Experimentation, Selection, and Imitation Imply Growth and Heterogeneity

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Spring 2024

three models of growth and firm heterogeneity

desiderata

- firm dynamics consistent with
  - a stable per-capita number of firms
  - a stable firm employment size distribution...
  - ...that is Pareto-like, close to Zipf's law
  - firm employment growth roughly in line with Gibrat's law
  - robust entry and exit of mostly small firms
- account for persistent productivity differences

the story here

- a traveling wave of productivities
  - one that emerges endogenously
- why does it generate Zipf's law?
- what determines the speed of this wave?
  - that is, what determines aggregate growth?

challenges

- geometric random walks are non-stationary
  - Gibrat's law seems to suggest we need them
  - then how do we keep things stationary?
- how to ensure that new entry remains possible?
  - surviving incumbents become more productive over time
  - how can entrants possibly compete?
- the log normal distribution has "thin" tails
  - how do we turn that into a power law?
  - what ensures the resulting power law has a finite mean?
- how to keep things interesting and tractable at the same time?
  - you know, the world is pretty complicated

the common elements

preferences and factor supplies

- the population is $H_t = He^{\eta t}$, with $\eta > 0$
- dynastic preferences over $\{C_t\}_{t \geq 0}$,

$$\mathcal{U}(C) = \int_0^\infty e^{-\rho t} H_t \ln(C_t/H_t) \mathrm{d}t$$

where

$$C_t = \left(\int e^{z/\varepsilon} C_{z,t}^{1-1/\varepsilon} N(t, \mathrm{d}z) \right)^{1/(1-1/\varepsilon)}$$

  - crucial parameter restrictions

$$\rho > \eta, \qquad \varepsilon > 1$$

- given primary factor prices (w_t, q_t), a Roy model gives
  - labor

$$\mathcal{L}(q_t/w_t) = \int x \iota \{w_t x > q_t y\} \, \mathrm{d}\mathcal{P}(x, y)$$

  - entrepreneurial services

$$\mathcal{E}(q_t/w_t) = \int y \iota \{w_t x < q_t y\} \, \mathrm{d}\mathcal{P}(x, y)$$

technology and product market equilibrium

- demand curves for differentiated goods

$$C_{z,t} = \left(\frac{p_{z,t}}{P_t}\right)^{-\varepsilon} e^z C_t, \qquad P_t = \left(\int e^z p_{z,t}^{1-\varepsilon} N(t, \mathrm{d}z)\right)^{1/(1-\varepsilon)}$$

- a linear labor-only technology with a unit productivity
  - monopolistic competition implies the Lerner price

$$p_{z,t} = \frac{w_t}{1 - 1/\varepsilon}$$

- together with the price index P_t, this implies

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right)\left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)}, \qquad e^{Z_t} = \frac{1}{N_t}\int e^z N(t, \mathrm{d}z)$$

  where $N_t = N(t, \infty)$

  - gains from variety via N_t
  - the quality distribution $N(t, z)/N_t$ will be a traveling wave

key product market implications

- firm profits and use of labor

$$\begin{bmatrix} v_{z,t} \\ w_t l_{z,t} \end{bmatrix} = \begin{bmatrix} 1/\varepsilon \\ 1 - 1/\varepsilon \end{bmatrix} e^{z-Z_t} \times \frac{P_t C_t}{N_t}$$

  - this is a "Red Queen environment"

- aggregate production labor L_t is

$$L_t = \int l_{z,t} N(t, \mathrm{d}z)$$

  - the definition of Z_t implies

$$w_t L_t = \left(1 - \frac{1}{\varepsilon}\right) P_t C_t, \quad l_{z,t} = e^{z-Z_t} \times \frac{L_t}{N_t}$$

- average profits in units of labor are

$$\frac{1}{w_t N_t} \int v_{z,t} N(t, \mathrm{d}z) = \frac{1/\varepsilon}{1 - 1/\varepsilon} \frac{1}{N_t} \int l_{z,t} N(t, \mathrm{d}z) = \frac{1/\varepsilon}{1 - 1/\varepsilon} \frac{L_t}{N_t}$$

productivity dynamics and firm values

- the fundamental assumption is

$$\mathrm{d}z_t = \theta \mathrm{d}t + \sigma \mathrm{d}W_t$$

  - firm-specific random walks, with a trend $\theta \in (-\infty, \infty)$
  - for example, $\theta = -\frac{1}{2}\sigma^2$, so that e^{z_t} is a positive martingale
  - key: there is always a *non-trivial* new set of *modifications* to try
  - of course, we could, instead, run out of ideas...

