

Bounded Learning from Incumbent Firms

Erzo G.J. Luttmer

University of Minnesota

November 26, 2020

the issue

- in my 2007 QJE paper, I showed

random firm-specific incremental productivity shocks }
entrants learn from surviving incumbents } ⇒

long-run aggregate growth

- icing on the cake: Pareto-like firm size distributions...
- but: the model has a continuum of steady state equilibria with distinct growth rates and firm size distributions
 - the paper had a heuristic argument to select one equilibrium
- this multiplicity issue has arisen again in more recent models of social learning and aggregate growth
- ▶ this paper: a *diagnosis* of the problem, and a new way to obtain a *unique prediction* for long-run growth

the easiest example

- agents randomly select others at rate β and copy if “better”

$$D_t P(t, z) = -\beta P(t, z)[1 - P(t, z)]$$

- ▶ the *unique* solution to this system of logistic ODE is

$$P(t, z) = \frac{1}{1 + \left(\frac{1}{P(0, z)} - 1\right) e^{\beta t}}$$

– but $P(0, z)$ matters a lot...

- ▶ *many stationary* solutions (note that κ is a free parameter)

– linear trends

$$\text{if } P(0, z) = \frac{1}{1 + \left(\frac{1}{P(0, 0)} - 1\right) e^{-(\beta/\kappa)z}} \quad \text{then } P(t, z) = P(0, z - \kappa t)$$

– exponential trends

$$\text{if } P(0, z) = \frac{1}{1 + \left(\frac{1}{P(0, 1)} - 1\right) z^{-\beta/\kappa}} \quad \text{then } P(t, z) = P(0, ze^{-\kappa t})$$

relation to extreme value theory

- suppose N_t follows a pure birth process with birth rate βN_t ,

$$\Pr [N_t = n] = e^{-\beta t} (1 - e^{-\beta t})^{n-1}, \quad n \in \mathbb{N}$$

- N_t independent draws from F ,

$$\Pr [\max \{z_1, \dots, z_{N_t}\} \leq z | N_t] = [F(z)]^{N_t}$$

► this implies

$$\begin{aligned} \Pr [\max \{z_1, \dots, z_{N_t}\} \leq z] &= \sum_{n=1}^{\infty} e^{-\beta t} (1 - e^{-\beta t})^{n-1} [F(z)]^n \\ &= \frac{e^{-\beta t} F(z)}{1 - (1 - e^{-\beta t}) F(z)} = \frac{1}{1 + \left(\frac{1}{F(z)} - 1\right) e^{\beta t}} \end{aligned}$$

– random imitation: recursive way to sample from $F(z) = P(0, z)$

a better model: social learning *and* individual discovery

- two independent standard Brownian motions $W_{1,t}, W_{2,t}$,

$$\mathbb{E} [\max \{ \sigma W_{1,t}, \sigma W_{2,t} \}] = \sigma \sqrt{t} \int_{-\infty}^{\infty} 2x\phi(x)\Phi(x)dx = \sigma \sqrt{t/\pi}$$

- reset to the max at random time $\tau_{j+1} > \tau_j$

$$z_{\tau_{j+1}} = z_{\tau_j} + \sigma \max \{ W_{1,\tau_{j+1}} - W_{1,\tau_j}, W_{2,\tau_{j+1}} - W_{2,\tau_j} \}$$

- reset times arrive randomly at rate $\alpha = 2\beta$

$$\begin{aligned} \mathbb{E} \left[\frac{z_{\tau_{n+1}} - z_{\tau_n}}{\tau_{n+1} - \tau_n} \mid z_{\tau_n} \right] &= \int_0^{\infty} \frac{\sigma \sqrt{\tau/\pi}}{\tau} \times \alpha e^{-\alpha\tau} d\tau \\ &= \sigma \sqrt{\alpha} \\ &= \sigma^2 \times \sqrt{\frac{\beta}{\sigma^2/2}} \end{aligned}$$

what's next

- in this example
 - not just learning from each other but also trying new things
 - research is cumulative, with random increments
 - rather than more draws from the same old distribution
 - and successful improvements are shared
 - no multiplicity issues anywhere in sight
- unlike in a large economy, with only two agents there can be no thick right tail of possible gains from social learning
- the idea in this paper
 - a simple cap on how much entrants can learn from incumbents is enough to get rid of the multiplicity in a large economy
 - this has a well-behaved limit as the cap becomes large
 - the selective replication logic survives and produces long-run growth
- will need to be careful to collect *all* the equilibrium conditions

preferences and factor supplies

- population

$$H_t = H e^{\eta t}, \quad \eta > 0$$

- dynastic preferences over $\{C_t\}_{t \geq 0}$,

$$\mathcal{U}(C) = \int_0^{\infty} e^{-\rho t} H_t \ln(C_t/H_t) dt$$

where

$$C_t = \left(\int e^{z/\varepsilon} c_{z,t}^{1-1/\varepsilon} N(t, dz) \right)^{1/(1-1/\varepsilon)}$$

- crucial parameter restrictions

$$\rho > \eta, \quad \varepsilon > 1$$

- a Roy model for primary factors of production

– labor

$$\mathcal{L}(q_t/w_t) = \int x \iota \{w_t x > q_t y\} d\mathcal{P}(x, y)$$

– entrepreneurial services

$$\mathcal{E}(q_t/w_t) = \int y \iota \{w_t x < q_t y\} d\mathcal{P}(x, y)$$

product market equilibrium

- demand curves for differentiated goods

$$c_{z,t} = \left(\frac{p_{z,t}}{P_t} \right)^{-\varepsilon} e^z C_t, \quad P_t = \left(\int e^z p_{z,t}^{1-\varepsilon} N(t, dz) \right)^{1/(1-\varepsilon)}$$

- linear labor-only technology with a unit productivity
- monopolistic competition implies the Lerner price

$$p_{z,t} = \frac{w_t}{1 - 1/\varepsilon}$$

- together with the price index P_t , this implies

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon} \right) (e^{Z_t} N_t)^{1/(\varepsilon-1)}, \quad e^{Z_t} = \frac{1}{N_t} \int e^z N(t, dz)$$

where $N_t = N(t, \infty)$

- gains from variety via N_t
- productivity distribution $N(t, z)/N_t$ will be a traveling wave

key product market implications

- firm profits and use of labor

$$\begin{bmatrix} v_{z,t} \\ w_t l_{z,t} \end{bmatrix} = \begin{bmatrix} 1/\varepsilon \\ 1 - 1/\varepsilon \end{bmatrix} e^{z-Z_t} \times \frac{P_t C_t}{N_t}$$

– ... this is a Red Queen environment

- aggregate production labor L_t is

$$L_t = \int l_{z,t} N(t, dz)$$

- aggregate value of production labor

$$w_t L_t = \left(1 - \frac{1}{\varepsilon}\right) P_t C_t$$

- average profits in units of labor

$$\frac{1}{w_t N_t} \int v_{z,t} N(t, dz) = \frac{1}{\varepsilon - 1} \frac{L_t}{N_t}$$

productivity dynamics and firm values

- the fundamental assumption—

$$dz_t = \theta dt + \sigma dW_t$$

- firm-specific random walks, with a trend $\theta \in (-\infty, \infty)$
 - for example, $\theta = -\frac{1}{2}\sigma^2$, so that e^{z_t} is a positive martingale
 - there is always a *non-trivial* new set of *modifications* to try
 - of course, we could, instead, run out of ideas...
- firm continuation requires $\phi > 0$ units of labor per unit of time
 - given $z_t = z$, the value of a firm is

$$\frac{V(t, z)}{P_t} = \max_{\tau} E_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \times \frac{C_t/H_t}{C_s/H_s} \left(\frac{v_{z_s, s}}{P_s} - \frac{\phi w_s}{P_s} \right) ds \right]$$

