An Assignment Model of Knowledge Diffusion and Income Inequality

Erzo G.J. Luttmer

University of Minnesota

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`www.luttmer.org`

introduction

- ○ why is there so much inequality?
- ○ why are aggregate growth rates so stable?

▶ stuff grows, but not all at the same time

- a model of knowledge diffusion and growth with
  1. randomness in individual discovery
  2. randomness in who learns from whom
  3. randomness in social learning delays
  4. heterogeneity in ability to learn from others

▶ mechanism for growth and inequality:

  – individual discoveries generate and preserve heterogeneity

  – with selective replication, aggregate growth *emerges*

issue 1: how does useful knowledge spread?

▶ two ways in which an idea can travel *without* scale effects

- s_t = number of senders
- r_t = number of receivers

1. random meetings with imitation

$$s_{t+1} - s_t = r_t \times \frac{s_t}{r_t + s_t} \qquad \text{(CES with } \varepsilon = 1/2\text{)}$$

2. assignment with random learning

$$s_{t+1} - s_t = \min\{r_t, s_t\} \qquad \text{(CES with } \varepsilon = 0\text{)}$$

▶ will show: differences in ability magnified by (2)

issue 2: multiplicity of balanced growth paths

- learning from others with delay
- productivity distribution with a *thick* right tail

▶ implies *fast* growth

  – thick tail provides inexhaustible source of ideas to be copied

  – growth rate pinned down by choice of initial distribution

▶ but, *every* initial distribution with

finite support implies the *same* long-run growth rate

▶ for this, randomness in individual discoveries is essential

closely related models of idea flows

- ▶ Jovanovic and Rob 1989
- ▷ Kortum 1997
- ▷ Eaton and Kortum 1999
- • Luttmer 2007: *ideas embodied in firms, imitation by entrants*
- ∘ Alvarez, Buera and Lucas 2008
- ∘ Lucas 2009
- • Staley 2011
- • Luttmer 2012 (*JET*: *unique balanced growth path*)
- ∘ Lucas and Moll 2014
- ∘ Perla and Tonetti 2014
- • König, Lorenz, Zilibotti 2012
- • Luttmer 2012 (*Fed working paper*, *"Eventually, Noise and Imitation ..."*)
- • this paper, and Le 2014 (UMN senior thesis) for the Markov chain case

outline of these slides

1. basic math of individual discovery and social learning

2. an analytically tractable economy

  a. many balanced growth paths

  b. how to predict outcomes

3. quantitative implications

random imitation

- agents randomly select others at rate β and copy if "better"

$$\mathrm{D}_t P(t,z) = -\beta P(t,z)[1 - P(t,z)]$$

▶ the *unique* solution is

$$P(t,z) = \frac{1}{1 + \left(\frac{1}{P(0,z)} - 1\right) e^{\beta t}}$$

– $P(0,z)$ matters a lot…

▶ *many* logistic and log-logistic *stationary* solutions

$$P(0,z) = \frac{1}{1 + \left(\frac{1}{P(0,0)} - 1\right) e^{-(\beta/\kappa) z}} \quad \textit{implies} \quad P(t,z) = P(0, z - \kappa t)$$

$$P(0,z) = \frac{1}{1 + \left(\frac{1}{P(0,1)} - 1\right) z^{-\beta/\kappa}} \quad \textit{implies} \quad P(t,z) = P(0, z e^{-\kappa t})$$

easy to construct these stationary solutions

$$\mathrm{D}_t P(t,z) = -\beta P(t,z)[1 - P(t,z)] \tag{*}$$

- $P(t,z) = F(z - \kappa t)$ yields

$$\kappa \mathrm{D} F(z) = \beta F(z)[1 - F(z)] \tag{1}$$

– exponential tail index

$$\lim_{z\to\infty} \frac{\mathrm{D}F(z)}{1 - F(z)} = \frac{\beta}{\kappa}$$

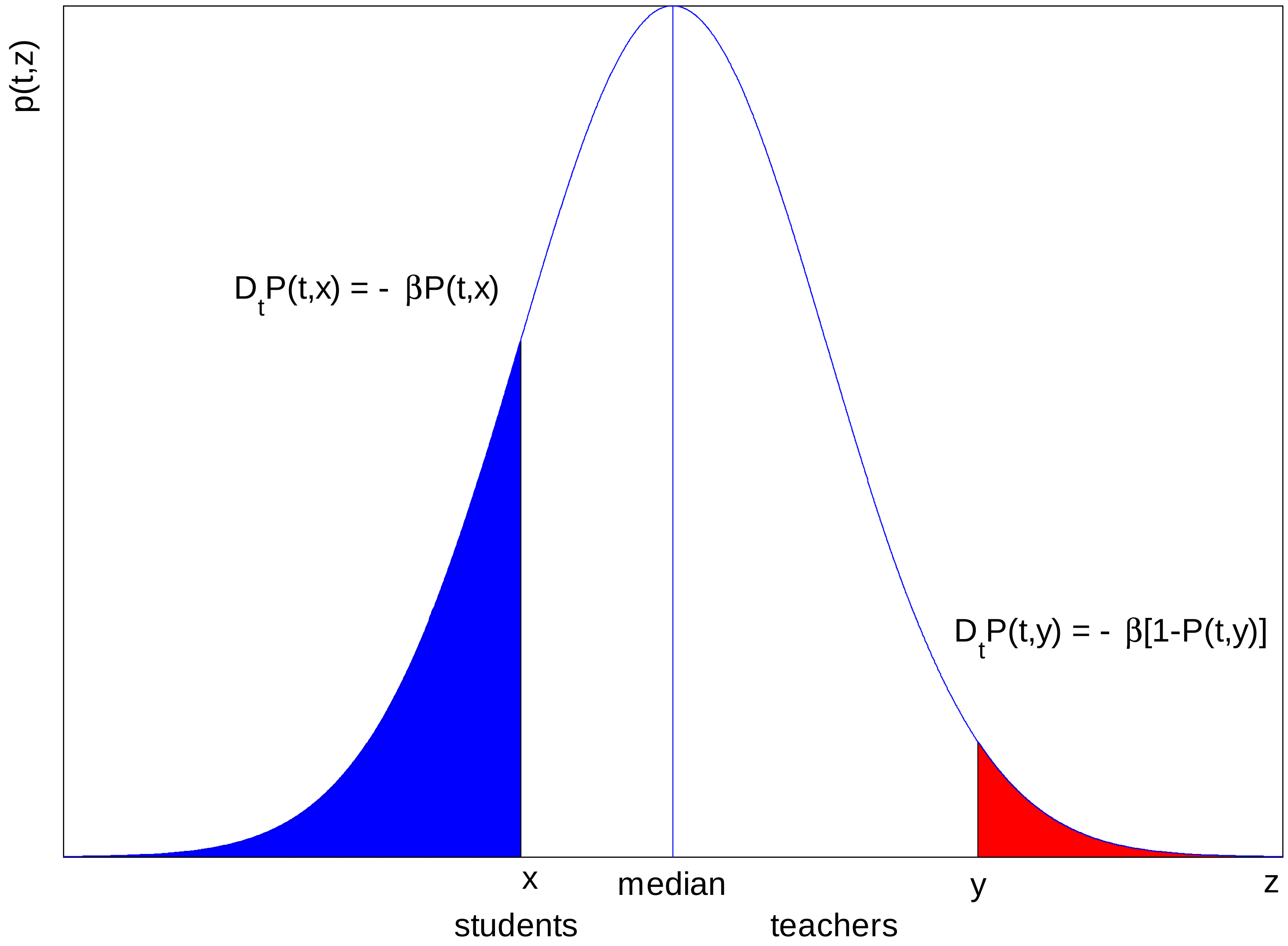
- $P(t,z) = F(ze^{-\kappa t})$ yields

$$\kappa z \mathrm{D} F(z) = \beta F(z)[1 - F(z)] \tag{2}$$

– power tail index

$$\lim_{z\to\infty} \frac{z\mathrm{D}F(z)}{1 - F(z)} = \frac{\beta}{\kappa}$$

one-on-one knowledge transfer



one-on-one knowledge transfer

- below-median student learns from above-median teacher at a rate β,

$$\mathrm{D}_t P(t,z) \;=\; -\beta \min\left\{P(t,z), 1-P(t,z)\right\}$$

- implied median x_t

$$\frac{1}{2} = P(t,x_t) = e^{\beta t}\left[1 - P(0,x_t)\right] \qquad (!)$$

  – which shows the role of the right tail

▶ the solution is

$$P(t,z) = \begin{cases} e^{-\beta t} P(0,z) & z \in (-\infty, x_0) \\ \frac{1}{2}\frac{1/2}{e^{\beta t}[1-P(0,z)]} & z \in (x_0, x_t) \\ 1 - e^{\beta t}[1-P(0,z)] & z \in (x_t, \infty) \end{cases}$$

