Bounded Learning from Incumbent Firms

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the issue

- in my 2007 QJE paper, I showed

$$\left.\begin{array}{l}(1)\ \texttt{small random firm-specific productivity shocks}\\ \\ (2)\ \texttt{entrants learn from surviving incumbents}\end{array}\right\}\Rightarrow$$

`long-run aggregate growth, at an endogenous rate`

  - icing on the cake: Pareto-like firm size distributions
- but: the model has a continuum of steady state equilibria with distinct growth rates and firm size distributions
  - the paper had a heuristic argument to select one equilibrium
- this multiplicity issue has arisen again in more recent models of social learning and aggregate growth

▶ this paper: a diagnosis of the problem, and a new way to obtain a *unique prediction* for long-run growth

idea flows

- some early work

  Iwai 1984, Jovanovic and Rob 1989, Chari and Hopenhayn 1991, Kortum 1997, Eaton and Kortum 1999

- social learning only

  Alvarez, Buera and Lucas 2008, Lucas 2009,
  Lucas and Moll 2014, Perla and Tonetti 2014

- individual discovery and social learning

  Luttmer 2007, Staley 2011, König, Lorenz, Zilibotti 2012,
  Luttmer 2015 (*Fed*)

- unique stationary distribution and balanced growth path

  Luttmer 2012 (*JET*), this paper

► see *Fed w.p. 724, "Four Models ..."* for a survey of technical issues

the easiest example

- agents randomly select others at rate β and copy if "better"

$$\mathrm{D}_t P(t,z) = -\beta P(t,z)[1 - P(t,z)]$$

▶ the *unique* solution to this system of logistic ODE is

$$P(t,z) = \frac{1}{1 + \left(\frac{1}{P(0,z)} - 1\right) e^{\beta t}}$$

– but $P(0,z)$ matters a lot…

▶ *many stationary* solutions (note that κ is a free parameter)

– linear trends

if $P(0,z) = \dfrac{1}{1 + \left(\frac{1}{P(0,0)} - 1\right) e^{-(\beta/\kappa) z}}$ then $P(t,z) = P(0, z - \kappa t)$

– exponential trends

if $P(0,z) = \dfrac{1}{1 + \left(\frac{1}{P(0,1)} - 1\right) z^{-\beta/\kappa}}$ then $P(t,z) = P(0, z e^{-\kappa t})$

a better model: social learning *and* individual discovery

- two independent standard Brownian motions $W_{1,t}, W_{2,t}$,

$$\mathrm{E}\left[\max\left\{\sigma W_{1,t}, \sigma W_{2,t}\right\}\right] = \sigma\sqrt{t}\int_{-\infty}^{\infty} 2x\phi(x)\Phi(x)\mathrm{d}x = \sigma\sqrt{t/\pi}$$

- reset to the max at random time $\tau_{j+1} > \tau_j$

$$z_{\tau_{j+1}} = z_{\tau_j} + \sigma\max\left\{W_{1,\tau_{j+1}} - W_{1,\tau_j}, W_{2,\tau_{j+1}} - W_{2,\tau_j}\right\}$$

- reset times arrive randomly at rate $\alpha = 2\beta$

$$\begin{aligned}\mathrm{E}\left[\frac{z_{\tau_{n+1}} - z_{\tau_n}}{\tau_{n+1} - \tau_n} \,\Big|\, z_{\tau_n}\right] &= \int_0^{\infty} \frac{\sigma\sqrt{\tau/\pi}}{\tau} \times \alpha e^{-\alpha\tau}\mathrm{d}\tau \\ &= \sigma\sqrt{\alpha} \\ &= \sigma^2 \times \sqrt{\frac{\beta}{\sigma^2/2}}\end{aligned}$$

what's next

- in this example
  - not just learning from each other but also trying new things
  - research is cumulative, with random increments
  - rather than more draws from the same old distribution
  - and successful improvements are shared
  - no multiplicity issues anywhere in sight
- unlike in a large economy, with only two agents there can be no thick right tail of possible gains from social learning
- the idea in this paper
  - a simple cap on how much entrants can learn from incumbents is enough to get rid of the multiplicity in a large economy
  - this has a well-behaved limit as the cap becomes large
  - the selective replication logic survives and produces long-run growth
- will need to be careful to collect *all* the equilibrium conditions

preferences, factor supplies, a bit of technology

- the population is $H_t = He^{\eta t}$, with $\eta > 0$
- dynastic preferences over $\{C_t\}_{t\geq 0}$,

$$\mathcal{U}(C) = \int_0^\infty e^{-\rho t} H_t \ln(C_t/H_t)\mathrm{d}t$$

where

$$C_t = \left(\int e^{z/\varepsilon} c_{z,t}^{1-1/\varepsilon} N(t, \mathrm{d}z)\right)^{1/(1-1/\varepsilon)}$$

- crucial parameter restrictions

$$\rho > \eta, \qquad \varepsilon > 1$$

- a Roy model for primary factors of production
  - labor

$$\mathcal{L}(q_t/w_t) = \int x\iota\left\{w_t x > q_t y\right\} \mathrm{d}\mathcal{P}(x, y)$$

  - entrepreneurial services

$$\mathcal{E}(q_t/w_t) = \int y\iota\left\{w_t x < q_t y\right\} \mathrm{d}\mathcal{P}(x, y)$$

- a linear labor-only technology with a unit productivity

product market equilibrium

- demand curves for differentiated goods

$$c_{z,t} = \left(\frac{p_{z,t}}{P_t}\right)^{-\varepsilon} e^z C_t, \qquad P_t = \left(\int e^z p_{z,t}^{1-\varepsilon} N(t, \mathrm{d}z)\right)^{1/(1-\varepsilon)}$$

- monopolistic competition implies the Lerner price

$$p_{z,t} = \frac{w_t}{1 - 1/\varepsilon}$$

- together with the price index P_t, this implies

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right) \left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)}, \qquad e^{Z_t} = \frac{1}{N_t} \int e^z N(t, \mathrm{d}z)$$

  where $N_t = N(t, \infty)$

  – gains from variety via N_t

  – the quality distribution $N(t, z)/N_t$ will be a traveling wave

key product market implications

- firm profits and use of labor

$$\begin{bmatrix} v_{z,t} \\ w_t l_{z,t} \end{bmatrix} = \begin{bmatrix} 1/\varepsilon \\ 1 - 1/\varepsilon \end{bmatrix} e^{z-Z_t} \times \frac{P_t C_t}{N_t}$$

  – this is a "Red Queen environment"

- aggregate production labor L_t is

$$L_t = \int l_{z,t} N(t, \mathrm{d}z)$$

  – the definition of Z_t implies

$$w_t L_t = \left(1 - \frac{1}{\varepsilon}\right) P_t C_t$$

- average profits in units of labor are

$$\frac{1}{w_t N_t} \int v_{z,t} N(t, \mathrm{d}z) = \frac{1}{\varepsilon - 1} \frac{L_t}{N_t}$$

productivity dynamics and firm values

- the fundamental assumption is

$$\mathrm{d}z_t = \theta \mathrm{d}t + \sigma \mathrm{d}W_t$$

  - firm-specific random walks, with a trend $\theta \in (-\infty, \infty)$
  - for example, $\theta = -\frac{1}{2}\sigma^2$, so that e^{z_t} is a positive martingale
  - there is always a *non-trivial* new set of *modifications* to try
  - of course, we could, instead, run out of ideas…