- in units of composite consumption
 - this uses the logarithmic utility assumption
- optimal to exit when $z_t \leq b_t$, for some b_t to be determined

the knowledge diffusion assumption

- flow of entry opportunities $\mathcal{E}(q_t/w_t)H_t$
- an entry opportunity is
 - a random draw from the incumbent population
 - can copy the z of a randomly sampled firm
 - but only if $z \in [b_t, b_t + \Delta]$, for some $\Delta \in (0, \infty)$
 - implicit: “everyone knows” b_t
- alternative knowledge diffusion assumption in Luttmer [2015]:
 - amount of copying limited by “teaching capacity” of incumbents
- value of an entry opportunity
$$q_t = \left(\int_{b_t}^{\infty} N(t, dz) \right)^{-1} \int_{b_t}^{b_t + \Delta} V(t, z) N(t, dz)$$
 - draws from $(b_t + \Delta, \infty)$ go to waste
- the Roy model determines $\mathcal{E}(\cdot)$

the Kolmogorov forward equation

- at any $z \in (b_t, b_t + \Delta)$

$$\text{flow of entrants at } z = \mathcal{E} \left(\frac{q_t}{w_t} \right) H_t \times \frac{n(t, z)}{N_t} = \alpha_t n(t, z)$$

where α_t is the *attempted entry rate*

$$\alpha_t = \frac{\mathcal{E}(q_t/w_t)}{N_t/H_t}$$

- the Kolmogorov forward equation is

$$D_t n(t, z) = -\theta D_z n(t, z) + \frac{1}{2} \sigma^2 D_{zz} n(t, z) + \alpha_t n(t, z),$$

for $z \in (b_t, b_t + \Delta)$ and

$$D_t n(t, z) = -\theta D_z n(t, z) + \frac{1}{2} \sigma^2 D_{zz} n(t, z),$$

for $z \in (b_t + \Delta, \infty)$

- immediate exit at b_t means that

$$n(t, b_t) = 0$$

- the density should be differentiable at $b_t + \Delta$

constructing a BGP—a rough outline

- conjecture that Z_t grows at some equilibrium rate $\theta - \mu$
- a steady state supply of firms

$$\frac{N}{H} = \mathcal{S} \left(\frac{q}{w}; \mu \right) \quad (1)$$

– from entrepreneurial incentives, life cycle of firms

- a steady state demand for firms

$$\frac{N}{H} = \mathcal{D} \left(\frac{q}{w}; \mu \right) \quad (2)$$

– how many firms needed to employ all workers?

- a present-value condition

$$\frac{q}{w} = \mathcal{Q}(\mu) \quad (3)$$

- given μ ,

– there is a unique stationary distribution if $\Delta \in (0, \infty)$,

– but a continuum if $\Delta = \infty$

balanced growth—(A) aggregate conjectures

- conjecture a common growth rate for
 - (i) per-capita consumption
 - (ii) the real wage
 - (iii) average real variable profits

- since

$$\left[\int v_{z,t} N(t, dz), w_t L_t \right] = \left[\frac{1}{\varepsilon}, 1 - \frac{1}{\varepsilon} \right] P_t C_t$$

and

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon} \right) (e^{Z_t} N_t)^{1/(\varepsilon-1)}, \quad e^{Z_t} = \frac{1}{N_t} \int e^z N(t, dz),$$

this implies

$$\frac{L_t}{H_t} = \frac{L}{H'} \quad \frac{N_t}{H_t} = \frac{N}{H}$$

and

$$Z_t = Z + (\theta - \mu)t$$

for some L/H , N/H and μ to be determined

what does μ do?

- for individual firms, μ is the drift of $z_t - Z_t$,

$$d(z_t - Z_t) = \mu dt + \sigma dW_t$$

– and $l_{z,t} \propto e^{z_t - Z_t}$

- recall that

$$\left[\frac{L_t}{H_t}, \frac{N_t}{H_t} \right] = \left[\frac{L}{H}, \frac{N}{H} \right], \quad H_t = H e^{\eta t}$$

and

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right) (e^{Z_t} N_t)^{1/(\varepsilon-1)}, \quad \frac{w_t L_t}{P_t H_t} = \left(1 - \frac{1}{\varepsilon}\right) \frac{C_t}{H_t}$$

- so then $Z_t = Z + (\theta - \mu)t$ implies

$$\left[\frac{w_t}{P_t}, \frac{C_t}{H_t} \right] = \left[\frac{w}{P}, \frac{C}{H} \right] e^{\kappa t}$$

where

$$\kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1}$$

the implied value function

- recall that

$$\frac{v_{z,t}}{w_t} = e^{z-Z_t} \times \frac{L_t/N_t}{\varepsilon - 1}$$

and C_t/H_t and w_t/P_t grow at a common rate

- this yields

$$\frac{V(t, z)}{P_t} = \frac{\phi w_t}{P_t} \times \max_{\tau} E_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \left(\frac{e^{z_s - Z_s} L}{(\varepsilon - 1)\phi N} - 1 \right) ds \right],$$

where

$$z_s - Z_s = z - Z_t + \mu(s - t) + \sigma(W_s - W_t) \text{ for all } s \geq t$$

- this must be of the form

$$\frac{V(t, z)}{P_t} = \frac{\phi w_t}{P_t} \times U(y), \quad e^y = \frac{e^{z - Z_t} L}{(\varepsilon - 1)\phi N}$$

- ▶ will need the equilibrium μ to satisfy

$$\mu + \frac{1}{2}\sigma^2 < \rho$$

the solution for $V(t, z)$ is

$$\frac{V(t, z)}{P_t} = \frac{\phi w_t}{P_t} \times U(y), \quad e^y = \frac{e^{z-Z_t L}}{(\varepsilon - 1)\phi N}$$

• where

$$U(y) = \begin{cases} 0 & , y < a \\ \frac{1}{\rho} \frac{\xi}{1+\xi} \left(e^{y-a} - 1 - \frac{1-e^{-\xi(y-a)}}{\xi} \right) & , y > a \end{cases}$$

– and the exit threshold a is determined by

$$e^a = \frac{\xi}{1+\xi} \left(1 - \frac{1}{\rho} \left(\mu + \frac{1}{2}\sigma^2 \right) \right)$$

and

$$\xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} \right)^2 + \frac{\rho}{\sigma^2/2}}$$

• so the only equilibrium variable that affects $U(\cdot)$ is μ

key properties of the value function

Lemma *The value function is well defined if and only if $\mu + \frac{1}{2}\sigma^2 < \rho$. Given this restriction, it has the following properties:*

- (i) *The value function is strictly increasing and unbounded in $y > a$.*
- (ii) *The exit threshold is strictly decreasing in μ ,*

$$\lim_{\mu \rightarrow -\infty} a = 0, \text{ and } \lim_{\mu \uparrow \rho - \sigma^2/2} a = -\infty.$$

- (iii) *For any $u \in (0, \infty)$ or $y \in (-\infty, \infty)$,*

$$\lim_{\mu \rightarrow -\infty} U(a + u) = 0, \quad \lim_{\mu \uparrow \rho - \sigma^2/2} U(a + u) \in (0, \infty), \quad \lim_{\mu \uparrow \rho - \sigma^2/2} U(y) = \infty,$$

and $U(a + u)$ is increasing in μ .