for future use

- density

$$p(t,z) = \mathrm{D}_z P(t,z)$$

- differentiate

$$\mathrm{D}_t P(t,z) = -\beta \min\left\{P(t,z), 1 - P(t,z)\right\}$$

with respect to z

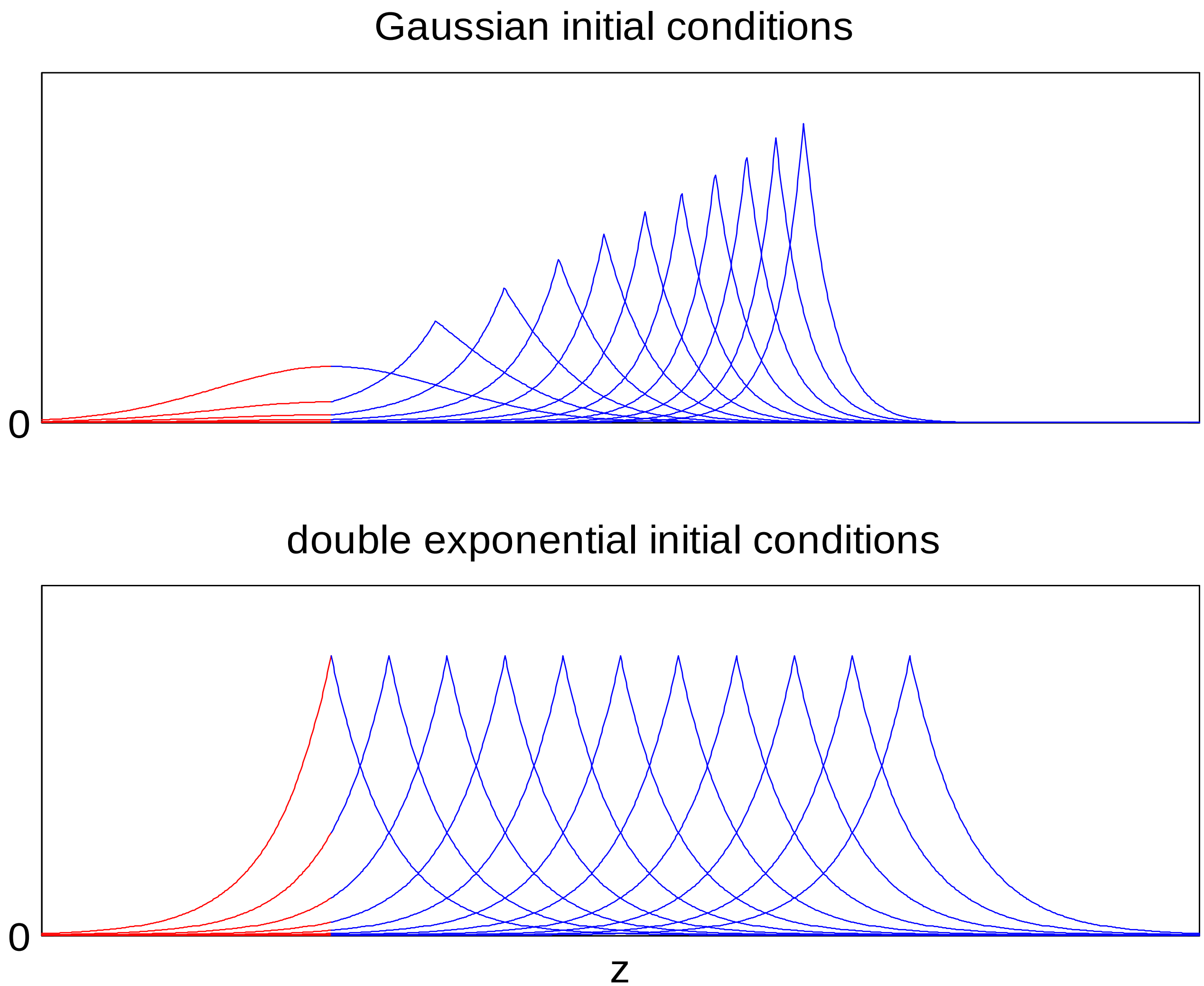
▶ this yields

$$\mathrm{D}_t p(t,z) = \begin{cases} -\beta p(t,z), & z < x_t \\ +\beta p(t,z), & z > x_t \end{cases}$$

where

$$\frac{1}{2} = \int_{-\infty}^{x_t} p(t,z)\mathrm{d}z$$

time lapse $\{p(j\Delta, z)\}_{j=1}^{J}$



the obvious problem

- NO long-run growth if the initial distribution has bounded support

- for example, if the population is finite

- everyone learns the most useful knowledge eventually...
  - someone had this knowledge already at some initial date
  - the only question is: how does it diffuse?

▶ it can't be all about catching up with some ancient geniuses

the solution

- two independent standard Brownian motions $B_{1,t}, B_{2,t}$,

$$\mathrm{E}\left[\max\left\{\sigma B_{1,t}, \sigma B_{2,t}\right\}\right] = \sigma\sqrt{t/\pi}$$

- reset to max at random time $\tau_{j+1} > \tau_j$

$$z_{\tau_{j+1}} = z_{\tau_j} + \sigma \max\left\{B_{1,\tau_{j+1}} - B_{1,\tau_j}, B_{2,\tau_{j+1}} - B_{2,\tau_j}\right\}$$

- reset times arrive randomly at rate β

$$\frac{\mathrm{E}\left[z_{\tau_{j+1}} - z_{\tau_j} | z_{\tau_j}\right]}{\mathrm{E}\left[\tau_{j+1} - \tau_j | z_{\tau_j}\right]} = \frac{1}{1/\beta}\int_0^\infty \sigma\sqrt{t/\pi}\beta e^{-\beta t}\mathrm{d}t = \frac{1}{2}\sigma\sqrt{\beta}$$

  – can also show

$$\mathrm{E}\left[\frac{z_{\tau_{j+1}} - z_{\tau_j}}{\tau_{j+1} - \tau_j} \mid z_{\tau_j}\right] = \sigma\sqrt{\beta}$$

- large populations

$$\texttt{trend} = \sigma^2\sqrt{\frac{\beta}{\sigma^2/2}} = \sigma\sqrt{2\beta} > \sigma\sqrt{\beta}\ldots$$

the economy

- dynastic preferences

$$\int_0^\infty e^{-\rho t} \ln(C_t) \mathrm{d}t$$

- generations pass randomly at the rate δ,

  1. replaced immediately
  2. perfect inheritance of learning ability $\lambda \in \Lambda$
  3. newborn individuals have no knowledge, begin as workers
  4. can acquire knowledge and become managers
  5. managers can quit and become workers again

     ▷ managerial knowledge then instantaneously obsolete

- complete markets..., interest rate $r_t = \rho + \mathrm{D}C_t/C_t$

production of consumption goods

- a manager with knowledge z and l units of labor produce

$$y = \left(\frac{e^z}{1-\alpha}\right)^{1-\alpha}\left(\frac{l}{\alpha}\right)^{\alpha}$$

  – as in Lucas [1978]

- continuation as manager requires ϕ units of overhead labor

factor supplies

- there is a unit measure of managers and workers
- type distribution $\{M(\lambda) : \lambda \in \Lambda\}$
- workers supply one unit of labor
- $M_t(\lambda, z)$ = time-t measure of type-λ managers with knowledge up to z

factor prices and consumption

- managerial profit maximization

$$v_t e^z = \max_l \left\{ \left(\frac{e^z}{1-\alpha} \right)^{1-\alpha} \left(\frac{l}{\alpha} \right)^{\alpha} - w_t l \right\}$$

so that $v_t^{1-\alpha} w_t^{\alpha} = 1$.

- consumption and wages

$$C_t = \left(\frac{H_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{1-(1+\phi)N_t}{\alpha} \right)^{\alpha}, \qquad w_t = \frac{\alpha C_t}{1-(1+\phi)N_t}$$

where

$$H_t = \sum_{\lambda \in \Lambda} \int e^z M_t(\lambda, \mathrm{d}z), \qquad N_t = \sum_{\lambda \in \Lambda} M_t(\lambda, \infty)$$

knowledge creation and diffusion

- type-λ manager in state z_{t-} matched with manager in state $\widetilde{z}_{t-} > z_{t-}$,

$$\mathrm{d}z_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t + (\widetilde{z}_{t-} - z_{t-})^+ \mathrm{d}J_t$$

  – B_t is a standard Brownian motion

  – J_t is a Poisson process with arrival rate $\lambda \in \Lambda$

- type-λ workers can also learn from managers in state z at rate λ

▶ knowledge state does *not* affect learning speed

  – learning ability λ may be determined in part by prior general education

▶ z_t measures how useful knowledge is

  – *not* how difficult it is to learn

nature of the assignment problem

- pairwise matching of students and teachers
- everyone can be a student, every manager can be a teacher
- individuals characterized by (λ, z)
  - learning ability $\lambda \in \Lambda$, a finite subset of $(0, \infty)$
  - type-λ workers know $z = -\infty$,
  - type-λ managers know $z \in (b_t(\lambda), \infty)$

- $V_t(z|\lambda)$ is value of a manager, $W_t(\lambda) = \min_z\{V_t(z|\lambda)\}$ is value of a worker
- expected gain of a match of the "student" (λ, z_{t-}) and "teacher" $(\widetilde{\lambda}, \widetilde{z}_{t-})$

$$\lambda \left[V_{t-}(\widetilde{z}_{t-}|\lambda) - V_{t-}(z_{t-}|\lambda) \right]$$

when $\widetilde{z}_{t-} \geq z_{t-}$

the market for students and teachers

- a manager in state z charges flow tuition $T_t(z) \geq 0$
  - when a student "gets it," he or she enjoys a capital gain

- define "surplus" values,

$$S_t(\lambda) = \sup_{\tilde{z}} \{\lambda V_t(\tilde{z}|\lambda) - T_t(\tilde{z})\}$$

▶ flow gains for type-λ managers in state z,

$$\max \{T_t(z), S_t(\lambda) - \lambda V_t(z|\lambda)\}$$

▶ if $T_t(z) = 0$ for z low enough,

$$S_t(\lambda) - \lambda W_t(\lambda) \geq 0$$

equilibrium tuition schedules

- recall

$$S_t(\lambda) = \sup_{\tilde{z}} \{\lambda V_t(\tilde{z}|\lambda) - T_t(\tilde{z})\}$$

  – hence

$$T_t(z) \;\geq\; \lambda V_t(z|\lambda) - S_t(\lambda), \qquad \text{for all } (\lambda, z)$$

  with equality if type-λ students select teachers at z

▶ if there are teachers at z, market clearing requires

$$T_t(z) = \max_{\lambda \in \Lambda} \{\lambda V_t(z|\lambda) - S_t(\lambda)\}$$

▶ type-μ managers at z choose to teach if

$$T_t(z) \geq S_t(\mu) - \mu V_t(z|\mu)$$

the "price system" $\{S_t(\lambda) : \lambda \in \Lambda\}$

Lemma 1 *The tuition schedule can be taken to be of the form*

$$T_t(z) = \max_{\lambda \in \Lambda} \left\{ [\lambda V_t(z|\lambda) - S_t(\lambda)]^+ \right\},$$

without loss of generality.