- firm continuation requires $\phi > 0$ units of labor per unit of time
- given $z_t = z$, the value of a firm is

$$\frac{V(t,z)}{P_t} = \max_{\tau \geq 0} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \times \frac{C_t/H_t}{C_s/H_s} \left(\frac{v_{z_s,s}}{P_s} - \frac{\phi w_s}{P_s} \right) \mathrm{d}s \right]$$

  - where τ is a stopping time
  - the use of logarithmic utility is not essential

- optimal to exit when $z_t \leq b_t$, for some b_t to be determined

the knowledge diffusion assumption

- entrepreneurs produce a flow of entry opportunities $\mathcal{E}(q_t/w_t)H_t$
- an entry opportunity is
  - a random draw from the incumbent population
  - then, may copy the z of the randomly sampled firm
  - but only if $z \in [b_t, b_t + \Delta]$, for some $\Delta \in (0, \infty)$
  - interpretation: "everyone knows" b_t and can learn up to Δ more
- the value of an entry opportunity is

$$q_t = \left(\int_{b_t}^{\infty} N(t, \mathrm{d}z) \right)^{-1} \int_{b_t}^{b_t+\Delta} V(t, z) N(t, \mathrm{d}z)$$

  - draws from $(b_t + \Delta, \infty)$ go to waste
- the Roy model determines $\mathcal{E}(\cdot)$

the Kolmogorov forward equation

- for any $z \in (b_t, b_t + \Delta)$

$$\texttt{the flow of entrants at } z \texttt{ is } \mathcal{E}\left(\frac{q_t}{w_t}\right) H_t \times \frac{n(t,z)}{N_t} = \alpha_t n(t,z)$$

where α_t is the *attempted entry rate*, defined as

$$\alpha_t = \frac{\mathcal{E}\left(q_t/w_t\right)}{N_t/H_t}$$

- the Kolmogorov forward equation is

$$\mathrm{D}_t n(t,z) = -\theta \mathrm{D}_z n(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} n(t,z) + \alpha_t n(t,z),$$

for $z \in (b_t, b_t + \Delta)$ and

$$\mathrm{D}_t n(t,z) = -\theta \mathrm{D}_z n(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} n(t,z),$$

for $z \in (b_t + \Delta, \infty)$

- immediate exit at b_t means that

$$n(t, b_t) = 0$$

- the density should be smooth at $b_t + \Delta$

constructing a BGP—an outline

- conjecture that Z_t grows at some equilibrium rate $\theta - \mu$
- given μ, we will show that
  - there is a unique stationary distribution if $\Delta \in (0, \infty)$,
  - but a continuum if $\Delta = \infty$

▶ a steady state supply of firms

$$\frac{N}{H} = \mathcal{S}\left(\frac{q}{w}; \mu\right) \tag{1}$$

- from entrepreneurial incentives, life cycle of firms

▶ a steady state demand for firms

$$\frac{N}{H} = \mathcal{D}\left(\frac{q}{w}; \mu\right) \tag{2}$$

- how many firms needed to employ all workers?

▶ a present-value condition

$$\frac{q}{w} = \mathcal{Q}(\mu) \tag{3}$$

aggregate conjectures

- conjecture a common growth rate for

  (i) per-capita consumption

  (ii) the real wage

  (iii) average real variable profits

- recall that

$$\left[\int v_{z,t} N(t, \mathrm{d}z), w_t L_t\right] = \left[\frac{1}{\varepsilon}, 1 - \frac{1}{\varepsilon}\right] P_t C_t$$

and

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right) \left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)}, \quad e^{Z_t} = \frac{1}{N_t} \int e^z N(t, \mathrm{d}z)$$

- this implies

$$\frac{L_t}{H_t} = \frac{L}{H}, \quad \frac{N_t}{H_t} = \frac{N}{H}$$

and

$$Z_t = Z + (\theta - \mu)t$$

for some L/H, N/H and μ to be determined

what does μ do?

- for individual firms, μ is the drift of $z_t - Z_t$,

$$\mathrm{d}\left(z_t - Z_t\right) = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

  – and $l_{z,t} \propto e^{z_t - Z_t}$

- recall that

$$\left[\frac{L_t}{H_t}, \frac{N_t}{H_t}\right] = \left[\frac{L}{H}, \frac{N}{H}\right], \quad H_t = He^{\eta t}$$

  and

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right)\left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)}, \qquad \frac{w_t}{P_t}\frac{L_t}{H_t} = \left(1 - \frac{1}{\varepsilon}\right)\frac{C_t}{H_t}$$

- so then $Z_t = Z + (\theta - \mu)t$ implies

$$\left[\frac{w_t}{P_t}, \frac{C_t}{H_t}\right] = \left[\frac{w}{P}, \frac{C}{H}\right] e^{\kappa t}, \qquad \kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1}$$

  – *fast* aggregate growth means *slow* firm growth

the implied value function

- recall that

$$\frac{v_{z,t}}{w_t} = e^{z-Z_t} \times \frac{L_t/N_t}{\varepsilon - 1}$$

and C_t/H_t and w_t/P_t grow at a common rate

- this yields

$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \left(\frac{e^{z_s - Z_s} L}{(\varepsilon - 1)\phi N} - 1 \right) \mathrm{d}s \right],$$

where

$$z_s - Z_s = z - Z_t + \mu(s-t) + \sigma(W_s - W_t) \text{ for all } s \geq t$$

▶ this must be of the form

$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times U(y), \qquad e^y = \frac{e^{z-Z_t} L}{(\varepsilon - 1)\phi N}$$

▶ will need the equilibrium μ to satisfy

$$\mu + \frac{1}{2}\sigma^2 < \rho$$

the solution for $V(t,z)$

- is given by

$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times U(y), \qquad e^y = \frac{e^{z-Z_t} L}{(\varepsilon - 1)\phi N}$$

- where

$$U(y) = \begin{cases} 0 & , y \leq a \\ \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{y-a} - 1 - \frac{1-e^{-\xi(y-a)}}{\xi}\right) & , y \geq a \end{cases}$$

  – and the exit threshold $a < 0$ is determined by

$$e^a = \frac{\xi}{1+\xi}\left(1 - \frac{1}{\rho}\left(\mu + \frac{1}{2}\sigma^2\right)\right)$$

  and

$$\xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}$$

  – note that $y = 0$ corresponds to zero flow profits

▶ so we have a mapping

$$\mu \mapsto [a, U(\cdot)]$$

the stationarity conjecture

- strengthen $Z_t - b_t = Z - b$ to time-invariance of the cross-sectional distribution of $z - b_t$,

$$n(t, z) = N_t f(z - b_t), \qquad z \in (b_t, \infty)$$

  – the definition of Z_t implies a consistency condition

$$e^{Z-b} = \int_0^\infty e^u f(u)\mathrm{d}u$$

▶ the value q_t/w_t of an entry opportunity now becomes

$$\frac{q_t}{w_t} = \phi \int_0^\Delta U\,(a + u)\, f(u)\mathrm{d}u$$

  – so q_t/w_t, $\mathcal{E}(q_t/w_t)$, and $\mathcal{L}(q_t/w_t)$ are constant over time

