}}{\beta-\alpha}}{\left(e^{\beta z}-e^{\alpha z}\right)^2} \times \left(\beta e^{\beta z}-\alpha e^{\alpha z}\right).\end{aligned}$$

Since $\beta > \alpha$, this is positive if and only if

$$\begin{aligned}&\left(\frac{(\beta+u)^2 e^{(\beta+u)z}-(\alpha-u)^2 e^{(\alpha-u)z}}{\beta+u-(\alpha-u)} - \frac{\beta^2 e^{\beta z}-\alpha^2 e^{\alpha z}}{\beta-\alpha}\right)\left(e^{\beta z}-e^{\alpha z}\right) \\ > &\left(\frac{(\beta+u) e^{(\beta+u)z}-(\alpha-u) e^{(\alpha-u)z}}{\beta+u-(\alpha-u)} - \frac{\beta e^{\beta z}-\alpha e^{\alpha z}}{\beta-\alpha}\right)\left(\beta e^{\beta z}-\alpha e^{\alpha z}\right).\end{aligned}$$

Note that

$$\left(\frac{\beta e^{\beta z}-\alpha e^{\alpha z}}{\beta-\alpha}\right)\left(\beta e^{\beta z}-\alpha e^{\alpha z}\right) - \left(\frac{\beta^2 e^{\beta z}-\alpha^2 e^{\alpha z}}{\beta-\alpha}\right)\left(e^{\beta z}-e^{\alpha z}\right) = (\beta-\alpha)\, e^{(\alpha+\beta)z}.$$

The desired inequality is therefore equivalent to

$$\begin{aligned}0 &< (\beta+u-(\alpha-u))\,(\beta-\alpha)e^{(\alpha+\beta)z} \\ &\quad + \left((\beta+u)^2 e^{(\beta+u)z}-(\alpha-u)^2 e^{(\alpha-u)z}\right)\left(e^{\beta z}-e^{\alpha z}\right) \\ &\quad - \left((\beta+u) e^{(\beta+u)z}-(\alpha-u) e^{(\alpha-u)z}\right)\left(\beta e^{\beta z}-\alpha e^{\alpha z}\right)\end{aligned}$$

By collecting like powers of e^z and scaling by $e^{-(\alpha+\beta)z}$, this can also be written as

$$\begin{aligned}0 &< (\beta+u-(\alpha-u))\,(\beta-\alpha) + u\left((\beta+u)e^{(\beta+u-\alpha)z}-(\alpha-u)e^{-(\beta+u-\alpha)z}\right) \\ &\quad -(\beta+u)(\beta+u-\alpha)e^{uz} + (\alpha-u)(\beta+u-\alpha)e^{-uz}\end{aligned}$$

Setting $z = 0$ on the right-hand side gives

$$0 = (\beta+u-(\alpha-u))\,(\beta-\alpha) + u\,((\beta+u)-(\alpha-u)) - (\beta+u)(\beta+u-\alpha) + (\alpha-u)(\beta+u-\alpha).$$

Subtracting this from the preceding inequality and dividing by $uz > 0$ gives

$$\begin{aligned}0 &< (\beta+u)\left(\frac{e^{(\beta+u-\alpha)z}-1}{z}\right) + (\alpha-u)\left(\frac{1-e^{-(\beta+u-\alpha)z}}{z}\right) \\ &\quad -(\beta+u)(\beta+u-\alpha)\left(\frac{e^{uz}-1}{uz}\right) - (\alpha-u)(\beta+u-\alpha)\left(\frac{1-e^{-uz}}{uz}\right)\end{aligned}$$