Lemma 2 *Given numbers* $\{S_t(\lambda) : \lambda \in \Lambda\}$*, define*

$$T_t(z) = \max_{\lambda \in \Lambda} \left\{ [\lambda V_t(z|\lambda) - S_t(\lambda)]^+ \right\}, \quad S_t^*(\lambda) = \sup_z \left\{ \lambda V_t(z|\lambda) - T_t(z) \right\}$$

Then

$$T_t(z) \; = \; \max_{\lambda \in \Lambda} \left\{ [\lambda V_t(z|\lambda) - S_t^*(\lambda)]^+ \right\}.$$

The $S_t^*(\lambda)/\lambda$ *are weakly increasing in* $\lambda \in \Lambda$.

present values

- ▶ fix factor prices $[v_t, w_t]$ and $\{S_t(\lambda) : \lambda \in \Lambda\}$

  - type-λ workers

$$r_t W_t(\lambda) = w_t + \max\{0, S_t(\lambda) - \lambda W_t(\lambda)\} + \mathrm{D}_t W_t(\lambda)$$

  - type-λ managers

$$r_t V_t(z|\lambda) \;=\; v_t e^z - \phi w_t + \max\{T_t(z), S_t(\lambda) - \lambda V_t(z|\lambda)\} + \mathrm{D}_t V_t(z|\lambda)$$

$$+\mu \mathrm{D}_z V_t(z|\lambda) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} V_t(z|\lambda) + \delta\left[W_t(\lambda) - V_t(z|\lambda)\right]$$

    – where

$$T_t(z) = \max_{\lambda \in \Lambda}\left\{[\lambda V_t(z|\lambda) - S_t(\lambda)]^+\right\}$$

- ▶ piecewise linear!

balanced growth

- conjecture that the cross-section of $z_t - \kappa t$ is time invariant
  - growth rate κ to be determined...

- managerial human capital and consumption

$$H_t = He^{\kappa t},\ C_t = Ce^{(1-\alpha)\kappa t}$$

  - interest rates $r_t = \rho + (1-\alpha)\kappa$
  - factor prices,

$$[w_t, v_t] \ = \ \left[we^{(1-\alpha)\kappa t}, ve^{-\alpha\kappa t}\right]$$

- value functions,

$$[W_t(\lambda), V_t(z+\kappa t|\lambda), S_t(\lambda), T_t(z+\kappa t)] \ = \ [W(\lambda), V(z|\lambda), S(\lambda), T(z)]\, e^{(1-\alpha)\kappa t}$$

bellman equations

- type-λ workers

$$\rho W(\lambda) = w + \max\{0, S(\lambda) - \lambda W(\lambda)\}$$

– note

$$S(\lambda) - \lambda W(\lambda) \ > \ 0 \quad \Leftrightarrow \quad W(\lambda) > \frac{w}{\rho} \quad \Leftrightarrow \quad \frac{S(\lambda)}{w} > \frac{\lambda}{\rho}$$

- type-λ managers

$$\rho V(z|\lambda) \ = \ ve^z - \phi w + \max\{T(z), S(\lambda) - \lambda V(z|\lambda)\}$$
$$+(\mu - \kappa)\mathrm{D}V(z|\lambda) + \frac{1}{2}\sigma^2\mathrm{D}^2 V(z|\lambda) + \delta\left[W(\lambda) - V(z|\lambda)\right]$$

– where

$$T(z) = \max_{\lambda \in \Lambda}\left\{[\lambda V(z|\lambda) - S(\lambda)]^+\right\}$$

ability rent scenarios

- if sufficiently many fast learners

$$S(\gamma) - \gamma W(\gamma) = S(\beta) - \beta W(\beta) = 0$$

- if not too many fast learners

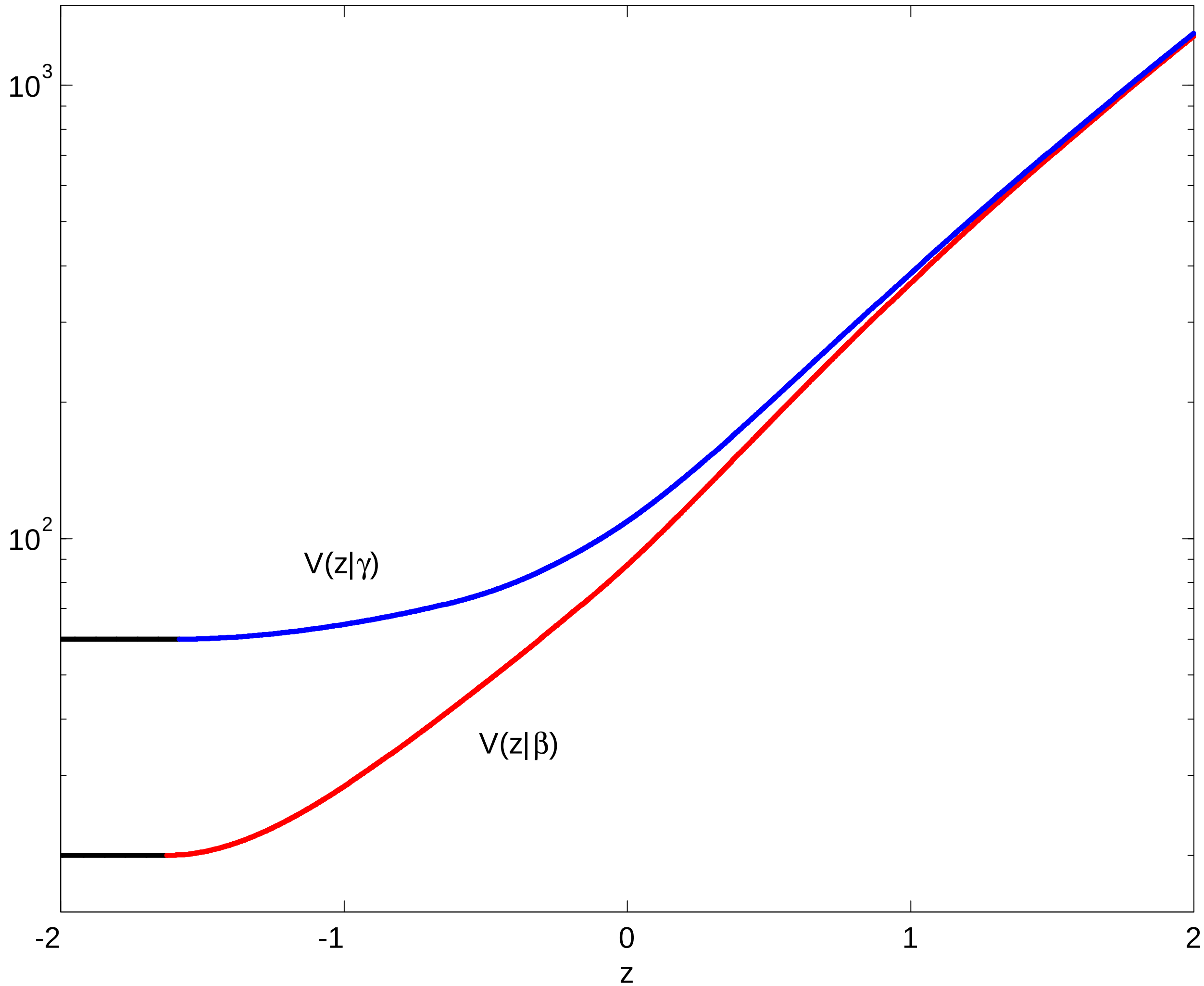
$$S(\gamma) - \gamma W(\gamma) \ > \ S(\beta) - \beta W(\beta) \geq 0$$

  – but if $S(\beta) - \beta W(\beta) > 0$ then
    - ◦ all workers and some managers are students
    - ◦ one-on-one teaching implies half the population is a teacher
    - ◦ this would imply more than half the population is a manager