▶ since $N_t/H_t = N/H$, this means that $\alpha_t = \alpha$, and hence

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right)$$

  – this is the *steady state supply* of firms as a function of q/w

clearing the labor market

- the employment size of firms scales with $e^u = e^{y-a} = e^{z-b}$
- recall the consistency condition

$$e^{Z-b} = \int_0^\infty e^u f(u)\mathrm{d}u$$

and that the threshold b for z is determined by the threshold a for y via

$$e^a = \frac{e^{b-Z}L}{(\varepsilon-1)\phi N}$$

▶ the labor market clearing condition

$$\mathcal{L}\left(\frac{q}{w}\right)H = \phi N + L$$

can therefore be written as

$$\frac{N}{H} = \frac{1}{\phi}\frac{\mathcal{L}\left(q/w\right)}{1+(\varepsilon-1)\int_0^\infty e^{a+u}f(u)\mathrm{d}u}$$

– the *steady state demand* for firms as a function of q/w

the stationary KFE

- recall
$$z - b_t = z - Z_t + Z_t - b_t = z - Z_t + Z - b$$
and
$$\mathrm{d}\left(z_t - Z_t\right) = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$
- the Kolmogorov forward equation for $n(t, z) = N_t f(z - b_t)$ becomes
$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u) + \alpha f(u)$$
for $u \in (0, \Delta)$ and
$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u)$$
for $u \in (\Delta, \infty)$
- the boundary conditions are
  - $f(0) = 0 = f(\infty)$
  - differentiability at Δ
- and $f(\cdot)$ is supposed to integrate to 1

solving the KFE—1: characteristic roots

- KEY RESULT for $\Delta \in (0, \infty)$
  - for any $\mu \in (-\infty, \infty)$, there is *precisely one* attempted entry rate $\alpha > 0$ for which it is possible to solve the KFE
  - so α and $f(\cdot)$ are pinned down jointly as a function of μ

- on $(0, \Delta)$, a solution of the form $e^{-\chi z}$ implies $\chi \in \{\chi_-, \chi_+\}$,

$$\chi_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 - \frac{\alpha - \eta}{\sigma^2/2}}$$

- on (Δ, ∞), a solution of the form $e^{-\zeta z}$ implies $\zeta \in \{\zeta_-, \zeta_+\}$,

$$\zeta_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- then $\eta > 0$ implies $\zeta_- < 0 < \zeta_+$, irrespective of the sign of μ
  - this forces $f(u) \propto e^{-\zeta_+ u}$ on (Δ, ∞), scale to be determined

- the $\chi_\pm$ may be real or complex, which obviously depends on α

solving the KFE—2: imposing differentiability at Δ

- FACT: no way to enforce differentiability at Δ if the $\chi_\pm$ are real
- suppose α large enough so that the roots $\chi_\pm$ are complex
  - let $\psi = \mathrm{Re}(\chi_+)$ and $\omega = \mathrm{Im}(\chi_+)$,

$$\psi = -\frac{\mu}{\sigma^2}, \qquad \omega = \sqrt{\frac{\alpha - \eta}{\sigma^2/2} - \psi^2}$$

  - requiring $f(u)$ to be real forces

$$f(u) = \left[A\cos(\omega u) + B\sin(\omega u)\right] e^{-\psi u}$$

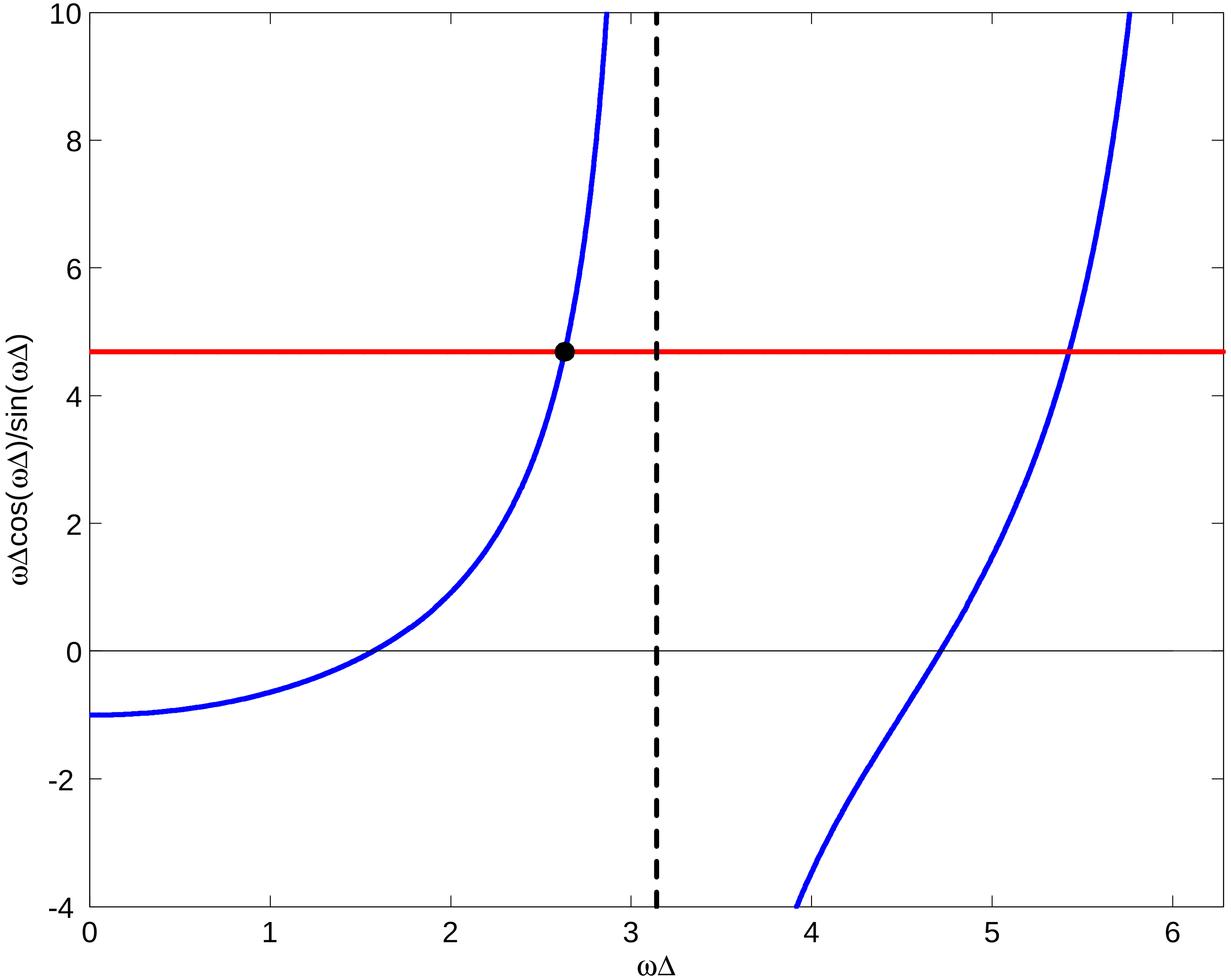
  - imposing $f(0) = 0$ forces $A = 0$
  - imposing continuity at $u = \Delta$ yields

$$f(u) = B \begin{cases} \sin(\omega u) e^{-\psi u}, & u \in [0, \Delta], \\ \sin(\omega\Delta) e^{-\psi\Delta} e^{-\zeta_+(u-\Delta)}, & u \in [\Delta, \infty) \end{cases}$$