The limit as $z \to 0$ of the right-hand side of this inequality is zero. Subtracting this $z \to 0$ limit from the above inequality and multiplying by z yields

$$\begin{aligned} 0 &< (\beta+u)\left(e^{(\beta+u-\alpha)z} - 1 - (\beta+u-\alpha)z\right) + (\alpha-u)\left(1 - e^{-(\beta+u-\alpha)z} - (\beta+u-\alpha)z\right) \\ &\quad -(\beta+u)(\beta+u-\alpha)z\left(\frac{e^{uz}-1}{uz} - 1\right) - (\alpha-u)(\beta+u-\alpha)z\left(\frac{1-e^{-uz}}{uz} - 1\right). \end{aligned}$$

This simplifies to

$$\begin{aligned} 0 &< (\beta+u)\left(e^{(\beta+u-\alpha)z} - 1 - (\beta+u-\alpha)z\left(\frac{e^{uz}-1}{uz}\right)\right) \\ &\quad +(\alpha-u)\left(1 - e^{-(\beta+u-\alpha)z} - (\beta+u-\alpha)z\left(\frac{1-e^{-uz}}{uz}\right)\right). \end{aligned}$$

This can also be written as

$$\begin{aligned} 0 &< (\beta+u-(\alpha-u))\left(e^{(\beta+u-\alpha)z} - 1 - (\beta+u-\alpha)z\left(\frac{e^{uz}-1}{uz}\right)\right) \\ &\quad +(\alpha-u)\left(e^{(\beta+u-\alpha)z} - e^{-(\beta+u-\alpha)z} - (\beta+u-\alpha)z\left(\frac{e^{uz}-e^{-uz}}{uz}\right)\right). \end{aligned}$$

The coefficients $\beta+u-(\alpha-u)$ and $\alpha-u$ are positive. The inequality will follow if we can prove that the functions multiplied by these two coefficients are positive. To show this, let $x = (\beta-\alpha)z$ and $y = uz$, and notice that these functions are given by

$$\begin{aligned} e^{x+y} - 1 - (x+y)\left(\frac{e^y-1}{y}\right) &= \left(\frac{e^x-1}{x} - \frac{1-e^{-y}}{y}\right)xe^y > 0, \\ e^{x+y} - e^{-(x+y)} - (x+y)\left(\frac{e^y-e^{-y}}{y}\right) &= \left(\frac{e^{x+y}-e^{-(x+y)}}{x+y} - \frac{e^y-e^{-y}}{y}\right)(x+y) > 0, \end{aligned}$$

respectively. The first inequality follows because because $e^x > 1+x$ and $e^{-y} > 1-y$ for all positive x and y. The second follows because the function $(e^a - e^{-a})/a$ is strictly increasing for $a > 0$.

6. The Density is Positive on $[y, \infty)$

Recall that $m(\beta, y+z) = m(\beta, y)e^{-\xi_+ z}$ and

$$\frac{m(\gamma, y+z)}{m(\beta, y)} = \frac{\left[\begin{array}{cc} e^{-\zeta_- z} & e^{-\zeta_+ z} \end{array}\right]}{m(\beta, y)} \left[\begin{array}{c} C_-(\gamma) \\ C_+(\gamma) \end{array}\right] - e^{-\xi_+ z}$$

for all $z > 0$. The coefficients $[C_-(\gamma), C_+(\gamma)]/m(\beta, y)$ are determined by

$$\begin{bmatrix} C_-(\gamma) \\ C_+(\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \zeta_- & \zeta_+ \end{bmatrix}^{-1} \begin{bmatrix} m(y) \\ -\mathrm{D}m(y) \end{bmatrix}.$$