▶ from hereon, focus on the case

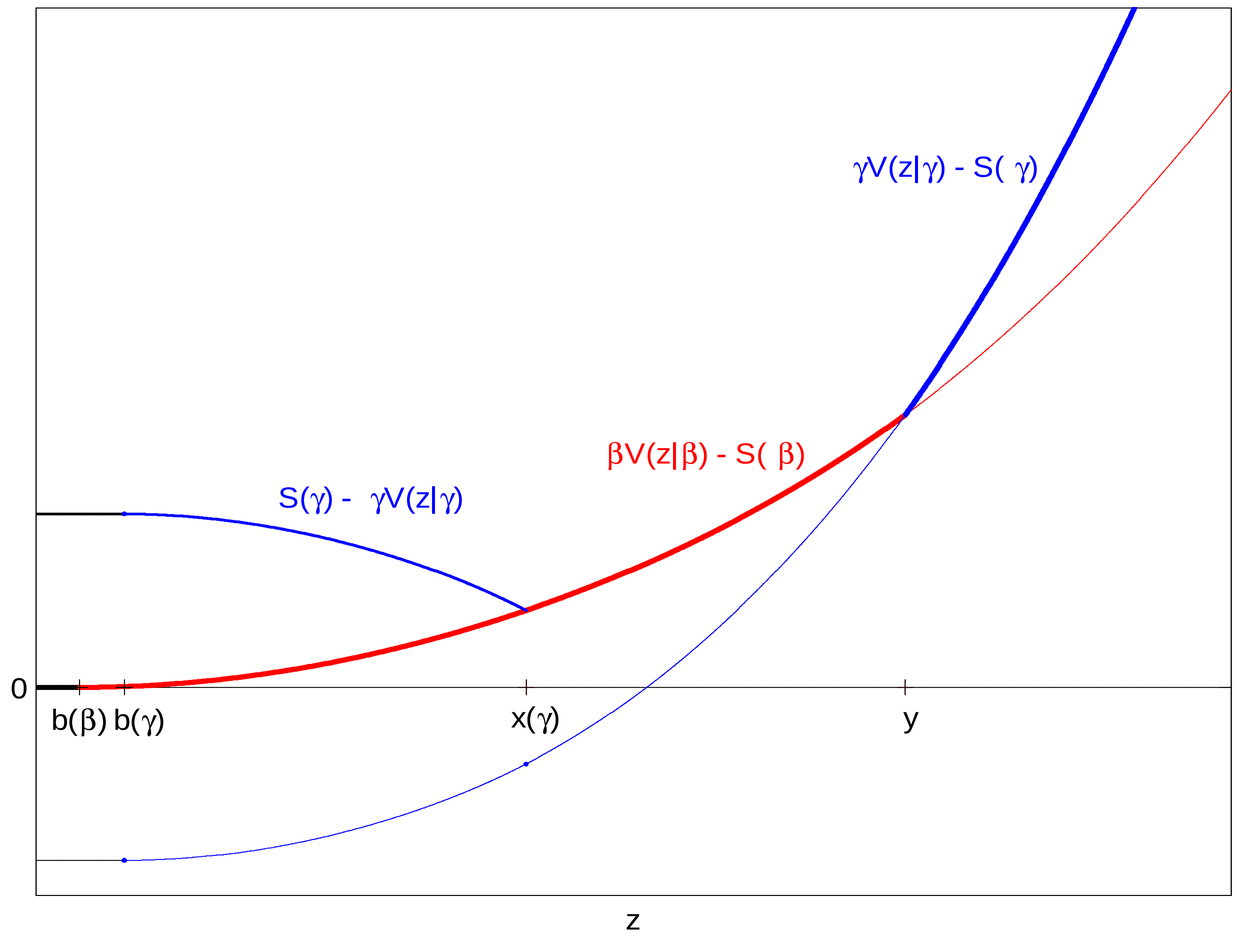
$$S(\gamma) - \gamma W(\gamma) \ > \ S(\beta) - \beta W(\beta) = 0$$

learning rates $\gamma > \beta > 0$



- note the log scale; next consider $\lambda V(z|\lambda) - S(\lambda)$

learning rates $\gamma > \beta > 0$



- expected learning gains $\lambda V(z|\lambda) - S(\lambda)$ satisfy a single-crossing property

the first equilibrium condition

• given "prices" $[v, w, S(\beta), S(\gamma)]$, the Bellman equations determine

$$[W(\beta), V(z|\beta), W(\gamma), V(z|\gamma)] \text{ and implied thresholds}$$

▷ indifferent slow learners scenario implies $S(\beta)/w = \beta W(\beta)/w = \beta/\rho$

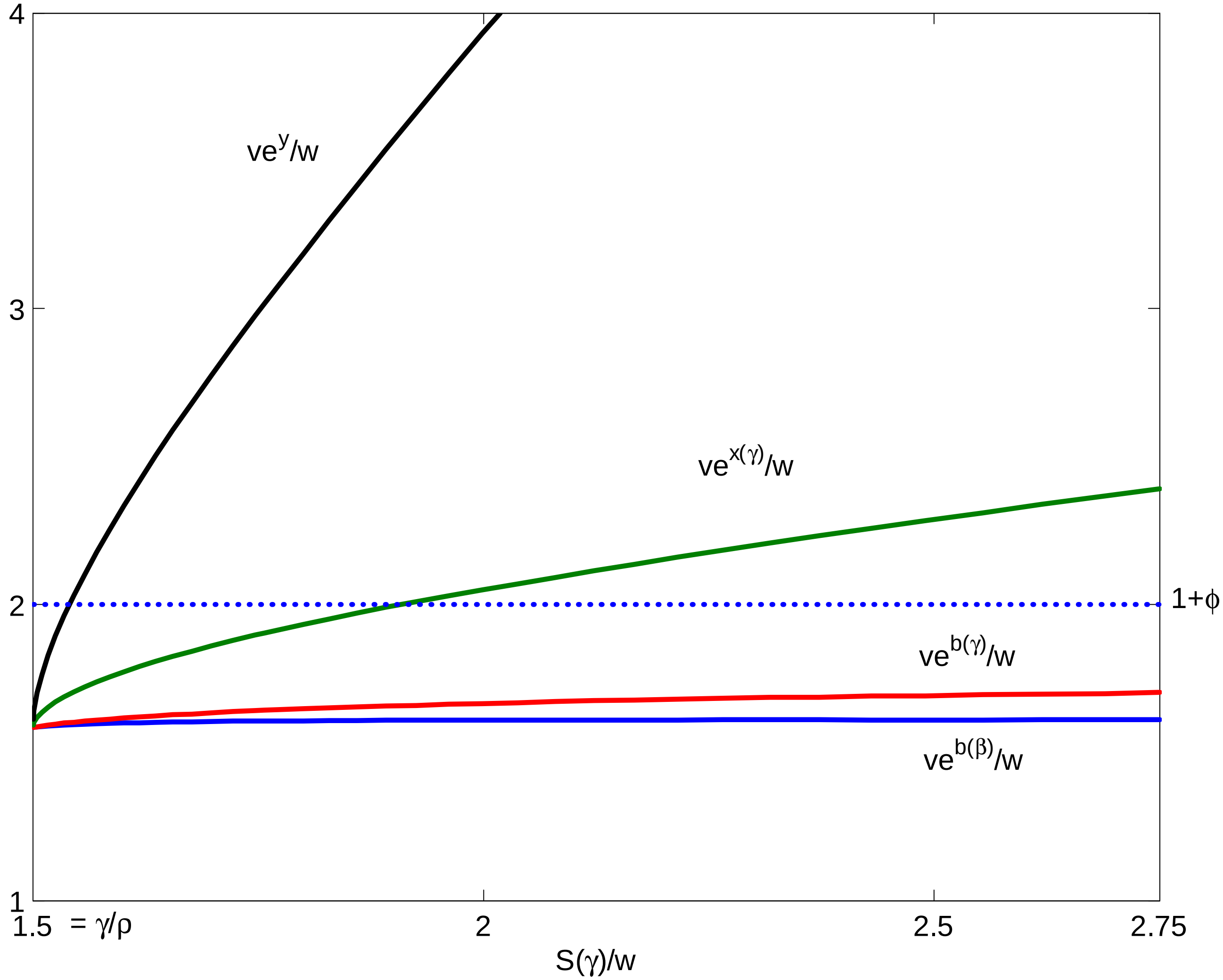
▷ eliminate dependence on v/w,

$$e^{\widehat{z}} = ve^{z}/w, \;\; \widehat{V}(\widehat{z}|\lambda) = [V(z|\lambda) - w/\rho]/w$$

▶ the Bellman equation therefore determines a curve

$$\frac{S(\gamma)}{w} \mapsto \frac{v}{w} \times \left[e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y}\right]$$

the first equilibrium condition



KFE intuition for $\mathrm{d}x_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$

- without noise, $f(t,x) = f(0, x - \mu t)$ implies

$$\mathrm{D}_t f(t,x) = -\mu \mathrm{D}_x f(0, x - \mu t) = -\mu \mathrm{D}_x f(t,x)$$

- without drift, random increments make population move downhill
  - CDF satisfies

$$\mathrm{D}_t F(t,x) = \frac{1}{2}\sigma^2 \mathrm{D}_x f(t,x)$$

  - differentiate

$$\mathrm{D}_t f(t,x) = \frac{1}{2}\sigma^2 \mathrm{D}_{xx} f(t,x)$$

▶ combine and add random death at rate δ

$$\mathrm{D}_t f(t,x) = -\mu \mathrm{D}_x f(t,x) + \frac{1}{2}\sigma^2 \mathrm{D}_{xx} f(t,x) - \delta f(t,x)$$

stationary densities

- forward equations ($\theta = \mu - \kappa$)

$$\delta m(\beta, z) = -\theta \mathrm{D} m(\beta, z) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(\beta, z) + \begin{cases} \beta m(\beta, z), & z \in (b(\beta), x(\gamma)) \\ \beta[m(\beta, z) + m(\gamma, z)], & z \in (x(\gamma), y) \\ 0, & z \in (y, \infty) \end{cases}$$

and

$$\delta m(\gamma, z) = -\theta \mathrm{D} m(\gamma, z) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(\gamma, z) + \begin{cases} -\gamma m(\gamma, z), & z \in (b(\gamma), x(\gamma)) \\ 0, & z \in (x(\gamma), y) \\ \gamma[m(\beta, z) + m(\gamma, z)], & z \in (y, \infty) \end{cases}$$