  - this is positive on $(0, \Delta)$ if and only if $\omega\Delta \in (0, \pi)$
  - imposing differentiability at $u = \Delta$ forces

$$-\frac{\cos(\omega\Delta)}{\sin(\omega\Delta)/(\omega\Delta)} = \Delta\sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}.$$

the solution for ω



solving the KFE—3: the implied attempted entry rate

- recall $\psi = \mathrm{Re}(\chi_+)$ and $\omega = \mathrm{Im}(\chi_+)$,

$$\psi = -\frac{\mu}{\sigma^2}, \qquad \omega = \sqrt{\frac{\alpha - \eta}{\sigma^2/2} - \psi^2}$$

- differentiability at Δ forces

$$-\frac{\cos(\omega\Delta)}{\sin(\omega\Delta)/(\omega\Delta)} = \Delta\sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}$$

  - LHS is increasing in $\omega\Delta \in (0, \pi)$, ranging throughout $(-1, \infty)$
  - unique solution $\omega \in (0, \pi/\Delta)$
  - this solution is increasing in ψ^2, decreasing in Δ

- inverting the definition of ω delivers the attempted entry rate

$$\alpha = \eta + \frac{1}{2}\sigma^2\left(\omega^2 + \psi^2\right)$$

  - so α is increasing in $\psi^2 \propto \mu^2$
  - in particular, $\mu \to -\infty$ gives $\alpha \to \infty$

the large-Δ limiting distribution

Lemma *The stationary distribution function converges to*

$$\lim_{\Delta\to\infty} F(u) = \begin{cases} 0 & ,\ \psi \in (-\infty, 0] \\ 1-(1+\psi u)e^{-\psi u} & ,\ \psi \in (0,\infty) \end{cases}$$

for any $u \in [0,\infty)$. *The truncated mean of* e^u *behaves like*

$$\lim_{\Delta\to\infty} \int_0^{\Delta} e^u \mathrm{d}F(u) = \begin{cases} \infty & ,\ \psi \in (-\infty, 1] \\ \left(\frac{\psi}{\psi-1}\right)^2 & ,\ \psi \in (1,\infty) \end{cases}$$

The attempted entry rate satisfies

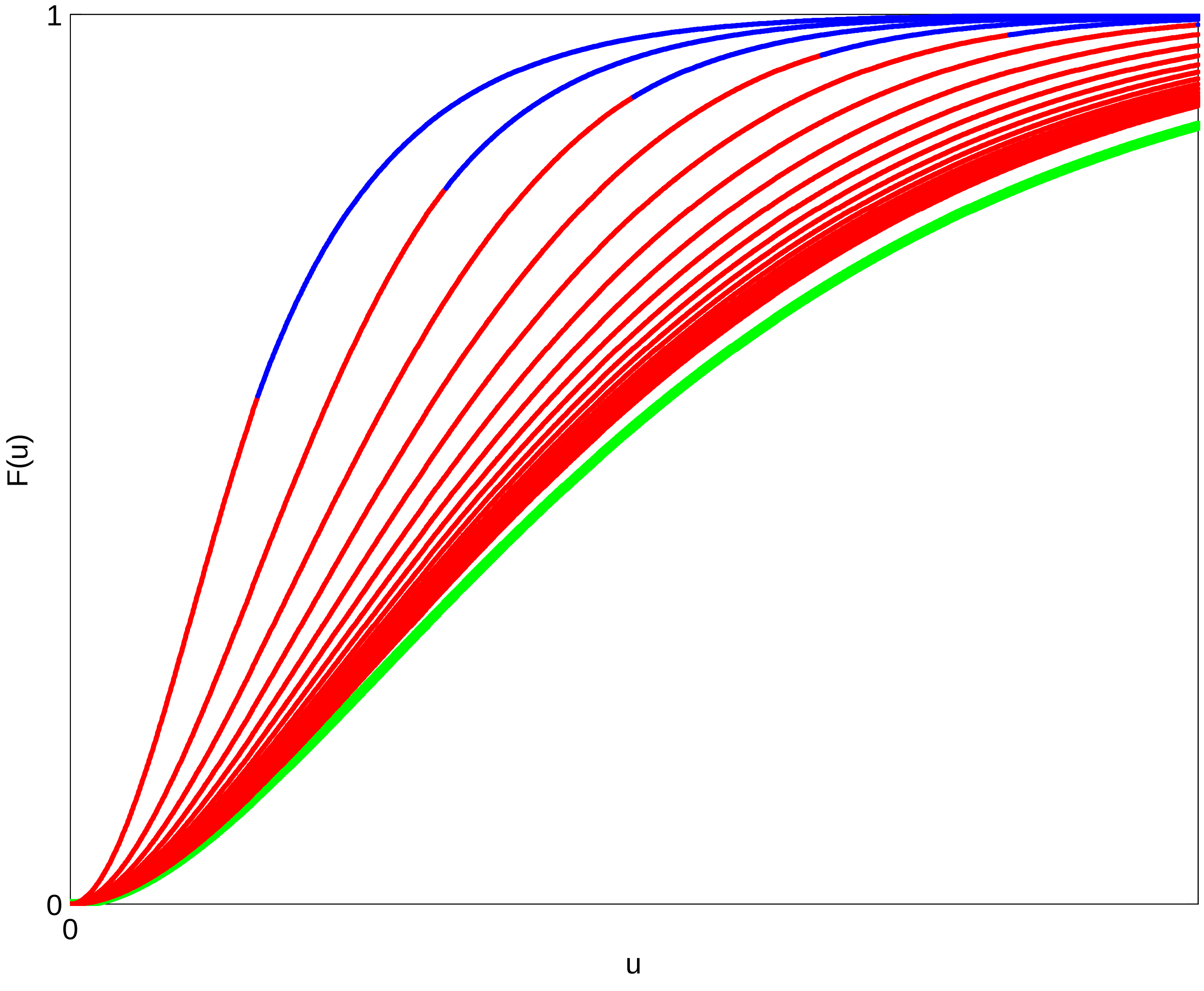
$$\lim_{\Delta\to\infty} \alpha = \eta + \frac{1}{2}\sigma^2\psi^2$$