- the time- t exit threshold for firm of type z must then be

$$b_t = b + (\theta - \mu)t, \quad e^a = \frac{e^{b-Z} L}{(\varepsilon - 1)\phi N}$$

- so the gap $Z_t - b_t = Z - b$ is constant over time

balanced growth—(B) the stationarity conjecture

- strengthen $Z_t - b_t = Z - b$ to time-invariance of the cross-sectional distribution of $z - b_t$,

$$n(t, z) = N_t f(z - b_t), \quad z \in (b_t, \infty)$$

- the definition of Z_t implies a consistency condition

$$e^{Z-b} = \int_0^\infty e^u f(u) du$$

- the value q_t/w_t of an entry opportunity becomes

$$\frac{q_t}{w_t} = \phi \int_0^\Delta U(a+u) f(u) du$$

– so q_t/w_t , $\mathcal{E}(q_t/w_t)$, and $\mathcal{L}(q_t/w_t)$ are constant over time

- since $N_t/H_t = N/H$, this means that $\alpha_t = \alpha$, and

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right)$$

– this is the *steady state supply* of firms as a function of q/w

– but α remains to be determined

the stationary KFE

- recall

$$z - b_t = z - Z_t + Z_t - b_t = z - Z_t + Z - b$$

and

$$d(z_t - Z_t) = \mu dt + \sigma dW_t$$

- the Kolmogorov forward equation for $n(t, z) = N_t f(z - b_t)$ becomes

$$\eta f(u) = -\mu Df(u) + \frac{1}{2}\sigma^2 D^2 f(u) + \alpha f(u)$$

for $u \in (0, \Delta)$ and

$$\eta f(u) = -\mu Df(u) + \frac{1}{2}\sigma^2 D^2 f(u)$$

for $u \in (\Delta, \infty)$

- and $f(\cdot)$ is supposed to integrate to 1
- the boundary conditions are $f(0) = 0 = f(\infty)$ and
 - continuity at Δ
 - differentiability at Δ

an accounting identity implied by the KFE

- integrating the differential equation over $(0, \infty)$ yields

$$\alpha \int_0^{\Delta} f(u) du = \eta + \frac{1}{2} \sigma^2 Df(0)$$

- need to use the boundary conditions
- this fails if $f(\cdot)$ not differentiable at Δ
- we also know that the exit rate is given by $\frac{1}{2} \sigma^2 Df(0)$
- this confirms the basic steady state accounting condition

successful entry rate

=

population growth rate + exit rate

- ▶ can infer α from $f(\cdot)$, without knowing μ

clearing the labor market

- the employment size of firms scales with $e^u = e^{z-b}$
- recall the consistency condition

$$e^{z-b} = \int_0^{\infty} e^u f(u) du$$

and that the threshold b for z is determined by

$$e^a = \frac{e^{b-z} L}{(\varepsilon - 1)\phi N}$$

- the labor market clearing condition

$$\mathcal{L}\left(\frac{q}{w}\right) H = \phi N + L$$

can then be written as

$$\frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}(q/w)}{1 + (\varepsilon - 1) \int_0^{\infty} e^{a+u} f(u) du}$$

- the *steady state demand* for firms as a function of q/w
- but a and $f(\cdot)$ remain to be determined

recap

► steady state “temporary equilibrium” conditions

- the supply of firms

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E} \left(\frac{q}{w} \right)$$

- the demand for firms

$$\frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}(q/w)}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) du}$$

- market clearing delivers q/w

► a perfect foresight present value condition

$$\frac{q}{w} = \phi \int_0^\Delta U(a+u) f(u) du$$

- the two versions of q/w must match $\Rightarrow \mu$

● in the background

- the Bellman equation gives a and $U(\cdot)$ as functions only of μ
- the Kolmogorov forward equation will show that *both* α and $f(\cdot)$ are functions only of μ , as long as $\Delta \in (0, \infty)$

solving the KFE—1: characteristic roots

- ▶ KEY RESULT—provided $\Delta < \infty$; not true when $\Delta = \infty$,
 - for any $\mu \in (-\infty, \infty)$, there is *precisely one* attempted entry rate $\alpha > 0$ for which it is possible to solve the KFE
 - so α and $f(\cdot)$ are pinned down jointly as a function of μ
- on $(0, \Delta)$, a solution of the form $e^{-\chi z}$ implies $\chi \in \{\chi_-, \chi_+\}$,

$$\chi_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 - \frac{\alpha - \eta}{\sigma^2/2}}$$

- on (Δ, ∞) , a solution of the form $e^{-\zeta z}$ gives $\zeta \in \{\zeta_-, \zeta_+\}$,

$$\zeta_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- then $\eta > 0$ implies $\zeta_- < 0 < \zeta_+$, irrespective of the sign of μ
 - this forces $f(u) \propto e^{-\zeta_+ u}$ on (Δ, ∞) , scale to be determined
- the χ_{\pm} may be real or complex; depends on attempted entry rate α

solving the KFE—2: imposing differentiability at Δ

- FACT: no way to enforce differentiability at Δ if the χ_{\pm} are real
- suppose α large enough so that the roots χ_{\pm} are complex

– let $\psi = \text{Re}(\chi_+)$ and $\omega = \text{Im}(\chi_+)$,

$$\psi = -\frac{\mu}{\sigma^2}, \quad \omega = \sqrt{\frac{\alpha - \eta}{\sigma^2/2} - \psi^2}$$