- students assigned to teachers by construction
  - but the number of type-λ workers choosing to study is left implicit
  - market clearing condition for type-γ students will determine scale

▶ piecewise linear!

market clearing conditions

- supplies $M(\lambda)$ of type-λ individuals are given

- supplies of type-λ students and teachers

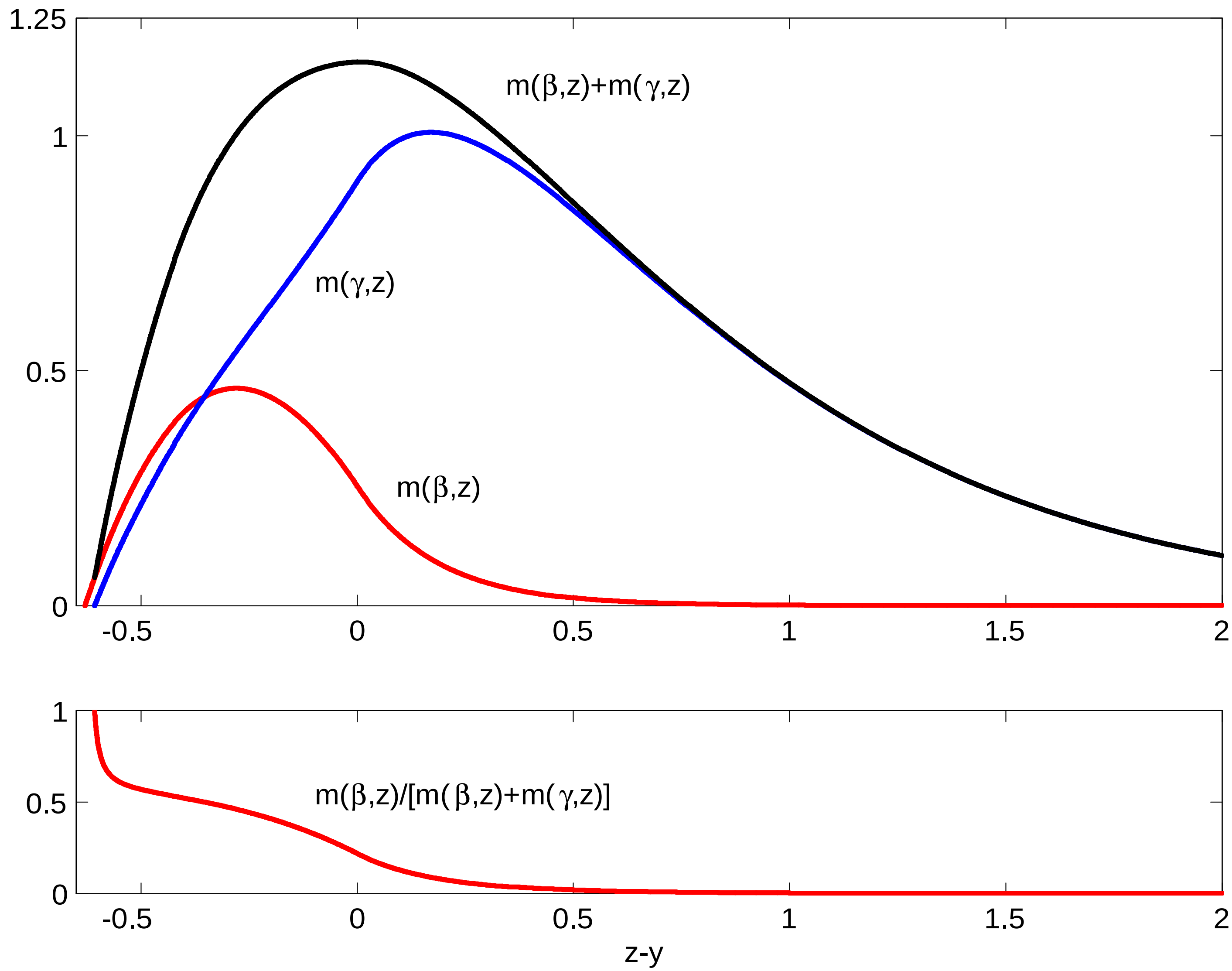
$$M(\beta) - \int_{b(\beta)}^{\infty} m(\beta, z)\mathrm{d}z \geq \int_{b(\beta)}^{y} m(\beta, z)\mathrm{d}z + \int_{x(\gamma)}^{y} m(\gamma, z)\mathrm{d}z$$

$$M(\gamma) - \int_{x(\gamma)}^{\infty} m(\gamma, z)\mathrm{d}z = \int_{y}^{\infty} [m(\beta, z) + m(\gamma, z)]\mathrm{d}z$$

  – these conditions depend only on $[y - b(\beta), y - b(\gamma), y - x(\gamma)]$
  – hence, function only of $S(\gamma)/w$

▶ not all type-β workers choose to study when $S(\beta) - \beta W(\beta) = 0$

▶ the type-γ condition determines the scale of m



1.25
1
0.5
0
m(β,z)+m(γ,z)
m(γ,z)
m(β,z)
-0.5
0
0.5
1
1.5
2
1
0.5
0
m(β,z)/[m(β,z)+m(γ,z)]
-0.5
0
0.5
1
1.5
2
z-y

the magnification effect—intro

- fast social learners accumulate knowledge more quickly
  - overrepresented in the right tail
  - even in economy with random assignment

- competitive assignment
  - sorting: fast learners assigned to most knowledgeable teachers
  - this magnifies the advantage of fast learners

▶ with two types, $\beta < \gamma$

1. right tail indices of $m(\beta, z)$ and $m(\gamma, z)$ do not depend on β
2. an infinitesimal gap $\gamma - \beta$ implies
   - very different outcome distributions
   - infinitesimal ex ante utility differences

right tails behave like $e^{-\zeta z}$

- with ζ determined by root(s) of a characteristic equation

▶ right tail slow learners

$$\delta m(\beta, z) \;=\; -(\mu - \kappa)\mathrm{D}m(\beta, z) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(\beta, z)$$

$$\Rightarrow \zeta_\beta = \frac{\kappa - \mu}{\sigma^2} + \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 + \frac{\delta}{\sigma^2/2}}$$

▶ right tail fast learners

$$\delta m(\gamma, z) \;=\; -(\mu - \kappa)\mathrm{D}m(\gamma, z) + \frac{1}{2}\sigma^2 \mathrm{D}^2 m(\gamma, z) + \gamma[m(\beta, z) + m(\gamma, z)]$$

$$\Rightarrow \zeta_{\gamma,\pm} = \frac{\kappa - \mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma - \delta}{\sigma^2/2}}$$

– where $\gamma > \delta > 0$

growth, inequality, the magnification effect

- recall

$$\zeta_\beta = \frac{\kappa-\mu}{\sigma^2} + \sqrt{\left(\frac{\kappa-\mu}{\sigma^2}\right)^2 + \frac{\delta}{\sigma^2/2}}$$

$$\zeta_{\gamma,\pm} = \frac{\kappa-\mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa-\mu}{\sigma^2}\right)^2 - \frac{\gamma-\delta}{\sigma^2/2}}$$

- need $\zeta_{\gamma,\pm}$ to be real

$$\frac{\kappa-\mu}{\sigma^2} \geq \sqrt{\frac{\gamma-\delta}{\sigma^2/2}} \qquad \text{(KPP)}$$

▶ will argue (KPP) should hold with equality, and thence

$$\kappa = \mu + \sigma^2 \times \zeta_\gamma, \qquad \zeta_\gamma = \sqrt{\frac{\gamma-\delta}{\sigma^2/2}} \qquad (!)$$

and

$$\zeta_\beta = \zeta_\gamma + \sqrt{\frac{\gamma}{\sigma^2/2}} \qquad (!!)$$

relation to Luttmer (2007)

- entrepreneurs try to copy randomly selected incumbents

$$\delta m(z) = -(\mu - \kappa)\mathrm{D}m(z) + \frac{1}{2}\sigma^2\mathrm{D}^2 m(z) + (\gamma E/N)m(z), \quad z > b$$

  - success rate of entrepreneurs $= \gamma$
  - number of entreprepreneurs $= E$
  - number of incumbent firms $= N$

- stationarity requires

$$\kappa \geq \mu + \sigma^2 \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}$$

- if with equality, $m(z) = \zeta(z-b)e^{-\zeta(z-b)}$, where

$$\zeta = \sqrt{\frac{(\gamma E/N) - \delta}{\sigma^2/2}}$$

▶ E/N endogenous, will depend on subjective discount rate

the second equilibrium condition

- the number of managers

$$N = \int_{b(\beta)}^{\infty} m(\beta, z)\mathrm{d}z + \int_{b(\gamma)}^{\infty} m(\gamma, z)\mathrm{d}z$$

- implied factor supplies

$$L = M(\beta) + M(\gamma) - (1+\phi)N$$

$$He^{-y} = \int_{b(\beta)}^{\infty} e^{z-y} m(\beta, z)\mathrm{d}z + \int_{b(\gamma)}^{\infty} e^{z-y} m(\gamma, z)\mathrm{d}z$$