- if $\psi > 0$ and $\Delta \in (0,\infty)$, then the right tail index is

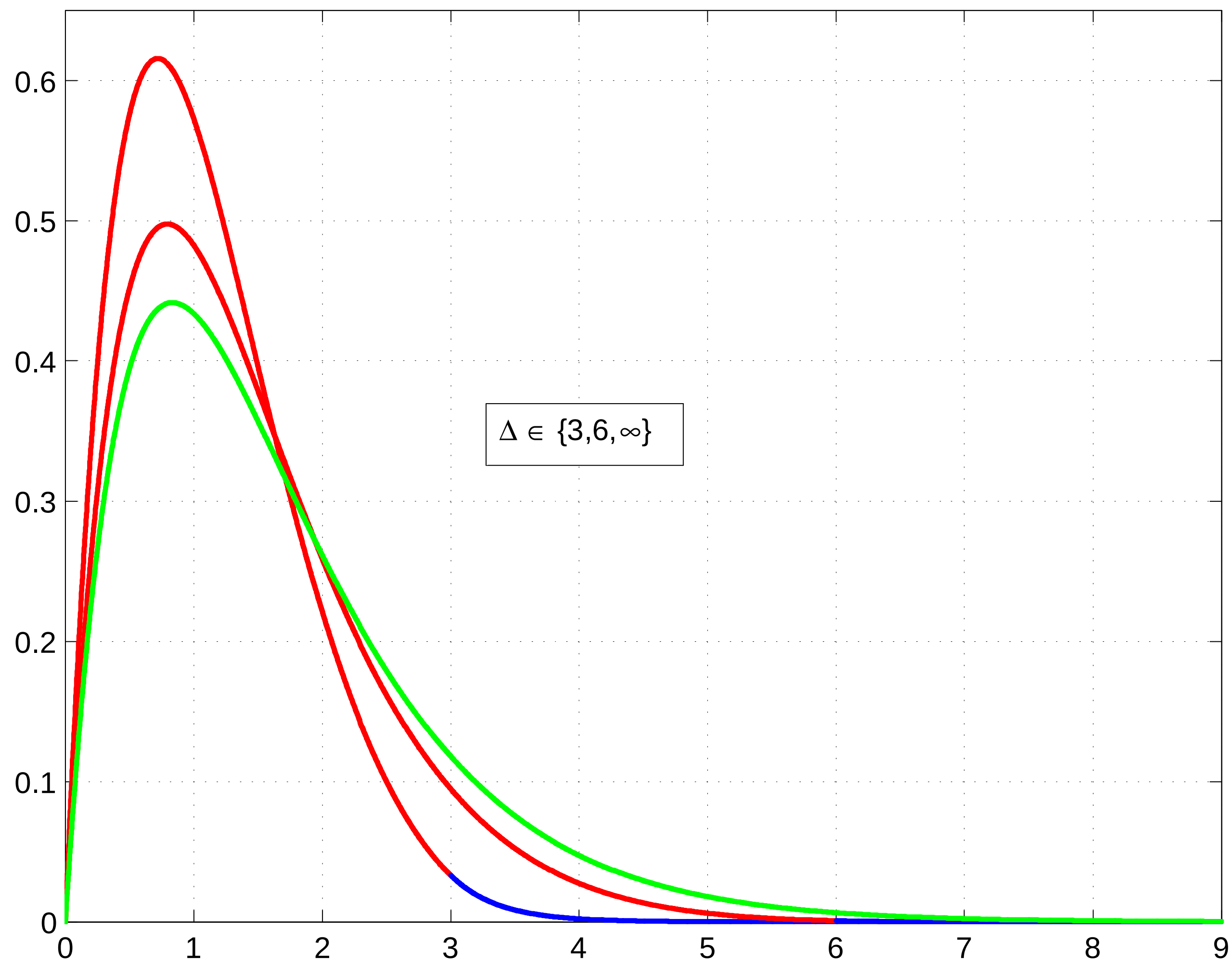
$$\zeta_+ = \psi + \sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}} > 2\psi \qquad (!)$$

▶ so the tail index is discontinuous at $\Delta = \infty$

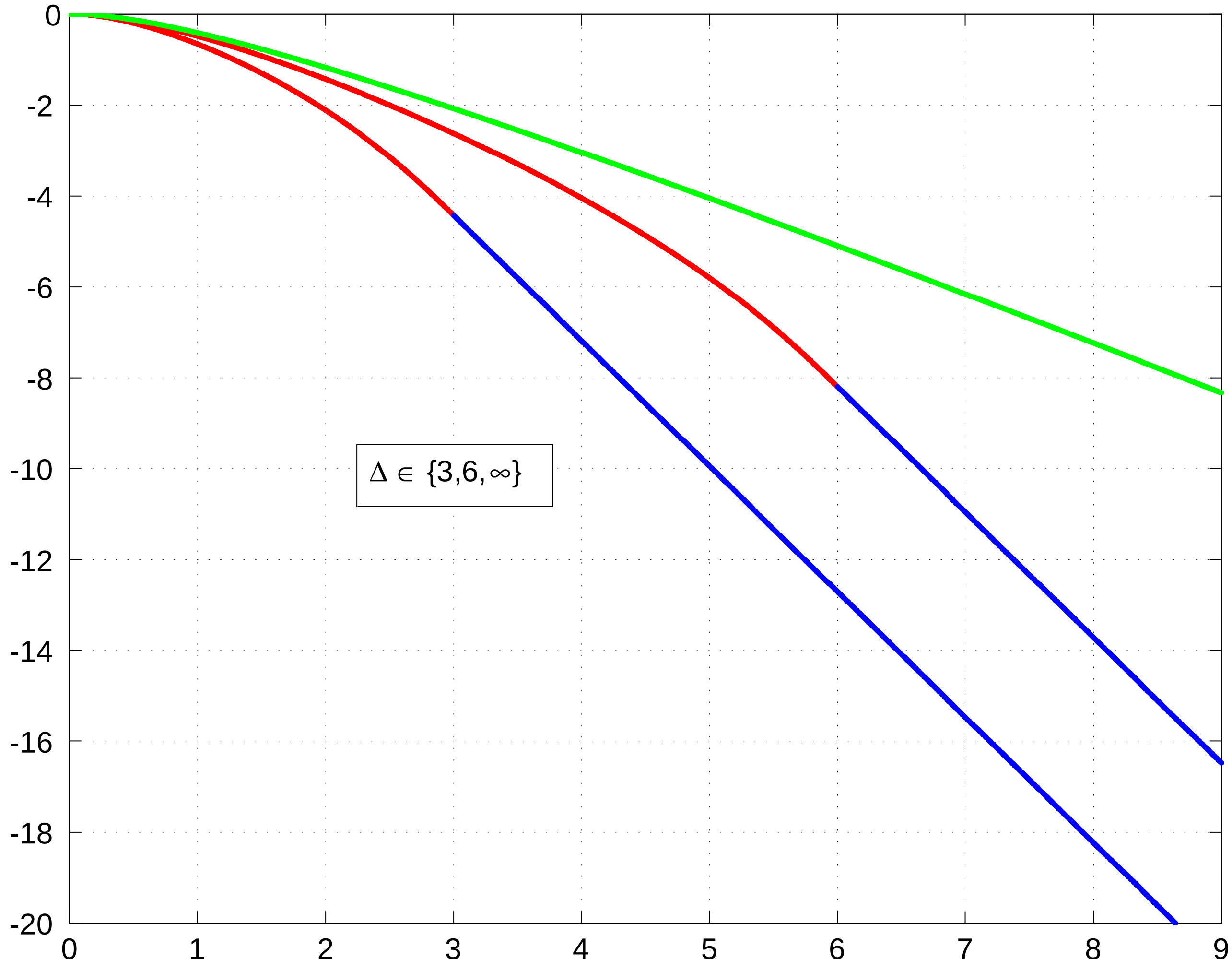
the convergence is monotone in the sense of first-order stochastic dominance



densities



log-log plot of distributions



recap

1. decision rules and steady state requirements imply

- the supply of firms

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right)$$

- the demand for firms

$$\frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}\left(q/w\right)}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) \mathrm{d}u}$$