- firm continuation requires $\phi > 0$ units of labor per unit of time
- given $z_t = z$, the value of a firm is

$$\frac{V(t,z)}{P_t} = \max_{\tau \geq 0} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \times \frac{C_t/H_t}{C_s/H_s} \left(\frac{v_{z_s,s}}{P_s} - \frac{\phi w_s}{P_s} \right) \mathrm{d}s \right]$$

  - where τ is a stopping time
  - the use of logarithmic utility is not essential

- optimal to exit when $z_t \leq b_t$, for some b_t to be determined

aggregate conjectures

- conjecture that

$$Z_t = Z + (\theta - \mu)t$$

  for some μ to be determined

  – note that $-\mu > 0$ means $\mathrm{D}Z_t > \theta$

- conjecture a common growth rate for

  (i) per-capita consumption C_t/H_t
  (ii) the real wage w_t/P_t
  (iii) average real variable profits

  • recall that

$$\begin{bmatrix} \int v_{z,t} N(t, \mathrm{d}z) \\ w_t L_t \end{bmatrix} = \begin{bmatrix} 1/\varepsilon \\ 1 - 1/\varepsilon \end{bmatrix} P_t C_t$$

  where

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right) \left(e^{Z_t} N_t\right)^{1/(\varepsilon - 1)}$$

- this implies

$$\frac{L_t}{H_t} = \frac{L}{H}, \qquad \frac{N_t}{H_t} = \frac{N}{H}$$

aggregate versus firm growth

▶ for individual firms, $\mu = \theta - (\theta - \mu)$ is the drift of $z_t - Z_t$,

$$\mathrm{d}\left(z_t - Z_t\right) = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

– and firm employment scales with $e^{z_t - Z_t}$

• recall that

$$\left[\frac{L_t}{H_t}, \frac{N_t}{H_t}\right] = \left[\frac{L}{H}, \frac{N}{H}\right], \quad H_t = He^{\eta t}$$

and

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right)\left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)}, \qquad \frac{w_t}{P_t} \times \frac{L_t}{H_t} = \left(1 - \frac{1}{\varepsilon}\right)\frac{C_t}{H_t}$$

▶ so then $Z_t = Z + (\theta - \mu)t$ implies

$$\left[\frac{w_t}{P_t}, \frac{C_t}{H_t}\right] = \left[\frac{w}{P}, \frac{C}{H}\right] e^{\kappa t}, \qquad \kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1}$$

– important: *fast* real wage growth means *slow* firm employment growth, because individual firm productivities cannot keep up with wage growth

the implied value function

- recall that
$$\frac{v_{z,t}}{w_t} = \frac{1/\varepsilon}{1-1/\varepsilon} \times l_{z,t}, \quad l_{z,t} = \frac{e^{z-Z_t} L_t}{N_t},$$
and that C_t/H_t and w_t/P_t grow at a common rate

- this yields
$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \left(\frac{1/\varepsilon}{1-1/\varepsilon} \frac{e^{z_s - Z_s} L}{\phi N} - 1 \right) \mathrm{d}s \right],$$
where
$$z_s - Z_s = z - Z_t + \mu(s-t) + \sigma(W_s - W_t) \text{ for all } s \geq t$$

▶ this must be of the form
$$V(t,z) = \phi w_t \times U(y), \qquad e^y = \frac{1/\varepsilon}{1-1/\varepsilon} \frac{e^{z-Z_t} L}{\phi N}$$
for some time-invariant function $U(\cdot)$ that depends on μ