– requiring $f(u)$ to be real forces

$$f(u) = [A \cos(\omega u) + B \sin(\omega u)] e^{-\psi u}$$

– imposing $f(0) = 0$ forces $A = 0$

– imposing continuity at $u = \Delta$ yields

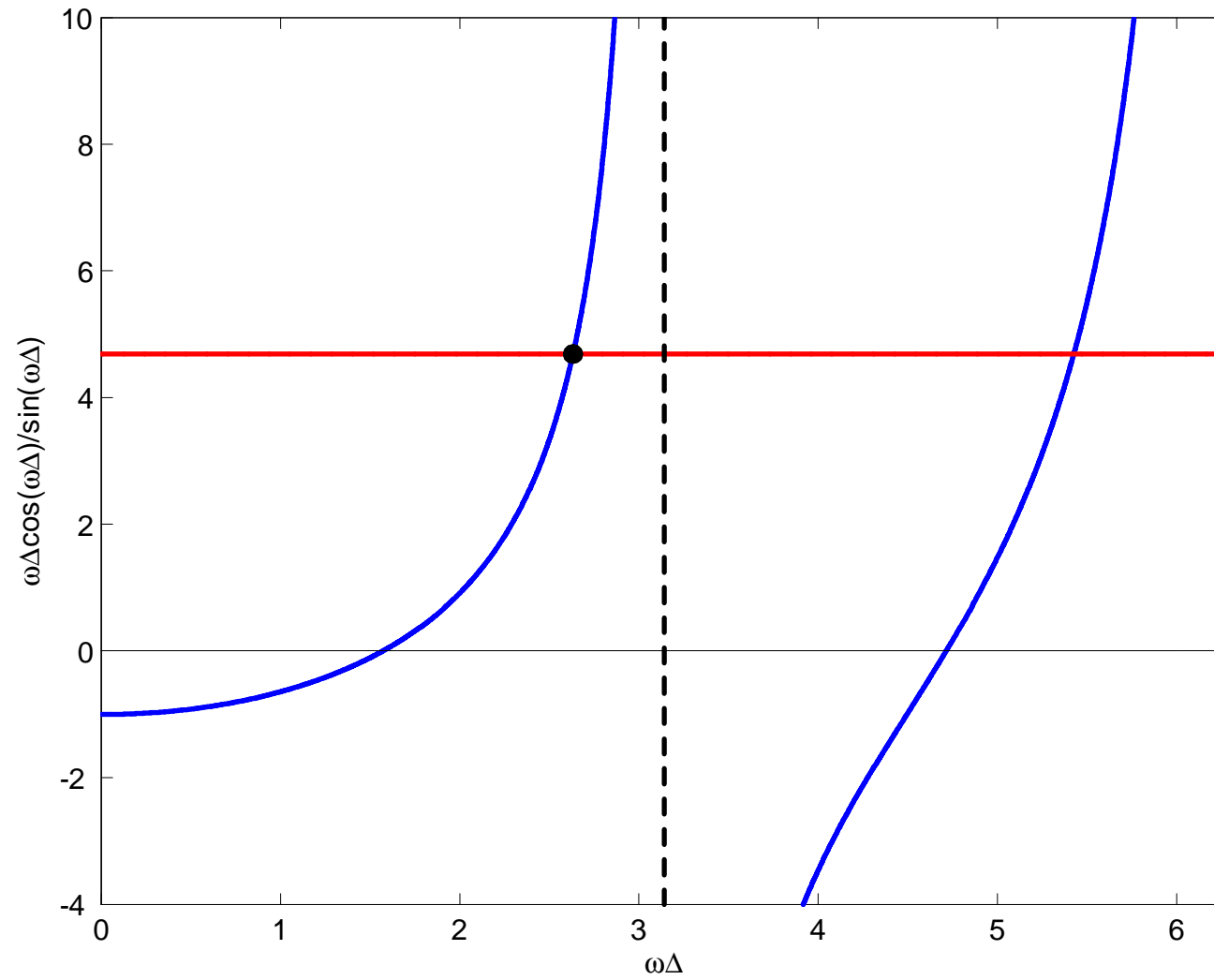
$$f(u) = B \begin{cases} \sin(\omega u) e^{-\psi u}, & u \in [0, \Delta], \\ \sin(\omega \Delta) e^{-\psi \Delta} e^{-\zeta_+(u-\Delta)}, & u \in [\Delta, \infty) \end{cases}$$

– this is positive on $(0, \Delta)$ if and only if $\omega \Delta \in (0, \pi)$

– imposing differentiability at $u = \Delta$ forces

$$-\frac{\cos(\omega \Delta)}{\sin(\omega \Delta)/(\omega \Delta)} = \Delta \sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}.$$

the solution for ω



solving the KFE—3: the implied attempted entry rate

- recall $\psi = \text{Re}(\chi_+)$ and $\omega = \text{Im}(\chi_+)$,

$$\psi = -\frac{\mu}{\sigma^2}, \quad \omega = \sqrt{\frac{\alpha - \eta}{\sigma^2/2} - \psi^2}$$

- differentiability at Δ forces

$$-\frac{\cos(\omega\Delta)}{\sin(\omega\Delta)/(\omega\Delta)} = \Delta \sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}$$

- LHS is increasing in $\omega\Delta$, ranging throughout $(-1, \infty)$
 - unique solution $\omega \in (0, \pi/\Delta)$
 - this solution is increasing in ψ^2 , decreasing in Δ
- inverting the definition of ω delivers the attempted entry rate

$$\alpha = \eta + \frac{1}{2}\sigma^2 (\omega^2 + \psi^2)$$

- so α is increasing in $\psi^2 \propto \mu^2$
- in particular, $\mu \rightarrow -\infty$ gives $\alpha \rightarrow \infty$

the large- Δ limiting distribution

Lemma *The stationary distribution function converges to*

$$\lim_{\Delta \rightarrow \infty} F(u) = \begin{cases} 0 & , \psi \in (-\infty, 0] \\ 1 - (1 + \psi u)e^{-\psi u} & , \psi \in (0, \infty) \end{cases}$$

for any $u \in [0, \infty)$. The truncated mean of e^u behaves like

$$\lim_{\Delta \rightarrow \infty} \int_0^\Delta e^u dF(u) = \begin{cases} \infty & , \psi \in (-\infty, 1] \\ \left(\frac{\psi}{\psi-1}\right)^2 & , \psi \in (1, \infty) \end{cases}$$