Recall that differentiability at y yields

$$\begin{bmatrix} m(\gamma, y) \\ -\mathrm{D}m(\gamma, y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix},$$

and that $[B_-(\beta), B_+(\beta)]$ is given by

$$\begin{bmatrix} B_-(\beta) \\ B_+(\beta) \end{bmatrix} = \begin{bmatrix} e^{-\theta_-\Theta} \\ -e^{-\theta_+\Theta} \end{bmatrix} A(\beta) + \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} \\ -e^{-\omega_+\Omega} \end{bmatrix} A(\gamma),$$

with $[A(\beta), A(\gamma)]/m(\beta, y)$ is determined by (22). Eliminating $m(y)$, $\mathrm{D}m(y)$, $[B_-(\beta), B_+(\beta)]$ and $[C_-(\gamma), C_+(\gamma)]$ gives

$$\frac{m(\gamma, y+z)}{m(\beta, y)} = \frac{\begin{bmatrix} e^{-\zeta_- z} & e^{-\zeta_+ z} \end{bmatrix}}{\zeta_+ - \zeta_-} \begin{bmatrix} \zeta_+ & -1 \\ -\zeta_- & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - e^{-\xi_+ z}$$

where

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \\ &\quad \times \left\{ \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Theta} \\ -e^{-\theta_+\Theta} \end{bmatrix} \frac{A(\beta)}{m(\beta, y)} + \begin{bmatrix} 1 & 1 \\ \omega_- & \omega_+ \end{bmatrix} \begin{bmatrix} e^{-\omega_-\Omega} \\ -e^{-\omega_+\Omega} \end{bmatrix} \frac{A(\gamma)}{m(\beta, y)} \right\} \end{aligned}$$

and $[A(\beta), A(\gamma)]/m(\beta, y)$ is determined by (22). We know that $m(\gamma, y)/m(\beta, y)$ is positive, and hence

$$a = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}}{\zeta_+ - \zeta_-} \begin{bmatrix} \zeta_+ & -1 \\ -\zeta_- & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} > 1$$

To ensure that $m(\gamma, y+z)/m(\beta, y)$ is positive for all $z > 0$ as well, we need

$$\frac{\begin{bmatrix} e^{-\zeta_- z} & e^{-\zeta_+ z} \end{bmatrix}}{\zeta_+ - \zeta_-} \begin{bmatrix} \zeta_+ & -1 \\ -\zeta_- & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} > e^{-\xi_+ z}.$$

This is the same as

$$\frac{(a\zeta_+ - b)e^{(\zeta_+ - \zeta_-)z} - (a\zeta_- - b)}{\zeta_+ - \zeta_-} \times e^{(\xi_+ - \zeta_+)z} > 1$$

for all positive z. This condition holds at $z = 0$ because the leading factor equals $a > 1$ and the second factor equals 1 at $z = 0$. The leading factor has a derivative equal to $(a\zeta_+ - b)e^{(\zeta_+ - \zeta_-)z}$ and $\xi_+ - \zeta_+$ is positive. It follows that $m(\gamma, y + z) > 0$ for all $z > 0$ if and only if $a\zeta_+ - b$ is positive.

6.1 The Condition $a\zeta_+ > b$

We need to show that

$$\begin{bmatrix} \zeta_+ & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} > 0$$

and $[a, b]$ can be written as

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \\ &\times \left\{ \begin{bmatrix} 1 \\ \theta_- - \left(\frac{e^{(\theta_+-\theta_-)\Theta}-1}{\theta_+-\theta_-}\right)^{-1} \end{bmatrix} H + \begin{bmatrix} 1 \\ \omega_- - \left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} \end{bmatrix} K \right\} \end{aligned}$$

where H and K are strictly positive coefficients given by

$$H = \frac{\left(e^{(\theta_+-\theta_-)\Theta} - 1\right) e^{-\theta_+\Theta} A(\beta)}{m(\beta, y)}, \quad K = \frac{\left(e^{(\omega_+-\omega_-)\Omega} - 1\right) e^{-\omega_+\Omega} A(\gamma)}{m(\beta, y)}.$$