▶ in equilibrium, we will have
$$\mu + \frac{1}{2}\sigma^2 < \eta < \rho$$

the solution for $V(t,z)$

- is given by

$$V(t,z) \;=\; \phi w_t \times U\left(y\right), \qquad e^y = \frac{1/\varepsilon}{1-1/\varepsilon}\frac{e^{z-Z_t}L}{\phi N}$$

- where

$$U(y) = \begin{cases} 0 & ,\; y \le a \\ \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{y-a} - 1 - \frac{1-e^{-\xi(y-a)}}{\xi}\right) & ,\; y \ge a \end{cases}$$

  – the exit threshold $a < 0$ is determined by

$$e^a = \frac{\xi}{1+\xi}\left(1 - \frac{1}{\rho}\left(\mu + \frac{1}{2}\sigma^2\right)\right)$$

  and

$$\xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}$$

  – note that $y = 0$ corresponds to zero flow profits

▶ so we have a mapping

$$\mu \mapsto [a, U(\cdot)]$$

key properties of the value function

Lemma 1 *The value function is well defined if and only if* $\mu + \frac{1}{2}\sigma^2 < \rho$*. Given this restriction, it has the following properties:*

(i) *The value function is strictly increasing and unbounded in* $y > a$*.*

(ii) *The exit threshold is strictly decreasing in* μ*,*

$$\lim_{\mu \to -\infty} a = 0, \text{ and } \lim_{\mu \uparrow \rho - \sigma^2/2} a = -\infty.$$

(iii) *For any* $u \in (0, \infty)$ *or* $y \in (-\infty, \infty)$*,*

$$\lim_{\mu \to -\infty} U(a+u) = 0, \quad \lim_{\mu \uparrow \rho - \sigma^2/2} U(a+u) \in (0, \infty), \quad \lim_{\mu \uparrow \rho - \sigma^2/2} U(y) = \infty,$$

(iv) *For any* $u \in (0, \infty)$*,* $U(a+u)$ *is increasing in* μ*.*

memo

- the profitability variable is

$$e^y = \frac{v_{z,t}}{\phi w_t} = \frac{1/\varepsilon}{1-1/\varepsilon}\frac{e^{z-Z_t}L}{\phi N} = \frac{1/\varepsilon}{1-1/\varepsilon}\frac{1}{\phi} \times l_{z,t}$$

  - employment of a type-z firm at time t is therefore

$$l_{z,t} = (\varepsilon - 1)\phi \times e^y$$

- the time-t exit threshold b_t for a firm of type z must then solve

$$e^a = \frac{1/\varepsilon}{1-1/\varepsilon}\frac{e^{b_t-Z_t}L}{\phi N}$$

  - so the gap $Z_t - b_t$ must be constant over time
  - since $Z_t = Z + (\theta - \mu)t$, this implies

$$b_t = b + (\theta - \mu)t$$

- at time t, a productivity state z implies

$$\underbrace{z - b_t}_{\text{productivity relative to exit}} = \underbrace{y - a}_{\text{profitability or employment relative to exit}} = u$$

learning from incumbents

the knowledge diffusion assumption

- an entry opportunity is
  - a random draw from the incumbent population

- then, attempt to imitate the randomly sampled firm

- for simplicity
  - no further entry cost
  - can enter with the sampled z (and a different good)
  - all entry attempts will be successful

- the value of an entry opportunity is

$$q_t = \left(\int_{b_t}^{\infty} N(t, \mathrm{d}z) \right)^{-1} \int_{b_t}^{\infty} V(t, z) N(t, \mathrm{d}z)$$

- entrepreneurs produce a flow of entry opportunities $\mathcal{E}(q_t/w_t)H_t$

the Kolmogorov forward equation

- define α_t to be the (attempted) entry rate

$$\alpha_t = \frac{\mathcal{E}\left(q_t/w_t\right)}{N_t/H_t}$$

- for any $z \in (b_t, \infty)$

$$\texttt{the flow of entrants at } z \texttt{ is } \mathcal{E}\left(\frac{q_t}{w_t}\right) H_t \times \frac{n(t,z)}{N_t}$$