$\Rightarrow$ market clearing delivers q/w

2. perfect foresight also delivers a present value condition

$$\frac{q}{w} = \phi \int_0^\Delta U\left(a+u\right) f(u) \mathrm{d}u$$

- in the background
  - the Bellman equation gives a function $\mu \mapsto [a, U(\cdot)]$
  - the KFE gives a *function* $\mu \mapsto [\alpha, f(\cdot)]$, provided $\Delta \in (0, \infty)$

▶ the two versions of q/w must match, producing a restriction on μ

the equations for balanced growth

- clearing the steady state market for firms gives

$$\frac{\mathcal{E}\left(q/w\right)}{\mathcal{L}\left(q/w\right)} = \frac{\alpha/\phi}{1+(\varepsilon-1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u} \tag{1}$$

- the relative price q/w must also satisfy

$$\frac{q}{w} = \phi \int_0^\Delta U\left(a+u\right) f(u)\mathrm{d}u \tag{2}$$

- in the background
  - the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

  - the KFE yields

$$\mu \mapsto \{\alpha, f(\cdot)\}$$

- if the initial density satisfies $n(0,z)/N = f(z-b)$ for some b, then the economy is on a balanced growth path

an *irregular* special case: perfectly elastic factor supplies

- this fixes q/w, and then μ is determined by

$$\frac{q}{w} = \phi \int_0^\Delta U\left(a+u\right) f(u) \mathrm{d}u \tag{PF}$$

  – the firm value $U(a+u)$ is finite if and only if

$$\rho > \mu + \frac{1}{2}\sigma^2$$

- market clearing still requires finite average employment

$$\int_0^\infty e^{a+u} f(u) \mathrm{d}u < \infty$$

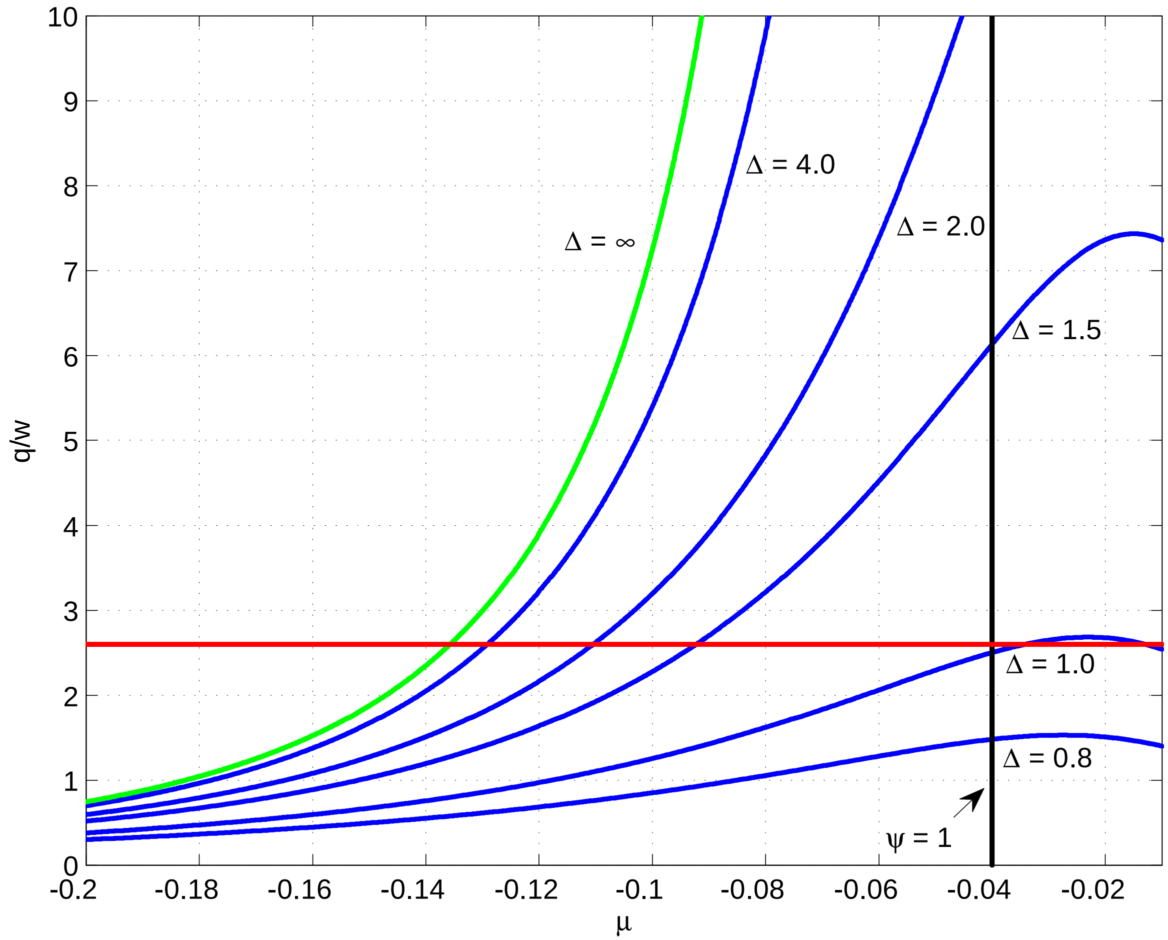
  – this is the same as $\zeta_+ > 1$, or

$$\eta > \mu + \frac{1}{2}\sigma^2$$

▶ may not have a BGP because

  – finite dynastic utility requires $\rho > \eta$, and then...
  – the RHS of (PF) may not reach q/w on $\left\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\right\}$

perfectly elastic factor supplies



- value of entry, bounded on the domain $\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\}$
- there may not be a BGP when $\Delta \in (0, \infty)$

a *regular* special case: perfectly inelastic factor supplies

- this fixes $\mathcal{E}/\mathcal{L}$, and μ is determined by

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{\alpha/\phi}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) \mathrm{d}u} \tag{MC}$$

  – the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

  – the KFE yields

$$\mu \mapsto \{\alpha, f(\cdot)\}$$

▶ the RHS of (MC) ranges throughout $(0, \infty)$ on $\left\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\right\}$