1. recall from the Bellman equations

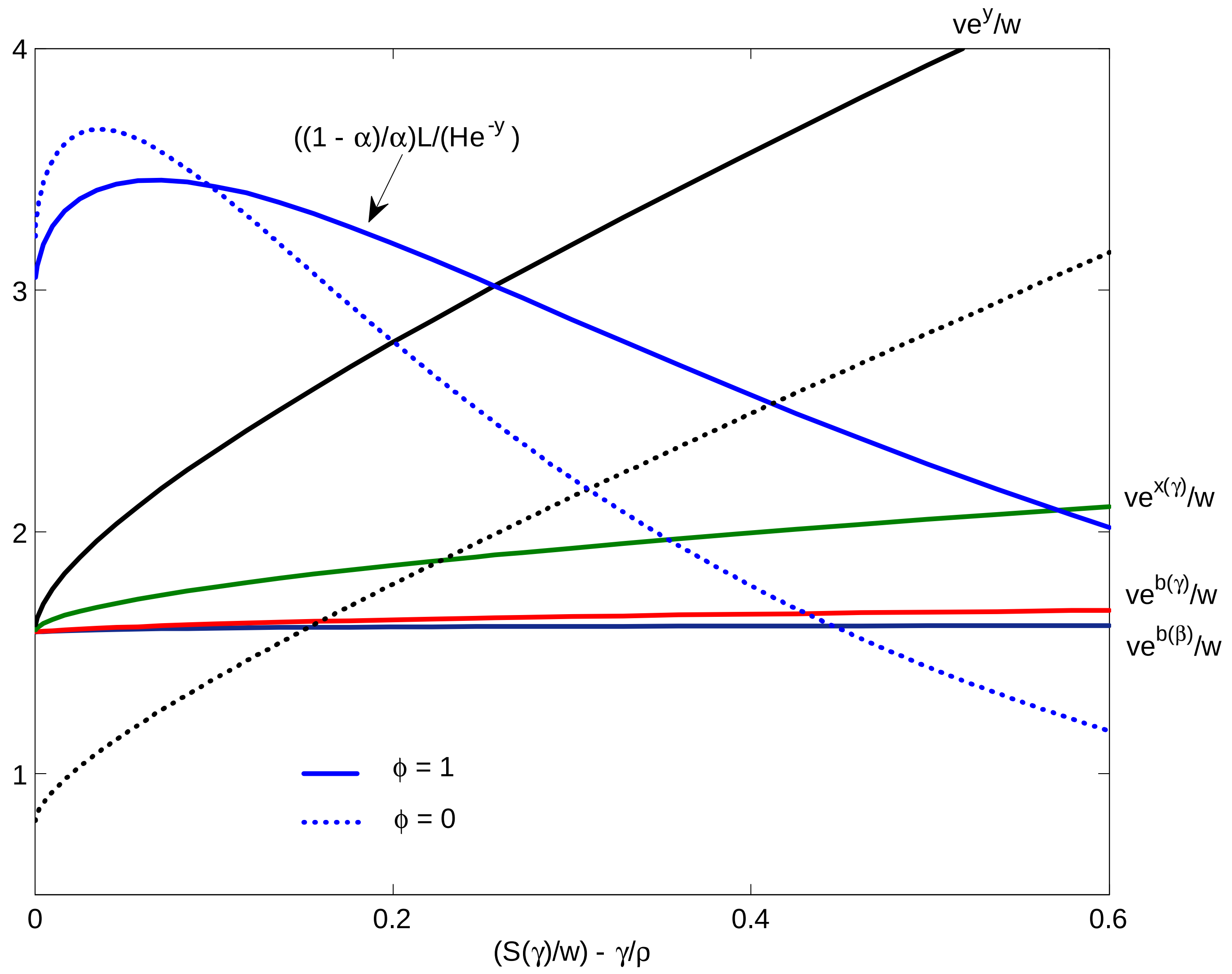
$$\frac{S(\gamma)}{w} \mapsto \frac{v}{w} \times \left[e^{b(\beta)}, e^{b(\gamma)}, e^{x(\gamma)}, e^{y}\right]$$

2. Cobb-Douglas

$$\frac{ve^y}{w} = \frac{1-\alpha}{\alpha} \frac{L}{He^{-y}}$$

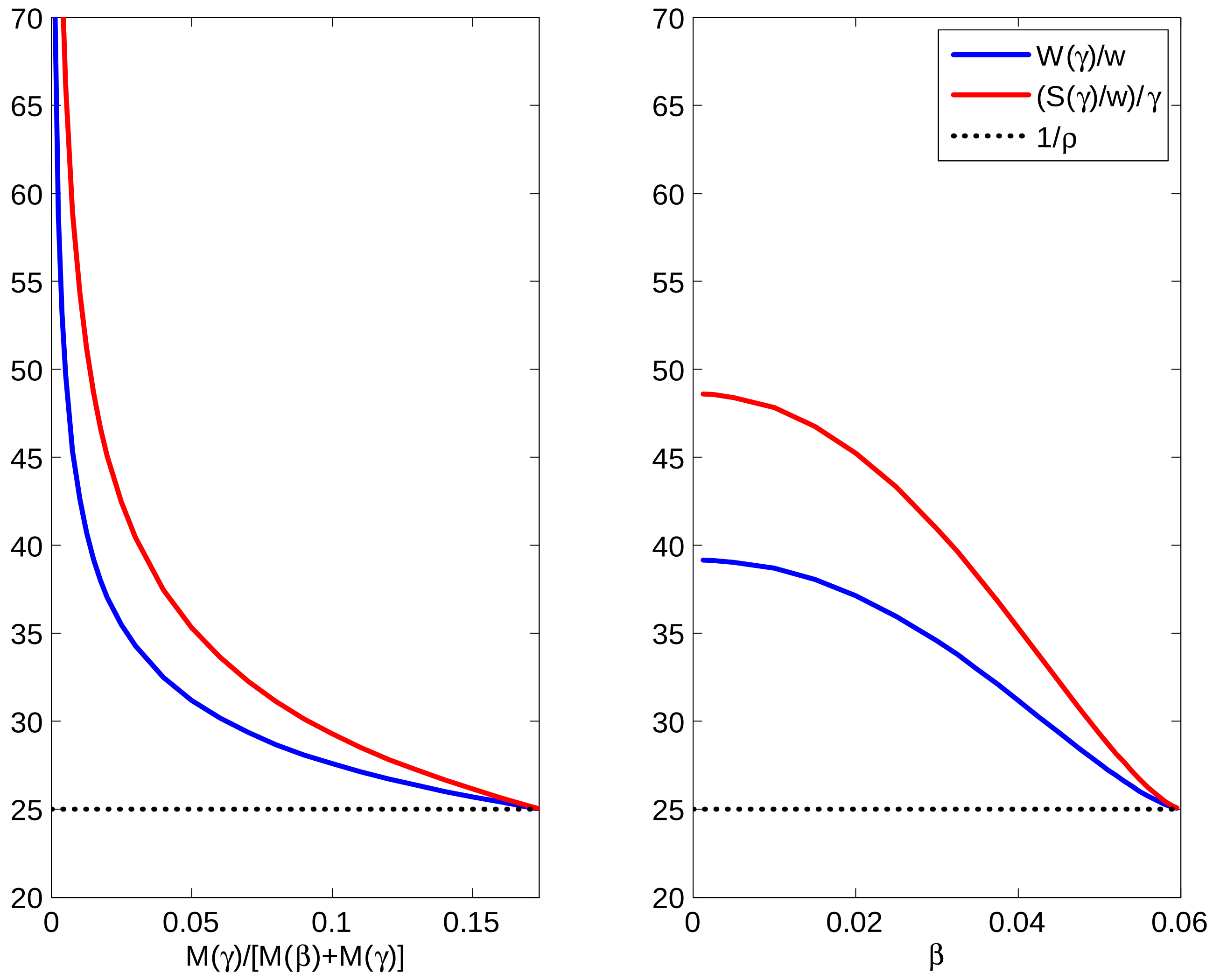
▶ (1) and (1)+(2): *two* ways to map $S(\gamma)/w$ into ve^y/w

the fixed point



- solid: $\phi = 1$; dots: $\phi = 0$

ability rents



(round) numbers used for these diagrams

technology	
α	ϕ
0.60	1

ability distribution
$M(\gamma)/[M(\beta)+M(\gamma)]$
0.10

rates				
ρ	δ	β	γ	σ
0.04	0.04	0.05	0.06	0.10

implications

- tail indices

$$\zeta_\gamma = \frac{\kappa - \mu}{\sigma^2} = \sqrt{\frac{\gamma - \delta}{\sigma^2/2}} = \sqrt{\frac{0.06 - 0.04}{(0.1)^2/2}} = 2$$

$$\zeta_\beta = \zeta_\gamma + \sqrt{\frac{\gamma}{\sigma^2/2}} = 2 + \sqrt{\frac{0.06}{(0.1)^2/2}} \approx 5.5$$

- growth

$$\kappa - \mu = \sigma^2 \zeta_\gamma = (0.1)^2 \times 2 = 0.02$$

- value of a worker

$$\frac{W(\beta)}{w} = \frac{1}{\rho} = 25$$

$$\frac{W(\gamma)}{w} = \frac{1}{\rho}\left(1 + \frac{S(\gamma) - \gamma W(\gamma)}{w}\right) = 25 \times (1 + 0.103) \approx 27.8$$

- managers

$$[N(\beta), N(\gamma)] \approx [0.011, 0.055]\,(M(\beta) + M(\gamma))$$

some measures of inequality

- for Pareto

$$\ln(\texttt{top share}) = \left(1 - \frac{1}{\zeta}\right) \times \ln(\texttt{top percentile})$$

▶ Piketty et al. income shares by percentile,

	10%	1%	.1%
1964	$32\% \rightarrow \zeta = 1.98$	$10\% \rightarrow \zeta = 2.00$	$2.0\% \rightarrow \zeta = 2.31$
2004	$48\% \rightarrow \zeta = 1.47$	$22\% \rightarrow \zeta = 1.49$	$8.8\% \rightarrow \zeta = 1.54$