- therefore

$$\mathcal{E}\left(\frac{q_t}{w_t}\right) H_t \times \frac{n(t,z)}{N_t} = \frac{\mathcal{E}\left(q_t/w_t\right)}{N_t/H_t} \times n(t,z) = \alpha_t n(t,z)$$

- the Kolmogorov forward equation is then

$$\mathrm{D}_t n(t,z) = -\theta \mathrm{D}_z n(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} n(t,z) + \alpha_t n(t,z),$$

  for $z \in (b_t, \infty)$

- immediate exit at b_t means that

$$n(t, b_t) = 0$$

now conjecture stationarity

- strengthen $Z_t - b_t = Z - b$ to time-invariance of the cross-sectional distribution of $z - b_t$,

$$n(t, z) = N_t f(z - b_t), \qquad z \in (b_t, \infty)$$

  – the definition of Z_t implies

$$e^{Z-b} = \int_0^\infty e^u f(u) \mathrm{d}u$$

▶ the value q_t/w_t of an entry opportunity now becomes

$$\frac{q_t}{w_t} = \phi \int_0^\infty U\left(a + u\right) f(u) \mathrm{d}u$$

  – so q_t/w_t, $\mathcal{E}(q_t/w_t)$, and $\mathcal{L}(q_t/w_t)$ are constant over time

▶ since $N_t/H_t = N/H$, this means that $\alpha_t = \alpha$, and hence

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right)$$

  – given α, this is the *steady state supply* of firms as a function of q/w

clearing the labor market

- the variable employment of a type-z firm at time t is

$$l_{z,t} \;=\; (\varepsilon - 1)\phi e^{a+u}, \quad u = z - b_t$$

- recall that

$$e^{Z-b} \;=\; \int_0^\infty e^u f(u)\mathrm{d}u, \quad e^a = \frac{1/\varepsilon}{1-1/\varepsilon}\frac{e^{b-Z}L}{\phi N}$$

► the labor market clearing condition

$$\mathcal{L}\left(\frac{q}{w}\right) H = \phi N + L$$

can therefore be written as

$$\frac{N}{H} = \frac{1}{\phi}\frac{\mathcal{L}\left(q/w\right)}{1 + (\varepsilon - 1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u}$$

– the *steady state demand* for firms

traveling waves

the stationary KFE

- recall

$$z - b_t = z - Z_t + Z_t - b_t = z - Z_t + Z - b$$

and

$$\mathrm{d}\left(z_t - Z_t\right) = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

- the Kolmogorov forward equation for $n(t,z) = N_t f(z - b_t)$ becomes

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u) + \alpha f(u)$$

for $u \in (0, \infty)$

- the boundary conditions are
  - $f(0) = 0 = f(\infty)$
- and $f(\cdot)$ is supposed to integrate to 1

outside the steady state

- consider $u = z - b_t$ and some arbitrary entry rates $\{\alpha_t\}_{t\geq 0}$,

$$\mathrm{D}_1 m(t,u) = -\mu \mathrm{D}_2 m(t,u) + \frac{1}{2}\sigma^2 \mathrm{D}_{22} m(t,u) + \alpha_t m(t,u)$$

  starting from an initial value $m(0,u)$

  - this is a complete description of how $m(t,u)$ will evolve
  - notice that η does not appear in this description

- integrating over $u \in (0,\infty)$ gives

$$\mathrm{D}N_t = -\mu\left[m(t,\infty) - m(t,0)\right] + \frac{1}{2}\sigma^2\left[\mathrm{D}_2 m(t,\infty) - \mathrm{D}_2 m(t,0)\right] + \alpha_t N_t$$

  - imposing $m(t,\infty) = 0 = m(t,0)$ and $\mathrm{D}_2 m(t,\infty) = 0$ gives

$$\mathrm{D}N_t = -\frac{1}{2}\sigma^2 \mathrm{D}_2 m(t,0) + \alpha_t N_t$$

  - so $\frac{1}{2}\sigma^2 \mathrm{D}_2 m(t,0)$ must be the exit rate
  - and $\mathrm{D}N_t/N_t$ is implied by $m(0,u)$ and the trajectory $\{\alpha_t\}_{t\geq 0}$