  – since $\rho > \eta$, and since mean employment must be finite

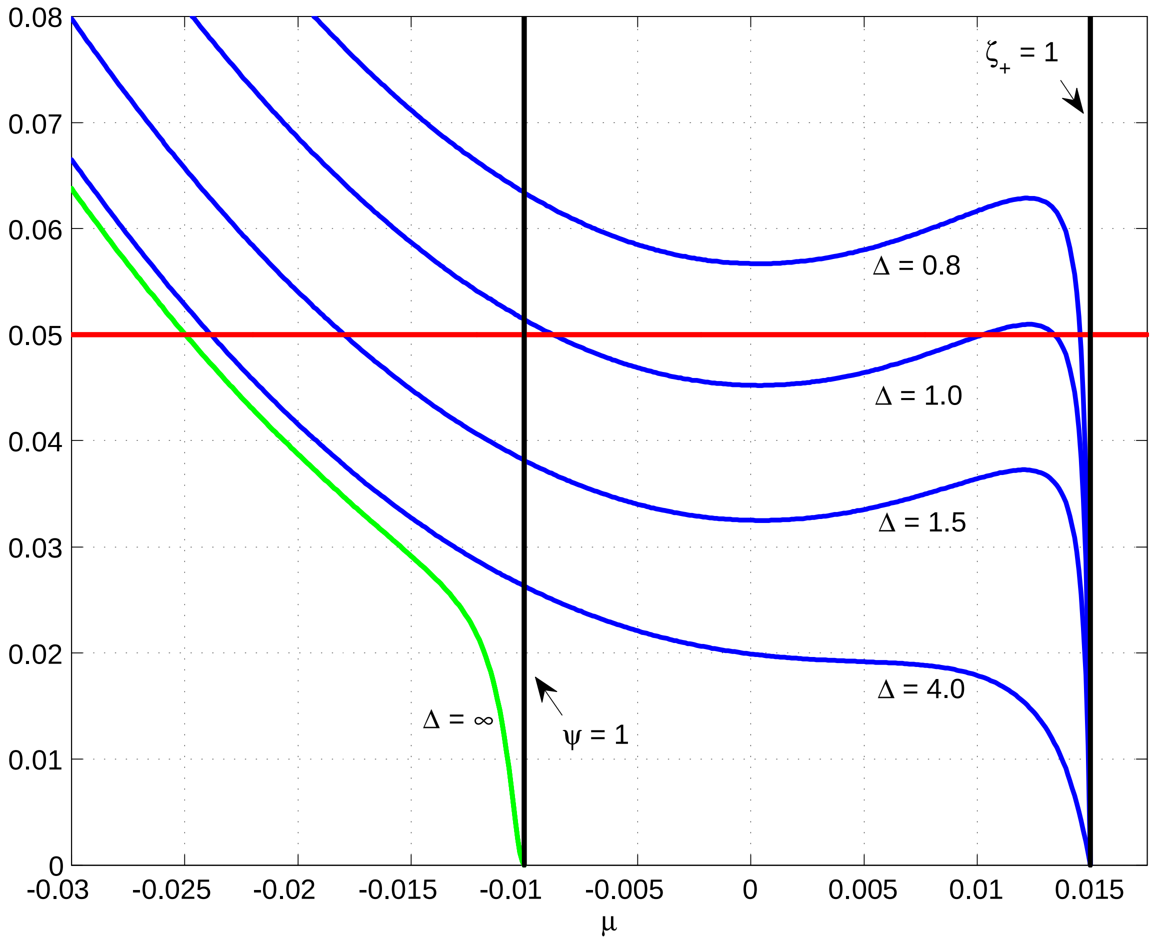
$$\mu + \frac{1}{2}\sigma^2 < \eta < \rho$$

  – the relative price q/w is determined by

$$\frac{q}{w} = \phi \int_0^\Delta U(a + u) f(u) \mathrm{d}u$$

  which is well defined by construction

perfectly inelastic factor supplies



- relative factor demands, on the domain $\{\mu : \zeta_+ > 1\} = \left\{\mu : \mu + \frac{1}{2}\sigma^2 < \eta\right\}$
- note that $\mu > 0$ is possible
  - the trend of $\ln(Z_t)$ may be below θ when $\Delta \in (0, \infty)$

existence of a BGP

Proposition A *Assume the relative factor supply curve* $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ *is continuous. When* Δ *is large enough, a finite-*Δ *economy must have at least one equilibrium, and every equilibrium must satisfy* $\psi_\Delta > 1$.

Proposition B *Assume the relative factor supply curve* $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ *is continuous. Consider the equilibrium conditions (1)-(2) with* $\psi > 1$, $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$, $f(u) = \psi^2 u e^{-\psi u}$, *and* $\Delta = \infty$. *With these restrictions, the economy has precisely one balanced growth path, denoted by* $\psi_\infty \in (1, \infty)$.

Proposition C *Assume the relative factor supply curves* $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ *are continuous and let* $E_\Delta \subset \{\psi : \zeta_+ > 1\}$ *be the set of equilibria for the* Δ *economy. Then* $\sup_{\psi \in E_\Delta} |\psi - \psi_\infty|$ *converges to zero as* Δ *becomes large.*

Corollary *Productivity grows faster than* θ *when* Δ *is large enough, since* $\psi_\infty = -\mu_\infty/\sigma^2 > 1$.

solving the KFE for $\Delta = \infty$

- the KFE simplifies to

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u) + \alpha f(u)$$

for all $u \in (0, \infty)$, with the boundary conditions $f(0) = 0 = f(\infty)$

- solved by linear combinations of $e^{-\chi_- u}$ and $e^{-\chi_+ u}$,

$$\chi_\pm = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \qquad \psi = -\frac{\mu}{\sigma^2}$$

  - complex $\chi_\pm$ yields a positive density only on a bounded interval
  - if $\alpha \in [0, \eta]$, then $\chi_- \le 0 \le \chi_+$, which rules out $f(0) = 0 = f(\infty)$

- need α to satisfy $0 < (\alpha - \eta)/(\sigma^2/2) \le \psi^2$ and $\psi > 0$, and then

$$f(u) = \frac{\chi_+ \chi_-}{\chi_+ - \chi_-} \times \left(e^{-\chi_- u} - e^{-\chi_+ u} \right),$$

for all $u \in [0, \infty)$

▶ this was the density obtained in Luttmer [2007]

  - if $(\alpha - \eta)/(\sigma^2/2) \uparrow \psi^2$ this matches the large-Δ limit $f(u) = \psi^2 u e^{-\psi u}$

balanced growth pathS

- steady state market clearing requires

$$\frac{\mathcal{E}\left(q/w\right)}{\mathcal{L}\left(q/w\right)} = \frac{\alpha/\phi}{1+(\varepsilon-1)\int_0^\infty e^{a+u}f(u)\mathrm{d}u}$$

- the relative price q/w must also satisfy

$$\frac{q}{w} = \phi\int_0^\infty U\left(a+u\right)f(u)\mathrm{d}u \tag{PF}$$

- in the background
  - the Bellman equation yields

$$\mu \mapsto \{a, U(\cdot)\}$$

  - the KFE yields

$$(\mu,\alpha) \mapsto f(\cdot) \tag{!}$$

  rather than $\mu \mapsto \{\alpha, f(\cdot)\}$

- aside: in Luttmer [2007], the factor supplies are perfectly elastic, and (PF) forces the mean of e^u to be finite

feasible α given μ

- recall

$$\chi_\pm = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \qquad \psi = -\frac{\mu}{\sigma^2}$$

- need $\chi_\pm$ real and $\chi_- > 1$, which is the same as

$$1 < \psi, \qquad 2\psi - 1 < \frac{\alpha - \eta}{\sigma^2/2} \leq \psi^2$$

- note that on this domain

$$\frac{\partial \chi_-}{\partial \psi} = \frac{\partial}{\partial \psi}\left(\psi - \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}\right) < 0$$

  - holding fixed α, a *lower* firm growth rate μ implies a thicker tail
  - *bootstrap logic*: a lower μ tends to generate more exit; without more entry, must have a thicker tail so it takes more firms longer to reach the exit barrier