The attempted entry rate satisfies

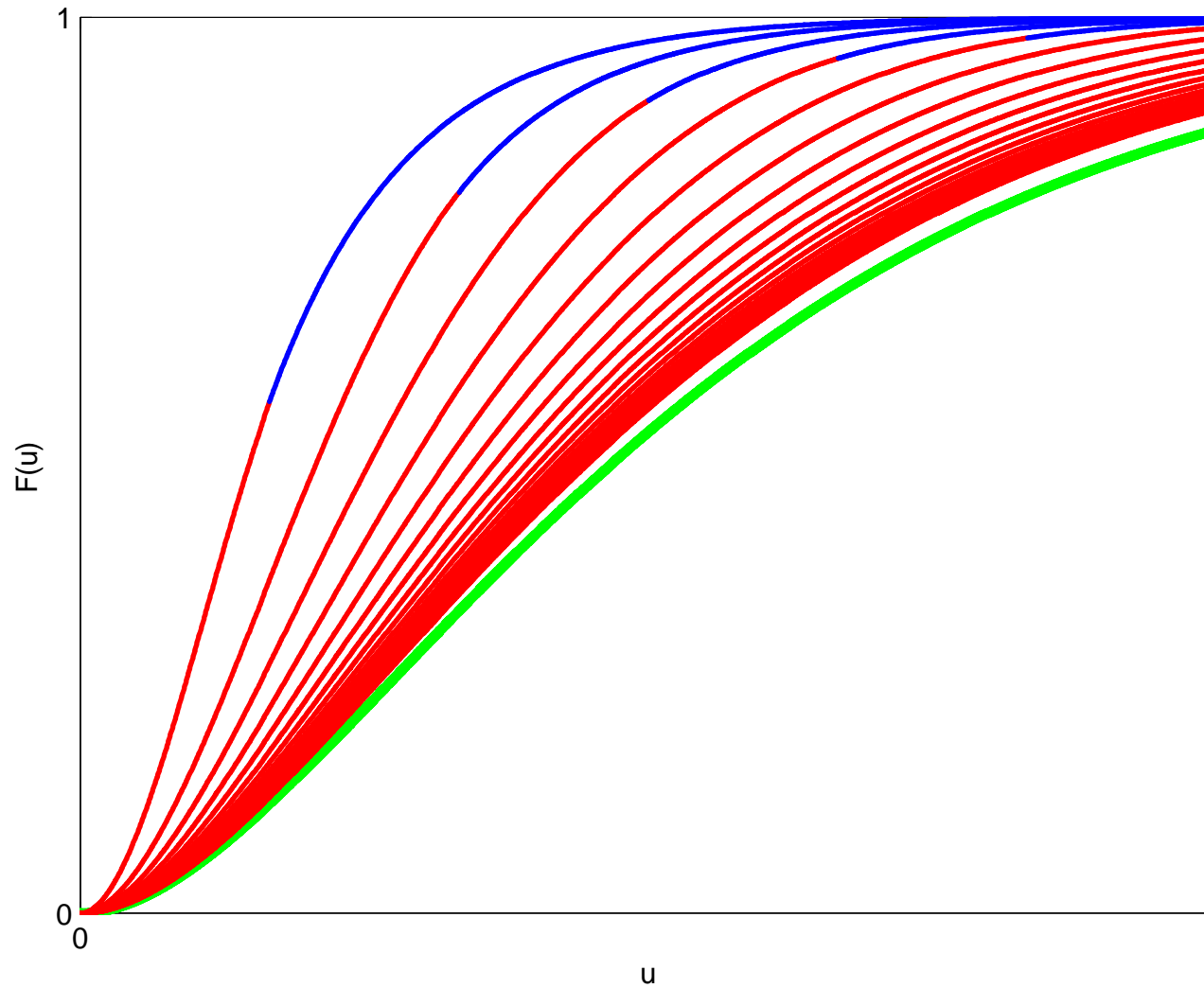
$$\lim_{\Delta \rightarrow \infty} \alpha = \eta + \frac{1}{2}\sigma^2\psi^2$$

- if $\psi > 0$ and $\Delta \in (0, \infty)$, then the right tail index is

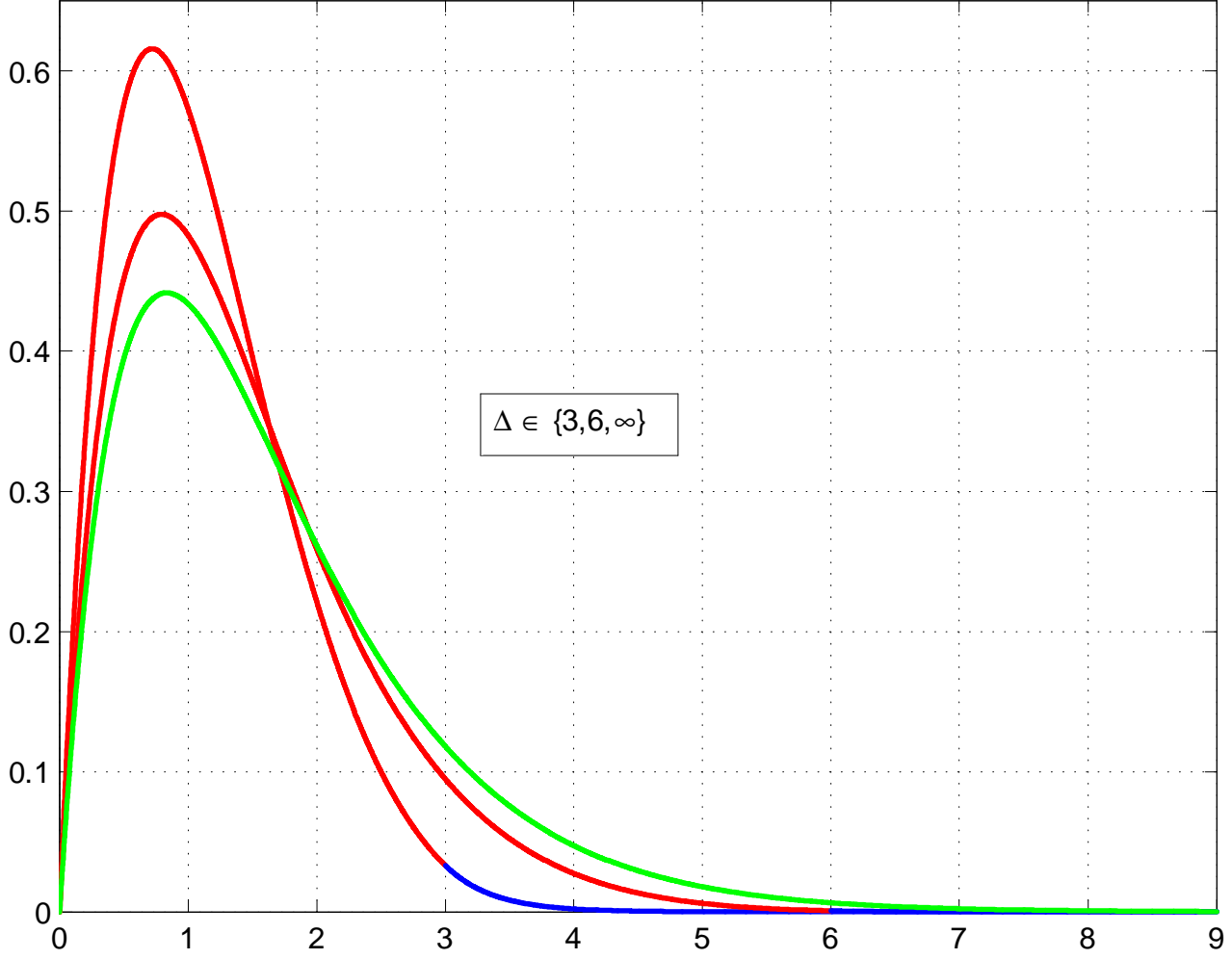
$$\zeta_+ = \psi + \sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}} > 2\psi \tag{!}$$