- but we need $\mathrm{D}N_t/N_t = \eta$ in the long run

the stationarity question

- consider exit at $u = 0$ and

$$\mathrm{D}_1 m(t,u) = -\mu \mathrm{D}_2 m(t,u) + \frac{1}{2}\sigma^2 \mathrm{D}_{22} m(t,u) + \alpha_t m(t,u),$$

  together with

$$\alpha_t = \eta + \frac{1}{N_t} \times \frac{1}{2}\sigma^2 \mathrm{D}_2 m(t,0), \quad N_t = \int_0^\infty m(t,u)\mathrm{d}u,$$

  starting from an initial value $m(0,u)$

  – that is, endogenize the entry rate so that $\mathrm{D}N_t/N_t = \eta$

- defining $f(t,u) = m(t,u)/N_t$ gives

$$\mathrm{D}_1 f(t,u) = -\mu \mathrm{D}_2 f(t,u) + \frac{1}{2}\sigma^2 \mathrm{D}_{22} f(t,u) + (\alpha_t - \eta) f(t,u),$$

  together with

$$\alpha_t = \eta + \frac{1}{2}\sigma^2 \mathrm{D}_2 f(t,0)$$

  – the parameters are μ, σ^2, and η

- now find solutions of the form $\alpha_t = \alpha$ and $f(t,u) = f(u)$

solving the KFE

- fix $\mu < 0$, $\sigma^2 > 0$, and $\eta \geq 0$, and consider

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u) + \alpha f(u)$$

for all $u \in (0, \infty)$, with the boundary conditions $f(0) = 0 = f(\infty)$

- possible solutions

$$f(u) = A_- e^{-\zeta_- u} + A_+ e^{-\zeta_+ u}, \quad u \in (0, \infty)$$

where ζ_- and ζ_+ solve

$$\zeta_\pm = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \quad \psi = -\frac{\mu}{\sigma^2}$$

  – if $\alpha \in [0, \eta]$, then $\zeta_- \leq 0 \leq \zeta_+$, which rules out $f(0) = 0 = f(\infty)$
  – complex $\zeta_\pm$ yield a positive density only on a bounded interval

- need α to satisfy $0 < (\alpha - \eta)/(\sigma^2/2) \leq \psi^2$, and then

$$f(u) = \frac{\zeta_+ \zeta_-}{\zeta_+ - \zeta_-} \times \left(e^{-\zeta_- u} - e^{-\zeta_+ u}\right),$$

for all $u \in [0, \infty)$

possible stationary densities and associated entry rates

- the possibilities are

$$f(u) = \frac{\zeta_+ \zeta_-}{\zeta_+ - \zeta_-} \times \left(e^{-\zeta_- u} - e^{-\zeta_+ u}\right)$$

  where

$$\zeta_\pm = \psi \pm \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \quad \psi = -\frac{\mu}{\sigma^2}$$

  and

$$0 < \frac{\alpha - \eta}{\sigma^2/2} \leq \psi^2$$

- taking $\alpha \uparrow \eta + \frac{1}{2}\sigma^2\psi^2$ gives

$$\zeta_- \uparrow \psi, \quad \zeta_+ \downarrow \psi$$

  and $f(u)$ converges to

$$f(u) = \psi^2 u e^{-\psi u}$$

  – this is a gamma density (note that $\frac{1}{2}\sigma^2 \mathrm{D} f(0) = \frac{1}{2}\sigma^2\psi^2$)

▶ bounded initial conditions (Luttmer [2007, 2015]) or bounded imitation (Luttmer [2020]) argues for $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$

growth and the tail index

- the right tail behaves like $e^{-\zeta_- u}$, where

$$\zeta_- = \psi - \sqrt{\psi^2 - \frac{\alpha - \eta}{\sigma^2/2}}, \quad \psi = -\frac{\mu}{\sigma^2} > 0$$

and

$$0 < \frac{\alpha - \eta}{\sigma^2/2} \leq \psi^2$$

- holding fixed an entry rate $\alpha < \eta + \frac{1}{2}\sigma^2\psi^2$,

$$\frac{\partial \zeta_-}{\partial \psi} = 1 - \frac{\psi}{\sqrt{\psi^2 - \frac{\alpha-\eta}{\sigma^2/2}}} < 0$$