- recall that the limiting BGP as $\Delta \to \infty$ is $\psi_\infty > 1$, and

  - the attempted entry rate is at its $(\Delta = \infty)$ upper bound

$$\alpha_\infty = \eta + \frac{1}{2}\sigma^2\psi_\infty^2$$

constructing alternative BGP

- can construct BGP for any $\psi > \psi_\infty$
- so there is no upper bound on how fast the economy can grow
- the key calculation is

$$\int_0^\infty e^u f(u)\mathrm{d}u = \frac{\chi_+\chi_-}{(\chi_+ - 1)(\chi_- - 1)} = \frac{\frac{\alpha-\eta}{\sigma^2/2}}{\frac{\alpha-\eta}{\sigma^2/2} - (2\psi - 1)}$$

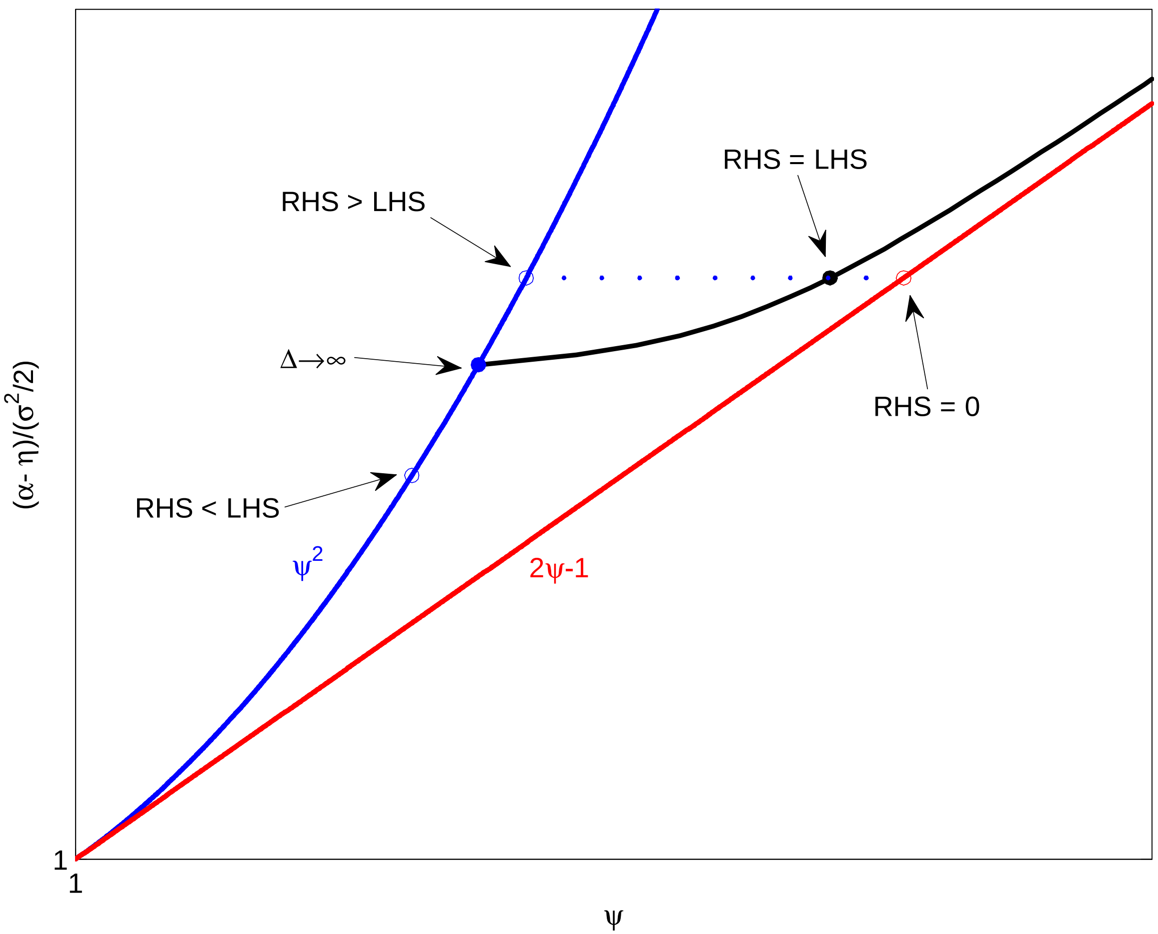
  – decreasing in α
  – increasing in ψ, reflecting the bootstrap logic
  – but at $\alpha = \eta + \frac{1}{2}\sigma^2\psi$, this mean equals $(\psi/(\psi-1))^2\ldots$
  – …which is decreasing in ψ
  – …as in the $\Delta \to \infty$ limit

- when factor supplies are inelastic, only need to consider

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{\alpha/\phi}{1 + (\varepsilon - 1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u}$$

  – and remember that the exit threshold e^a is increasing in $\psi = -\mu/\sigma^2$

miraculous growth in the $\Delta = \infty$ economy



- this construction is for an economy with inelastic factor supplies
  - first increase $\psi > \psi_\infty$ while $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2 > \alpha_\infty$
  - then fix α and increase ψ further to clear the market

concluding remark

- with continuous factor supplies, the equilibrium will satisfy

$$\mu + \frac{1}{2}\sigma^2 < \eta$$

- this implies a per-capita consumption growth rate

$$\kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1} > \frac{1}{\varepsilon - 1}\left(\theta + \frac{1}{2}\sigma^2\right)$$

- for individual firms

$$\mathrm{d}\left[e^{z_t}\right] = \left[e^{z_t}\right]\left(\left(\theta + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t\right)$$

- the scenario $\theta + \frac{1}{2}\sigma^2 = 0$ shows that
  - even if e^{z_t} is "only" a martingale for individual firms…
  - …the overall economy will grow

additional material

key properties of the value function

Lemma *The value function is well defined if and only if* $\mu + \frac{1}{2}\sigma^2 < \rho$*. Given this restriction, it has the following properties:*

(i) *The value function is strictly increasing and unbounded in* $y > a$*.*

(ii) *The exit threshold is strictly decreasing in* μ*,*

$$\lim_{\mu \to -\infty} a = 0, \text{ and } \lim_{\mu \uparrow \rho - \sigma^2/2} a = -\infty.$$

(iii) *For any* $u \in (0, \infty)$ *or* $y \in (-\infty, \infty)$*,*

$$\lim_{\mu \to -\infty} U(a+u) = 0, \qquad \lim_{\mu \uparrow \rho - \sigma^2/2} U(a+u) \in (0, \infty), \qquad \lim_{\mu \uparrow \rho - \sigma^2/2} U(y) = \infty,$$

and $U(a+u)$ *is increasing in* μ*.*

- the time-t exit threshold for firm of type z must then be

$$b_t = b + (\theta - \mu)t, \qquad e^a = \frac{e^{b-Z} L}{(\varepsilon - 1)\phi N}$$

- so the gap $Z_t - b_t = Z - b$ is constant over time

an accounting identity implied by the KFE

- integrating the differential equation over $(0, \infty)$ yields

$$\alpha \int_0^{\Delta} f(u)\mathrm{d}u = \eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)$$

  - need to use the above stated boundary conditions
  - this fails if $f(\cdot)$ not differentiable at Δ

- we also know that the exit rate at $z = b$ is given by $\frac{1}{2}\sigma^2 \mathrm{D}f(0)$
- this confirms the basic steady state accounting condition

`successful entry rate`

$=$

`population growth rate + exit rate`

▶ can infer α from $f(\cdot)$, without knowing μ