- so the tail index is discontinuous at $\Delta = \infty$

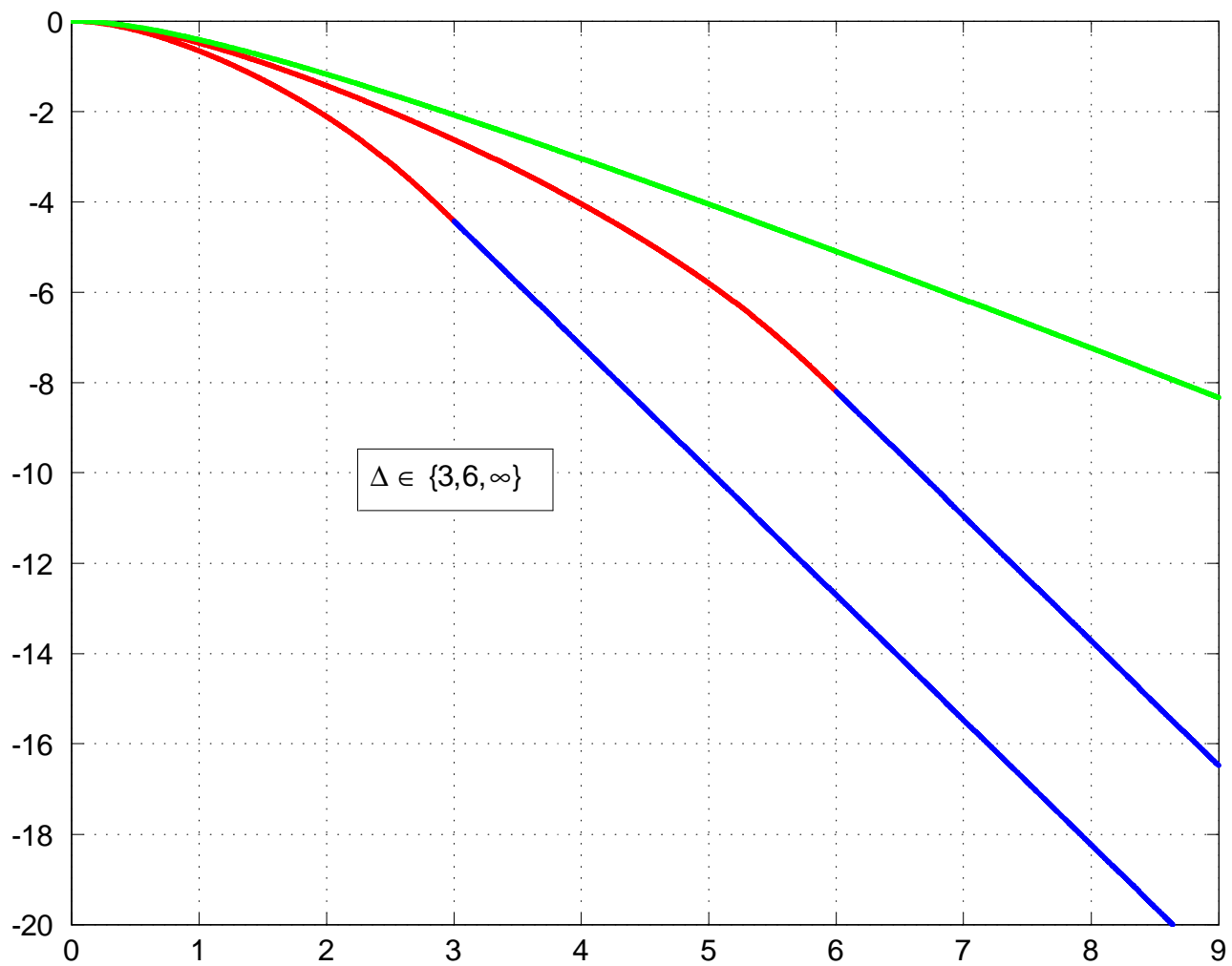
**the convergence is monotone in the sense
of first-order stochastic dominance**



densities



log-log plot of distributions



the equations for balanced growth

- clearing the steady state market for firms gives

$$\frac{\mathcal{E}(q/w)}{\mathcal{L}(q/w)} = \frac{\alpha/\phi}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) du} \quad (1)$$

- the relative price q/w must also satisfy

$$\frac{q}{w} = \phi \int_0^\Delta U(a+u) f(u) du \quad (2)$$

- in the background
 - the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

- the KFE yields

$$\mu \mapsto \{\alpha, f(\cdot)\}$$

- if the initial density satisfies $n(0, z)/H = (N/H)f(z - b)$ for some b , then the economy is on a balanced growth path

an *irregular* special case: perfectly elastic factor supplies

- this fixes q/w , and then μ is determined by

$$\frac{q}{w} = \phi \int_0^{\Delta} U(a+u) f(u) du \quad (*)$$

- the firm value $U(a+u)$ is finite if and only if

$$\rho > \mu + \frac{1}{2}\sigma^2$$

- market clearing still requires finite average employment

$$\int_0^{\infty} e^{a+u} f(u) du < \infty$$

- this is the same as $\zeta_+ > 1$, or

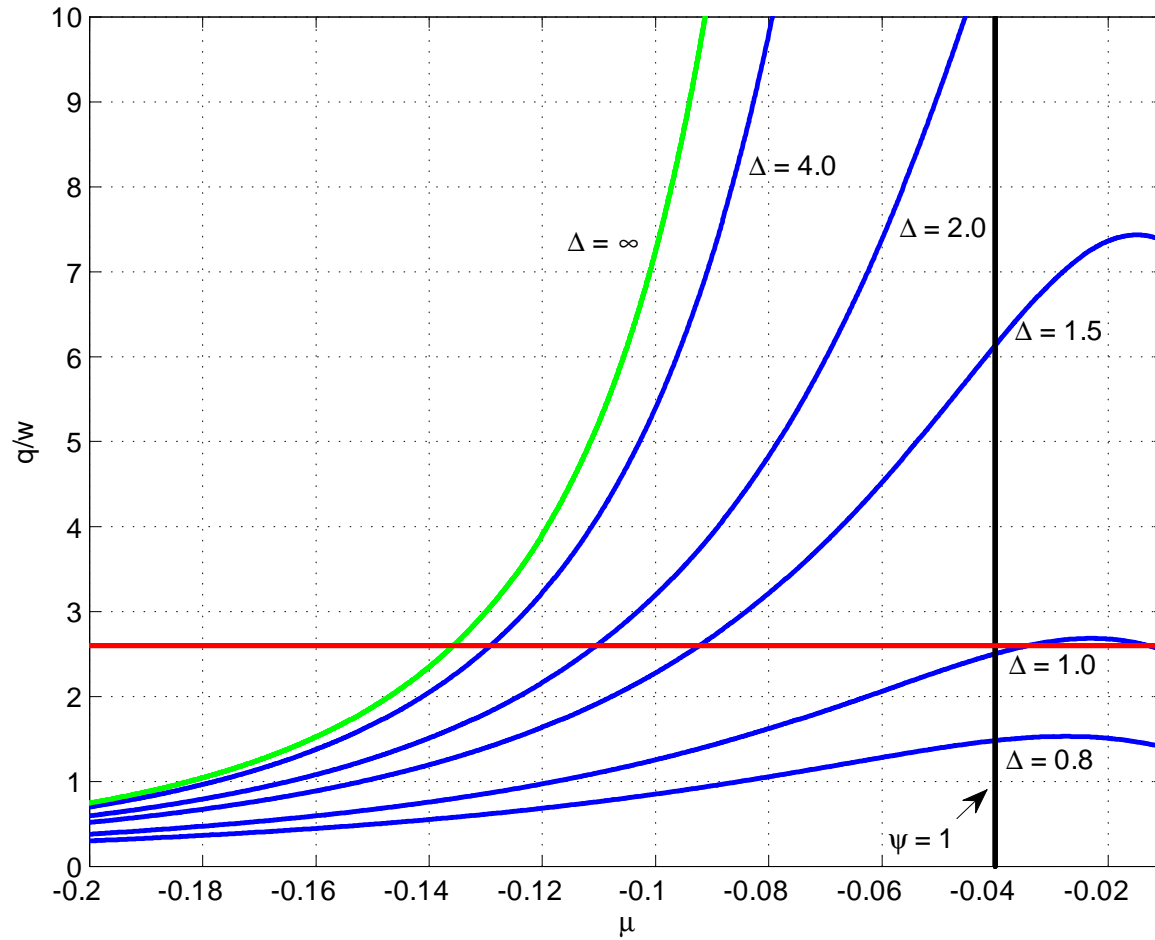
$$\eta > \mu + \frac{1}{2}\sigma^2$$

- may not have a BGP

- because finite dynastic utility requires $\rho > \eta$, and then...