  – more rapid growth $\Rightarrow \psi \uparrow \Rightarrow \ \zeta_- \downarrow \Rightarrow$ a *thicker* tail
  – bootstrap growth driven a thick-tailed initial value $n(0, z)$

- but if $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$, then

  – more rapid growth $\Rightarrow \psi \uparrow \ \Rightarrow \ f(u) = \psi^2 u e^{-\psi u}$ has a *thinner* tail
  – bottom up growth; explains history of thickening right tail

finite aggregates

- consider

$$f(u) = \psi^2 u e^{-\psi u}, \quad u \in (0, \infty)$$

- the mean of e^u is finite if and only if $\psi > 1$, and then

$$\int_0^\infty e^u f(u) \mathrm{d}u = \int_0^\infty \psi^2 u e^{-(\psi-1)u} \mathrm{d}u = \left(\frac{\psi}{\psi - 1} \right)^2$$

  – observe the $\psi = -\mu/\sigma^2 > 1$ ensures

$$\mu + \frac{1}{2}\sigma^2 < \mu + \sigma^2 < 0 \leq \eta$$

- the right tail probabilities are

$$R(u) = \int_{\psi u}^\infty s e^{-s} \mathrm{d}s = (1 + \psi u) e^{-\psi u}$$

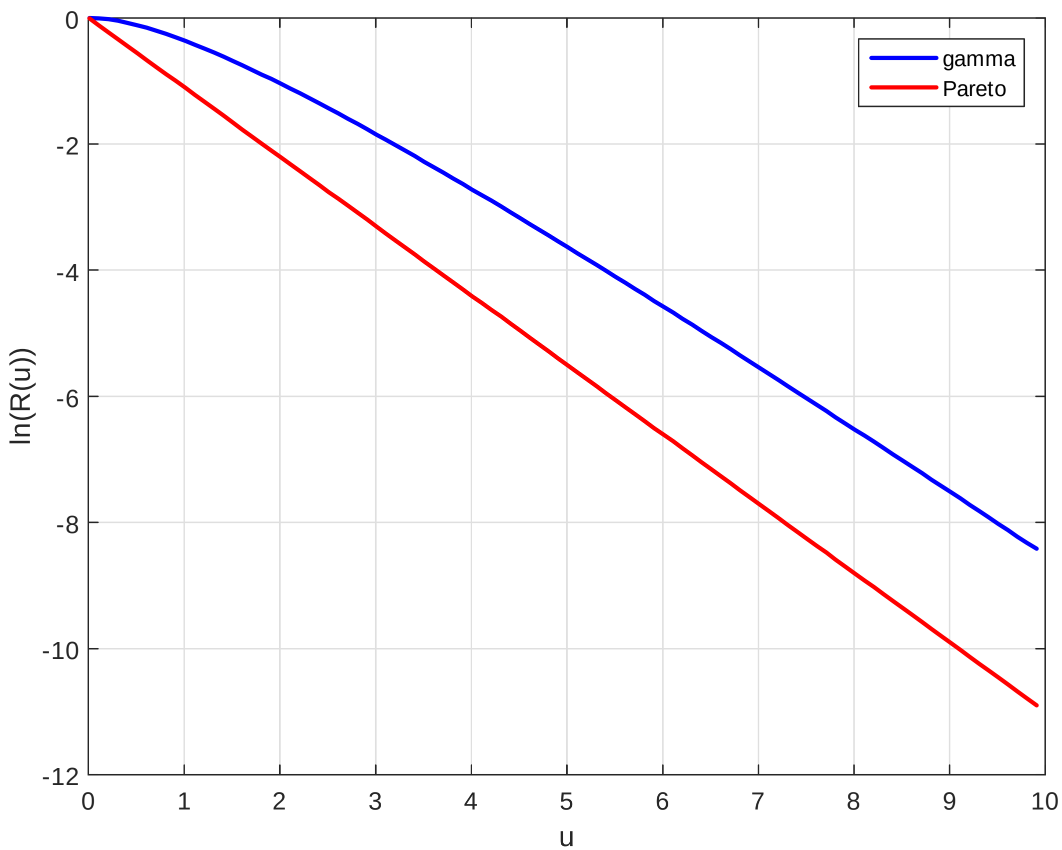
- this implies

$$\lim_{u \to \infty} \frac{\ln(R(u))}{u} = -\psi$$

  and also

$$\lim_{u \to \infty} \frac{\mathrm{d} \ln(R(u))}{\mathrm{d}u} = - \lim_{u \to \infty} \frac{f(u)}{R(u)} = - \lim_{u \to \infty} \frac{\psi^2 u e^{-\psi u}}{(1 + \psi u) e^{-\psi u}} = -\psi$$

the case $\psi = 1.1$



- the Pareto is $P(x) = 1 - x^{-\psi}$ for $x \in [1, \infty)$
- with $\ln(R(u)) = \ln(1 - P(x))$ plotted against $u = \ln(x)$

the balanced growth path

constructing a BGP

- the Bellman equation gives a function $\mu \mapsto [a, U(\cdot)]$
- the KFE (with a refinement) gives a function $\mu \mapsto [\alpha, f(\cdot)]$

▶ a present-value condition

$$\frac{q}{w} = \phi \int_0^\infty U(a+u) f(u) \mathrm{d}u \tag{1}$$

▶ a steady state supply of firms

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right) \tag{2}$$

▶ a steady state demand for firms

$$\frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}(q/w)}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) \mathrm{d}u} \tag{3}$$