▶ Cagetti and De Nardi report top 1% owns 30% of wealth in SCF

– this implies $\zeta = 1.35$

earnings growth

- cross-sectional variance of log earnings

$$\begin{aligned} \text{age } 25 &: \quad 0.60 \\ \text{age } 60 &: \quad 1.05 \end{aligned}$$

  – US social security records (Guvenen et al. [2015])

  – if pure random walk:

$$\text{annual standard deviation} = \sqrt{\frac{.45}{35}} \approx 0.11$$

- continuous part of managerial earnings growth in the model has $\sigma = 0.10$

$$\text{annual standard deviation} \approx \sqrt{(1-0.066)\times 0 + 0.066 \times (0.1)^2} \approx 0.026$$

  – the γ, β and δ shocks will have to do the heavy lifting...

an empirical difficulty

- income distribution:

$$\zeta = 2 \text{ in the 1960s}, \quad \zeta = 1.5 \text{ now}$$

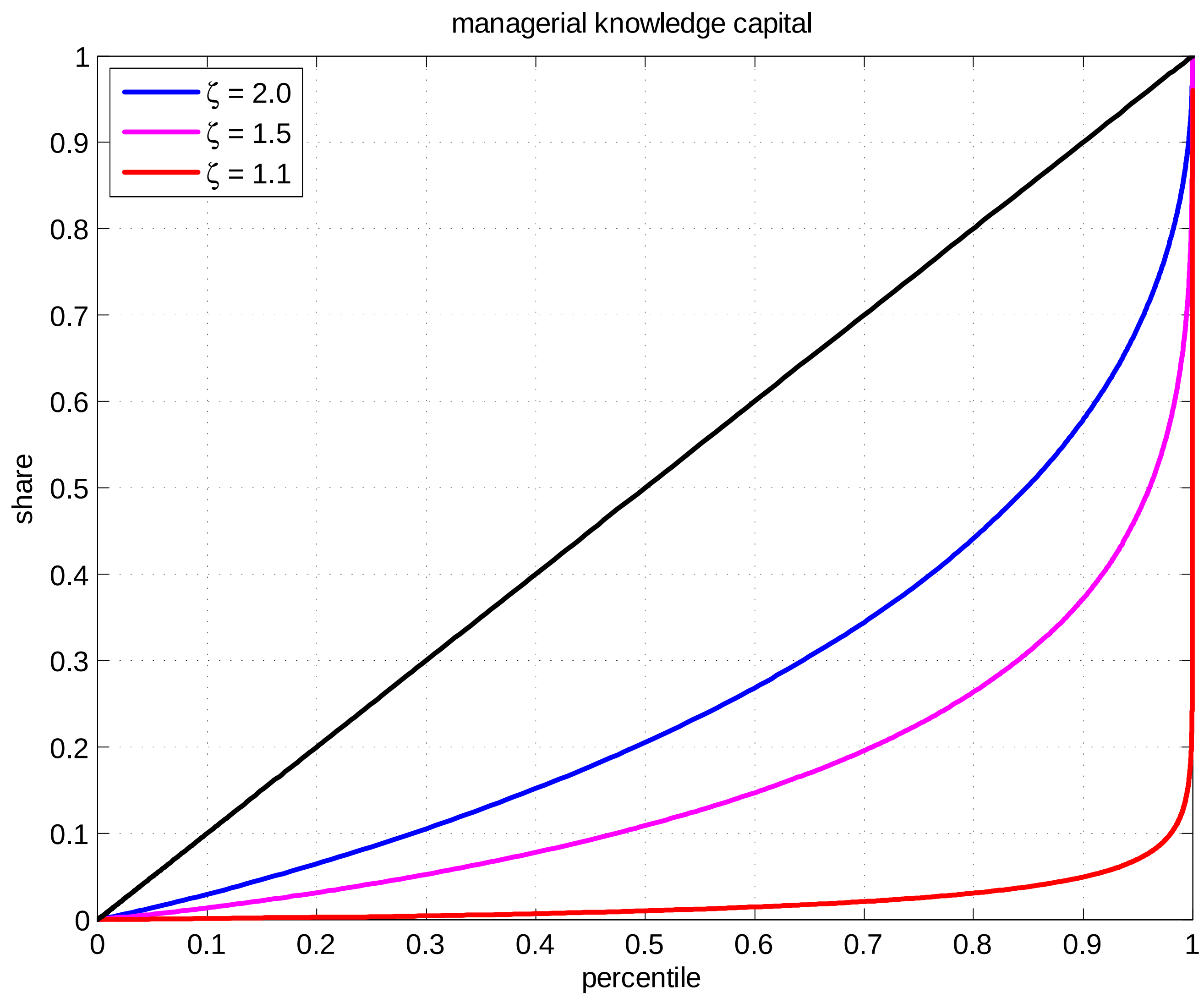
- employment size distribution of firms:

$$\zeta = 1.06$$

▶ these are very different distributions

▶ need to abandon Cobb-Douglas, or Lucas [1978]

Lorenz curves



but what determines κ?

- simplify to $\delta = 0$ and $\beta = \gamma$, and take z to be state without de-trending
- forward equation

$$\mathrm{D}_t p(t,z) = -\mu \mathrm{D}_z p(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} p(t,z) + \begin{cases} -\gamma p(t,z) & z < x_t \\ +\gamma p(t,z) & z > x_t \end{cases}$$

  – where x_t is the median

- then the right tail

$$R(t,z) = 1 - P(t,z)$$

satisfies

$$\mathrm{D}_t R(t,z) = -\mu \mathrm{D}_z R(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} R(t,z) + \gamma \min\{1 - R(t,z), R(t,z)\}$$

- in the case of random imitation

replace $\min\{1-R, R\}$ by $(1-R)R$

this is a new interpretation of an old equation

$$\mathrm{D}_t f(t,z) = \frac{1}{2}\sigma^2 \mathrm{D}_{zz} f(t,z) + \gamma f(t,z)[1 - f(t,z)]$$

- R.A. Fisher "The Wave of Advance of Advantageous Genes" (1937)
  - $f(t,z)$ is a population density at the location z
  - $\gamma f(t,z)[1 - f(t,z)]$ logistic growth of the population at z
  - random migration gives rise to a "diffusion" term $\frac{1}{2}\sigma^2 \mathrm{D}_{zz} f(t,z)$
- Cavalli-Sforza and Feldman (1981)
  - *Cultural Transmission and Evolution: A Quantitative Approach*
  - Section 1.9 applies Fisher's interpretation to memes (Dawkins [1976])
- these interpretations differ from random copying (f is a density)
  - Staley (2011) also has the random copying interpretation

an important theorem of KPP

- can construct stationary distribution for $z - \kappa t$, for any

$$\kappa \geq \mu + \sigma^2 \sqrt{\frac{\gamma}{\sigma^2/2}}$$

▶ Kolmogorov, Petrovskii, and Piskunov 1937

– and McKean 1975, Bramson 1981, many others

if support $P(0, z)$ *bounded then* $P(t, z - \kappa t)$ *converges for* $\kappa = \mu + \sigma^2 \sqrt{\frac{\gamma}{\sigma^2/2}}$

- right tail $R(t, z) \sim e^{-\zeta z}$, where

$$\zeta = \frac{\kappa - \mu}{\sigma^2} - \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma}{\sigma^2/2}} = \sqrt{\frac{\gamma}{\sigma^2/2}}$$

reaction-diffusion right tail ($\gamma = \beta$)

- forward equation for the right cumulative distribution

$$\mathrm{D}_t R(t,z) = -\mu \mathrm{D}_z R(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} R(t,z) + \gamma Q(R(t,z))$$