- the RHS of (*) is bounded on $\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\}$

perfectly elastic factor supplies



- value of entry, bounded on the domain $\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\}$
- there may not be a BGP when $\Delta \in (0, \infty)$

a regular special case: perfectly inelastic factor supplies

- this fixes \mathcal{E}/\mathcal{L} , and μ is determined by

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{\alpha/\phi}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) du} \quad (*)$$

- the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

- the KFE yields

$$\mu \mapsto \{\alpha, f(\cdot)\}$$

- the RHS of (*) ranges throughout $(0, \infty)$ on $\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\}$

- since $\rho > \eta$, and since mean employment must be finite

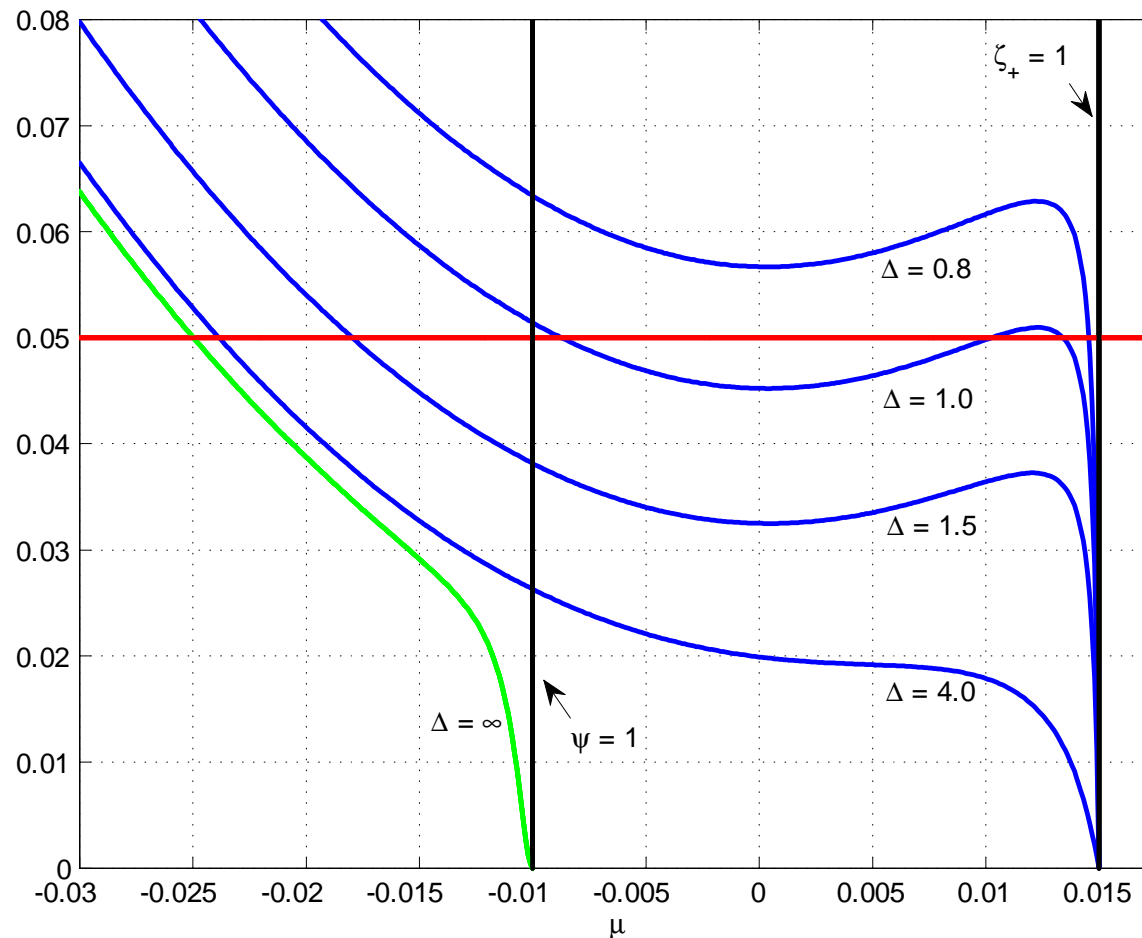
$$\mu + \frac{1}{2}\sigma^2 < \eta < \rho$$

- the relative price q/w is determined by

$$\frac{q}{w} = \phi \int_0^\Delta U(a+u) f(u) du$$

which is well defined by construction

perfectly inelastic factor supplies



- relative factor demands, on the domain $\{\mu : \zeta_+ > 1\} = \{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\}$
- note that $\mu > 0$ is possible: the trend of $\ln(Z_t)$ may be below θ when $\Delta \in (0, \infty)$

existence of a BGP

Proposition A *Assume the relative factor supply curve $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ is continuous. When Δ is large enough, a finite- Δ economy must have at least one equilibrium, and every equilibrium must satisfy $\psi_\Delta > 1$.*

Proposition B *Consider the equilibrium conditions (1)-(2) with $\psi > 1$, $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$, $f(u) = \psi^2ue^{-\psi u}$, and $\Delta = \infty$. The right-hand side of (1) is strictly increasing in $\psi \in (1, \infty)$ and maps onto $(0, \infty)$. The right-hand side of (2) is strictly decreasing in $\psi \in (1, \infty)$ and maps onto $(0, \infty)$. When the relative factor supply curve $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ is continuous, this implies a unique balanced growth path, denoted by $\psi_\infty \in (1, \infty)$.*

Proposition C *Assume the relative factor supply curves $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ are continuous and let $E_\Delta \subset \{\psi : \zeta_+ > 1\}$ be the set of equilibria for the Δ economy. Then $\sup_{\psi \in E_\Delta} |\psi - \psi_\infty|$ converges to zero as Δ becomes large.*

Corollary *The economy grows faster than θ when Δ is large enough.*

solving the KFE for $\Delta = \infty$

- the KFE simplifies to

$$\eta f(u) = -\mu Df(u) + \frac{1}{2}\sigma^2 D^2 f(u) + \alpha f(u)$$

for all $u \in (0, \infty)$, with the boundary conditions $f(0) = 0 = f(\infty)$

- solved by linear combinations of $e^{-\chi_- u}$ and $e^{-\chi_+ u}$,

$$\chi_{\pm} = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \quad \psi = -\frac{\mu}{\sigma^2}$$

- complex χ_{\pm} yields a positive density only on a bounded interval
- if $\alpha \in [0, \eta]$, then $\chi_- \leq 0 \leq \chi_+$, which rules out $f(0) = 0 = f(\infty)$

- need α to satisfy $0 < (\alpha - \eta)/(\sigma^2/2) \leq \psi^2$ and $\psi > 0$, and then

$$f(u) = \frac{\chi_+ \chi_-}{\chi_+ - \chi_-} \times (e^{-\chi_- u} - e^{-\chi_+ u}),$$

for all $u \in [0, \infty)$

- this was the density obtained in Luttmer [2007]