the equations for balanced growth

- clearing the steady state market for firms gives

$$\frac{\mathcal{E}\,(q/w)}{\mathcal{L}\,(q/w)} = \frac{1}{\phi}\frac{\alpha}{1+(\varepsilon-1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u} \tag{1}$$

- the relative price q/w must also satisfy

$$\frac{q}{w} = \phi \int_0^\infty U\,(a+u)\, f(u)\mathrm{d}u \tag{2}$$

- in the background
  - the Bellman equation yields

$$\mu \mapsto [a, U(\cdot)]$$

  - the KFE (with a refinement) yields

$$\mu \mapsto [\alpha, f(\cdot)]$$

▶ if $N_0/H_0 = N/H$ and the initial density satisfies $n(0,z)/N_0 = f(z-b)$ for some b, then the economy is on a balanced growth path inelastic factor supplies

inelastic factor supplies

- this fixes $\mathcal{E}/\mathcal{L} \in (0, \infty)$, and then μ is determined by

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{1}{\phi} \frac{\alpha}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u) \mathrm{d}u} \tag{$\mathcal{E}/\mathcal{L}$}$$

  - where the Bellman equation yields

$$\mu \mapsto [a, U(\cdot)]$$

  - and the KFE (with a refinement) yields

$$\mu \mapsto [\alpha, f(\cdot)]$$

▶ the RHS of ($\mathcal{E}/\mathcal{L}$) ranges throughout $(0, \infty)$ on $\left\{\mu : \mu < -\sigma^2\right\}$

  - since $\rho > \eta$, this implies

$$\mu + \frac{1}{2}\sigma^2 < \eta < \rho$$

  - the relative price q/w is determined by

$$\frac{q}{w} = \phi \int_0^\infty U(a + u) f(u) \mathrm{d}u,$$

    and this is well defined by construction

explicit calculations

- recall that $\psi = -\mu/\sigma^2$ and

$$\int_0^\infty e^u f(u)\mathrm{d}u = \left(\frac{\psi}{\psi - 1}\right)^2, \quad \frac{1}{2}\sigma^2 \mathrm{D}f(0) = \frac{1}{2}\sigma^2\psi^2$$

- the stopping problem gives

$$e^a = 1 + \frac{\sigma^2/2}{\rho} \times \psi - \sqrt{\left(\frac{\sigma^2/2}{\rho} \times \psi\right)^2 + \frac{\sigma^2/2}{\rho}}$$