Observe that

$$\begin{aligned} &\begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix} \begin{bmatrix} e^{-\theta_-\Delta} & 0 \\ 0 & e^{-\theta_+\Delta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \theta_- & \theta_+ \end{bmatrix}^{-1} \\ &= \frac{e^{-\theta_-\Delta} - e^{-\theta_+\Delta}}{\theta_+ - \theta_-} \begin{bmatrix} \frac{\theta_+ e^{(\theta_+-\theta_-)\Delta} - \theta_-}{e^{(\theta_+-\theta_-)\Delta}-1} & -1 \\ \theta_+\theta_- & \left(\frac{e^{(\theta_+-\theta_-)\Delta}-1}{\theta_+-\theta_-}\right)^{-1} - \theta_- \end{bmatrix} \end{aligned}$$

So we need to show that

$$\begin{aligned} &\begin{bmatrix} \zeta_+ & 1 \end{bmatrix} \begin{bmatrix} \frac{\theta_+ e^{(\theta_+-\theta_-)\Delta} - \theta_-}{e^{(\theta_+-\theta_-)\Delta}-1} & 1 \\ -\theta_+\theta_- & \left(\frac{e^{(\theta_+-\theta_-)\Delta}-1}{\theta_+-\theta_-}\right)^{-1} - \theta_- \end{bmatrix} \times \\ &\left\{ \begin{bmatrix} 1 \\ \left(\frac{e^{(\theta_+-\theta_-)\Theta}-1}{\theta_+-\theta_-}\right)^{-1} - \theta_- \end{bmatrix} H + \begin{bmatrix} 1 \\ \left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} - \omega_- \end{bmatrix} K \right\} > 0. \end{aligned}$$

More explicitly,

$$\begin{aligned} &\begin{bmatrix} \left(\frac{\theta_+ e^{(\theta_+-\theta_-)\Delta} - \theta_-}{e^{(\theta_+-\theta_-)\Delta}-1}\right)\zeta_+ - \theta_+\theta_- & \left(\frac{e^{(\theta_+-\theta_-)\Delta}-1}{\theta_+-\theta_-}\right)^{-1} + \zeta_+ - \theta_- \end{bmatrix} \times \\ &\left\{ \begin{bmatrix} 1 \\ \left(\frac{e^{(\theta_+-\theta_-)\Theta}-1}{\theta_+-\theta_-}\right)^{-1} - \theta_- \end{bmatrix} H + \begin{bmatrix} 1 \\ \left(\frac{e^{(\omega_+-\omega_-)\Omega}-1}{\omega_+-\omega_-}\right)^{-1} - \omega_- \end{bmatrix} K \right\} > 0. \end{aligned}$$

Note that the second entry of the leading vector is positive. Therefore, if the inequality holds for $\Theta = \Omega = \infty$, it will hold for all $\Theta \in (0, \infty)$ and $\Omega \in (0, \infty)$. It thus suffices to show

$$\left[\begin{array}{cc} \frac{\left(\frac{\theta_+ e^{(\theta_+ - \theta_-)\Delta} - \theta_-}{e^{(\theta_+ - \theta_-)\Delta} - 1}\right)\zeta_+ - \theta_+\theta_-}{\left(\frac{e^{(\theta_+ - \theta_-)\Delta} - 1}{\theta_+ - \theta_-}\right)^{-1} + \zeta_+ - \theta_-} & 1 \end{array} \right] \left\{ \left[\begin{array}{c} 1 \\ -\theta_- \end{array} \right] H + \left[\begin{array}{c} 1 \\ -\omega_- \end{array} \right] K \right\} > 0.$$

Since $\theta_- > \omega_-$, this will be true for any positive H and K if it is true for $H > 0$ and $K = 0$. This requires

$$\frac{\left(\frac{\theta_+ e^{(\theta_+ - \theta_-)\Delta} - \theta_-}{e^{(\theta_+ - \theta_-)\Delta} - 1}\right)\zeta_+ - \theta_+\theta_-}{\left(\frac{e^{(\theta_+ - \theta_-)\Delta} - 1}{\theta_+ - \theta_-}\right)^{-1} + \zeta_+ - \theta_-} > \theta_-.$$

It is easy to verify that this inequality holds simply because $\zeta_+ > \theta_-$.

References

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