- if $(\alpha - \eta)/(\sigma^2/2) \uparrow \psi^2$ this matches the large- Δ limit $f(u) = \psi^2 u e^{-\psi u}$

balanced growth paths

- steady state market clearing requires

$$\frac{\mathcal{E}(q/w)}{\mathcal{L}(q/w)} = \frac{\alpha/\phi}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) du}$$

- the relative price q/w must also satisfy

$$\frac{q}{w} = \phi \int_0^\infty U(a+u) f(u) du \quad (*)$$

- in the background

- the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

- the KFE yields

$$(\mu, \alpha) \mapsto f(\cdot) \quad (!)$$

rather than $\mu \mapsto \{\alpha, f(\cdot)\}$

- aside: in Luttmer [2007], the factor supplies are perfectly elastic, and (*) forces the mean of e^u to be finite

feasible α given μ

- recall

$$\chi_{\pm} = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \quad \psi = -\frac{\mu}{\sigma^2}$$

- need χ_{\pm} real and $\chi_- > 1$, which is the same as

$$1 < \psi, \quad 2\psi - 1 < \frac{\alpha - \eta}{\sigma^2/2} \leq \psi^2$$

- note that on this domain

$$\frac{\partial \chi_-}{\partial \psi} = \frac{\partial}{\partial \psi} \left(\psi - \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}} \right) < 0$$

- holding fixed α , a *lower* firm growth rate μ implies a thicker tail
 - *bootstrap logic*: a lower μ tends to generate more exit; without more entry, must have a thicker tail so it takes more firms longer to reach the exit barrier
- recall that the limiting BGP as $\Delta \rightarrow \infty$ is $\psi_{\infty} > 1$, and
 - the attempted entry rate is at its ($\Delta = \infty$) upper bound

$$\alpha_{\infty} = \eta + \frac{1}{2}\sigma^2\psi_{\infty}^2$$

constructing alternative BGP

- can construct BGP for any $\psi > \psi_\infty$
- so there is no upper bound on how fast the economy can grow
- the key calculation is

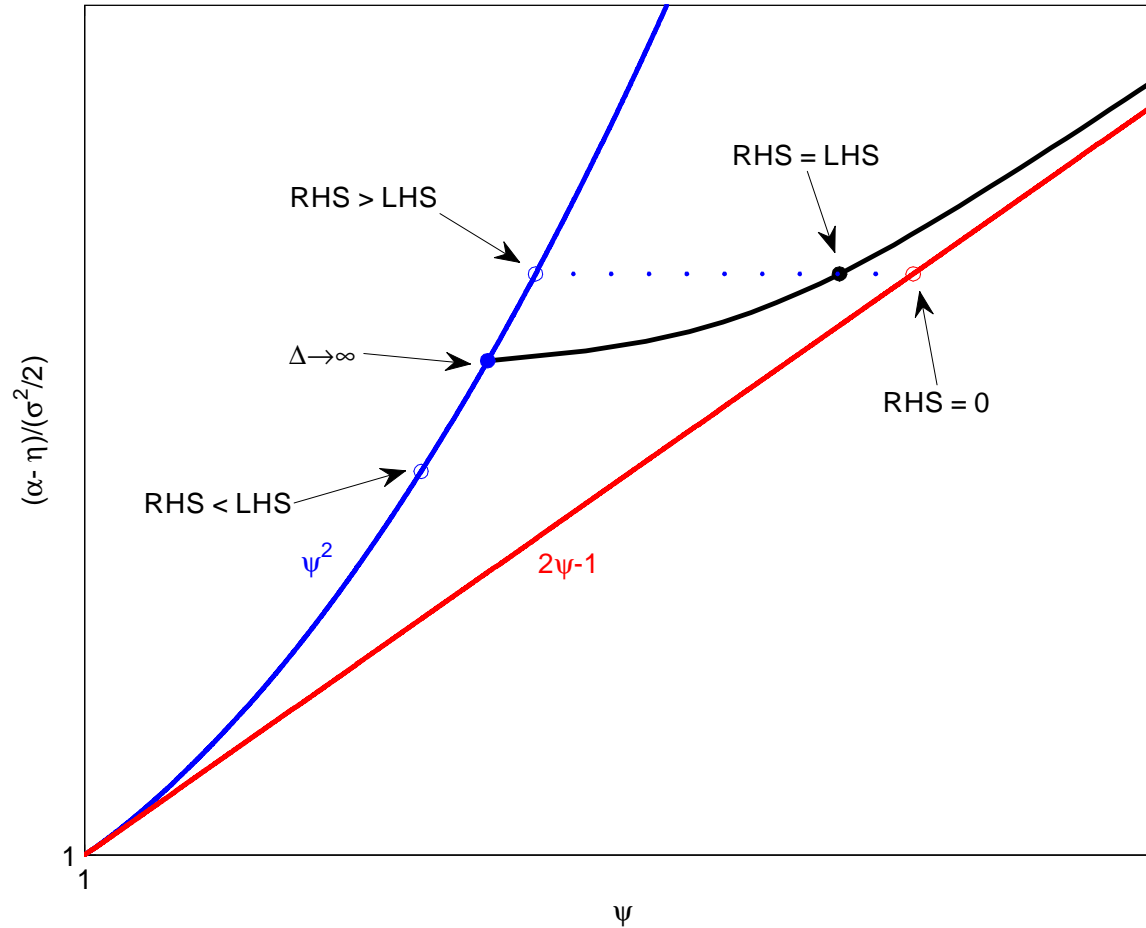
$$\int_0^\infty e^u f(u) du = \frac{\chi_+ \chi_-}{(\chi_+ - 1)(\chi_- - 1)} = \frac{\frac{\alpha - \eta}{\sigma^2/2}}{\frac{\alpha - \eta}{\sigma^2/2} - (2\psi - 1)}$$

- decreasing in α
 - increasing in ψ , reflecting the bootstrap logic
 - but at $\alpha = \eta + \frac{1}{2}\sigma^2\psi$, this mean equals $(\psi/(\psi - 1))^2 \dots$
 - ... which is decreasing in ψ
 - ... as in the $\Delta \rightarrow \infty$ limit
- when factor supplies are inelastic, only need to consider

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{\alpha/\phi}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) du}$$

- and remember that the exit threshold e^a is increasing in $\psi = -\mu/\sigma^2$

miraculous growth in the $\Delta = \infty$ economy



- this construction is for an economy with inelastic factor supplies
 - first increase $\psi > \psi_\infty$ while $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2 > \alpha_\infty$
 - then fix α and increase ψ further to clear the market

concluding remark

- with continuous factor supplies, the equilibrium will satisfy

$$\mu + \frac{1}{2}\sigma^2 < \eta$$

- this implies a per-capita consumption growth rate

$$\kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1} > \frac{1}{\varepsilon - 1} \left(\theta + \frac{1}{2}\sigma^2 \right)$$

- for individual firms

$$d[e^{z_t}] = [e^{z_t}] \left(\left(\theta + \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t \right)$$

- the scenario $\theta + \frac{1}{2}\sigma^2 = 0$ shows that
 - even if e^{z_t} is “only” a martingale for individual firms,
 - ... the overall economy will grow