  – random imitation

$$Q(R) = (1-R)\,R$$

  – random learning

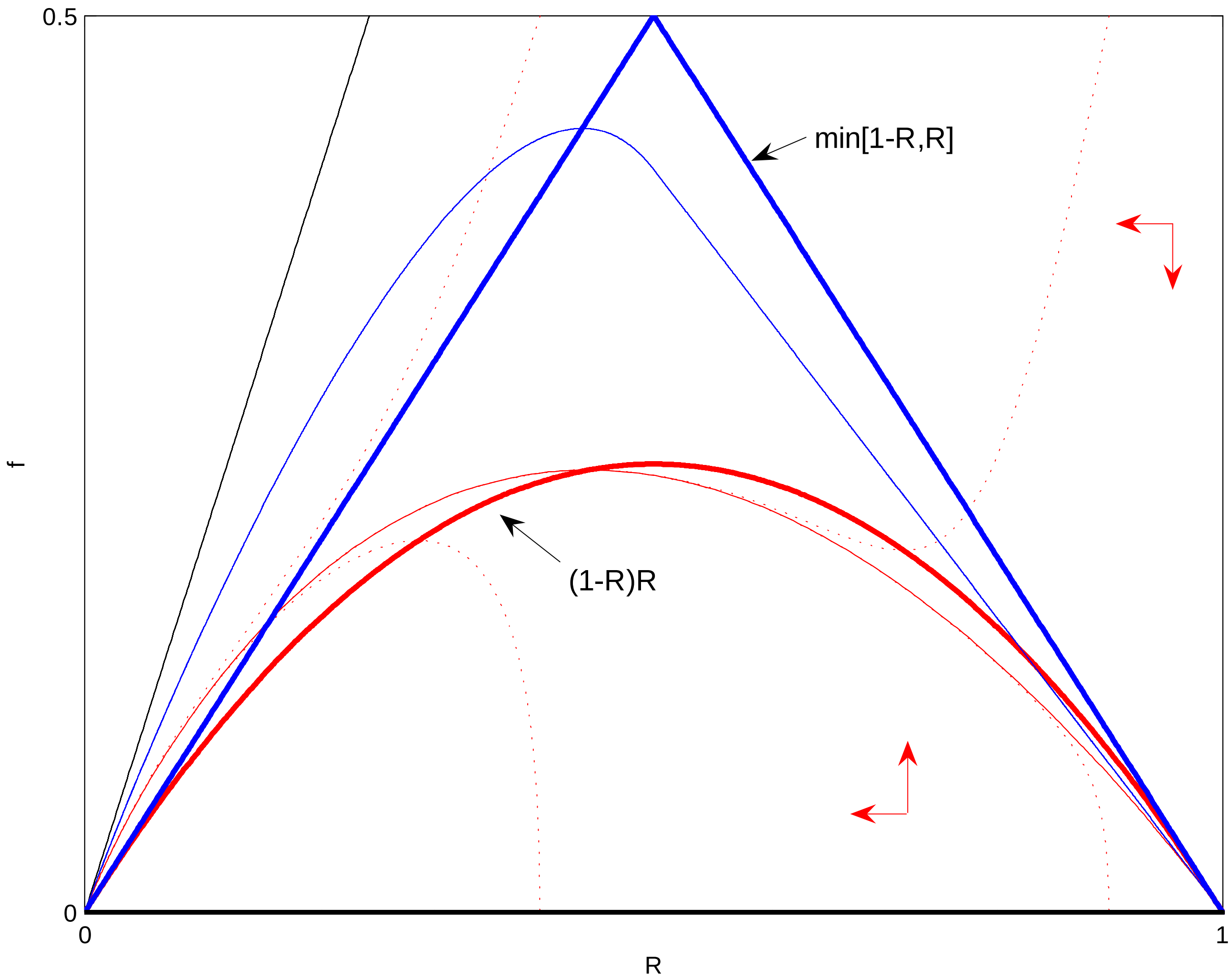
$$Q(R) \;=\; \min\{1-R, R\}$$

- stationary solutions $R(t,z) = R(z-\kappa t)$

$$\mathrm{D}R(z) \;=\; -f(z), \qquad \mathrm{D}f(z) = \frac{-(\kappa-\mu)f(z) + \gamma Q(R(z))}{\sigma^2/2}$$

▶ study phase diagram for $Q(0) = Q(1) = 0$, $\mathrm{D}Q(0) > 0$, and $\mathrm{D}Q(1) < 0$

the stationary distribution given some $\kappa \geq \mu + \sigma^2 \sqrt{\frac{\gamma}{\sigma^2/2}}$



linearize ODE near $R(z) = 0$

- random imitation

$$\left.\frac{\partial R(1-R)}{\partial R}\right|_{R=0} = 1$$

and thus

$$0 \;\approx\; -(\mu-\kappa)\mathrm{D}R(z) + \frac{1}{2}\sigma^2\mathrm{D}^2R(z) + \gamma R(z)$$

- random learning

$$\min\{R, 1-R\} = R \text{ near } R = 0$$

and thus

$$0 \;=\; -(\mu-\kappa)\mathrm{D}R(z) + \frac{1}{2}\sigma^2\mathrm{D}^2R(z) + \gamma R(z)$$

▶ same characteristic equation, with solutions $e^{-\zeta z}$

$$\zeta = \frac{\kappa-\mu}{\sigma^2} \pm \sqrt{\left(\frac{\kappa-\mu}{\sigma^2}\right)^2 - \frac{\gamma}{\sigma^2/2}}$$

summary on growth and inequality

- stationary distributions indexed by $\kappa \geq \mu + \sigma^2 \sqrt{\frac{\gamma}{\sigma^2/2}}$

- these have tail indices

$$\zeta = \frac{\kappa - \mu}{\sigma^2} - \sqrt{\left(\frac{\kappa - \mu}{\sigma^2}\right)^2 - \frac{\gamma}{\sigma^2/2}}$$

  – initial conditions with thicker tail $\Rightarrow$ faster growth

- initial conditions with bounded support select thinnest tail,

$$\kappa = \mu + \sigma^2 \zeta, \quad \zeta = \sqrt{\frac{\gamma}{\sigma^2/2}}$$

  – individual discovery more noisy $\Rightarrow$ faster growth and a thicker tail
  – more frequent learning $\Rightarrow$ faster growth and a thinner tail

- Luttmer (2007)

a misleading continuity

- a small-noise limit for the tail index

$$\zeta = \frac{1}{\sigma^2}\left(\kappa - \mu - \sqrt{(\kappa-\mu)^2 - 2\gamma\sigma^2}\right) \downarrow \frac{\gamma}{\kappa-\mu} \quad \text{as} \quad \sigma^2 \downarrow 0$$

  – same tail index as in economy without individual discovery

  – but this is for fixed $\kappa > \mu$

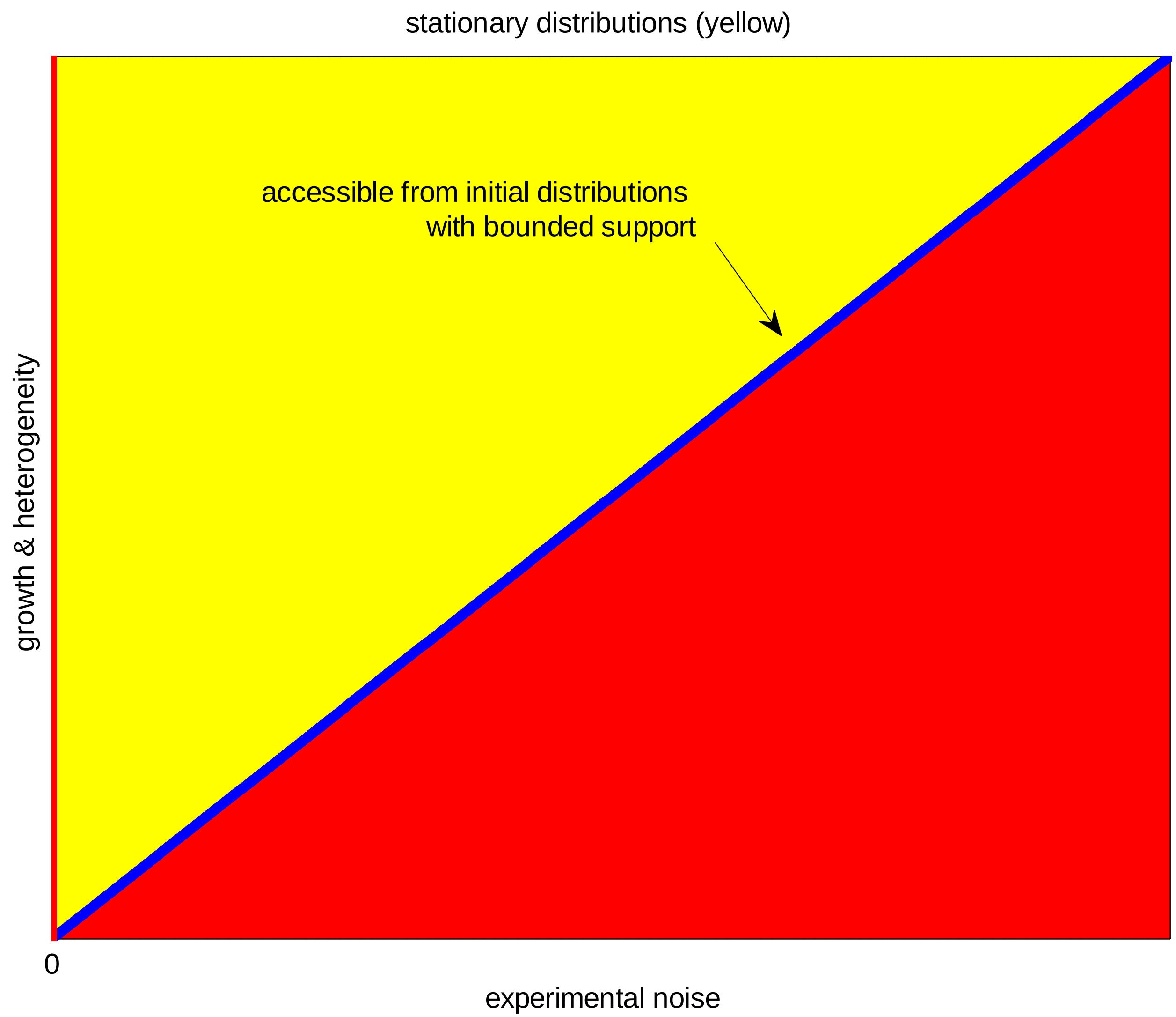
- and KPP implies $\kappa = \mu + \sigma^2\sqrt{\frac{\gamma}{\sigma^2/2}}$

  – hence

$$\lim_{\sigma^2\downarrow 0} \zeta = \lim_{\sigma^2\downarrow 0} \sqrt{\frac{\gamma}{\sigma^2/2}} = \infty$$

  ▶ thin, as in the bounded support example

accessible stationary distributions



concluding remarks

- small noise limit gives echo chamber, not logistic solution
- many stationary distributions and associated growth rates
  - initial conditions with bounded support select one
  - convergence question open for economy with fixed costs

1. random imitation

- more entry can increase growth rate,
- because there is no congestion as there is in teaching

2. one-on-one teaching

- hardwires flow into right tail, independent of entry cost parameters

$$\kappa = \mu + \sigma^2 \zeta_\gamma \qquad \zeta_\gamma = \sqrt{\frac{\gamma - \delta}{\sigma^2/2}} \qquad \zeta_\beta = \zeta_\gamma + \sqrt{\frac{\gamma}{\sigma^2/2}}$$

- the magnification effect