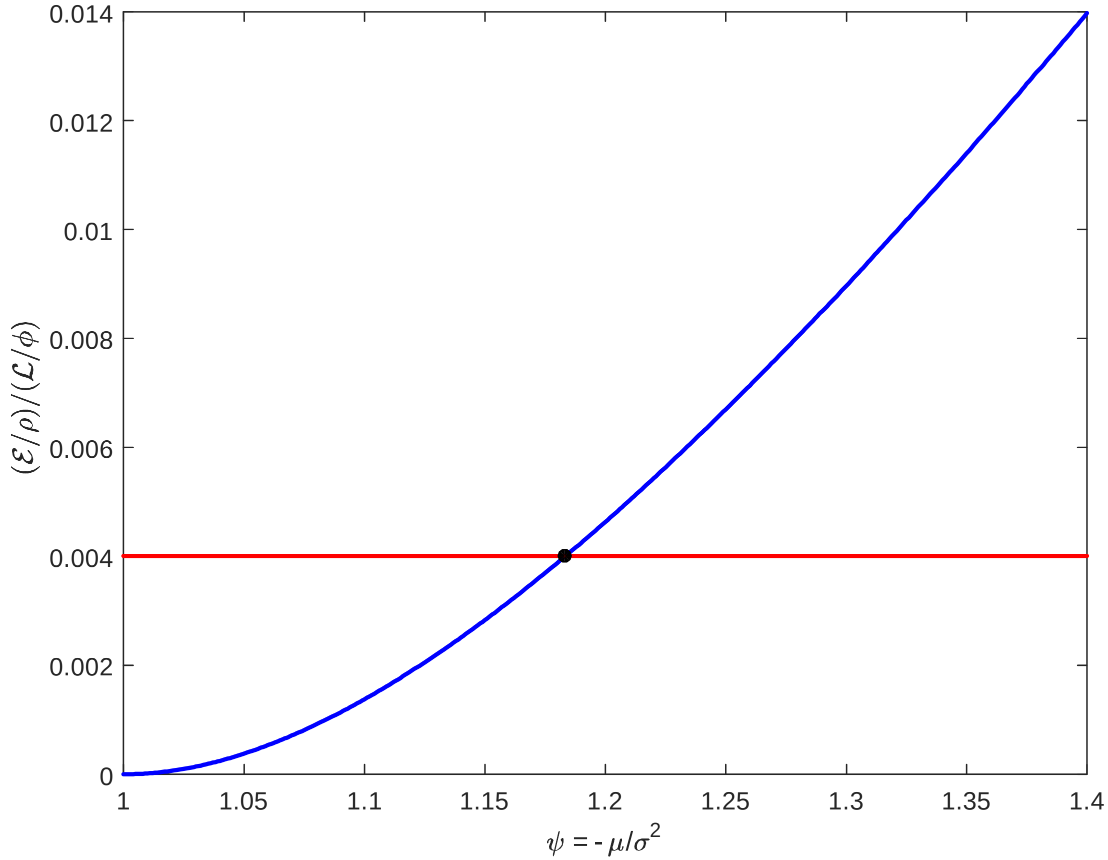
▶ therefore

$$\frac{\mathcal{E}/\rho}{\mathcal{L}/\phi} = \frac{\frac{\eta}{\rho} + \frac{\sigma^2/2}{\rho} \times \psi^2}{1 + (\varepsilon - 1)\left(1 + \frac{\sigma^2/2}{\rho} \times \psi - \sqrt{\left(\frac{\sigma^2/2}{\rho} \times \psi\right)^2 + \frac{\sigma^2/2}{\rho}}\right)\left(\frac{\psi}{\psi - 1}\right)^2}$$

▶ contribution of selection to consumption growth

$$\kappa - \frac{\eta + \theta}{\varepsilon - 1} = \frac{\mathrm{D}Z_t - \theta}{\varepsilon - 1} = -\frac{\mu}{\varepsilon - 1} = \frac{\sigma^2\psi}{\varepsilon - 1}$$

an example with $\rho = 0.05$, $\sigma = 0.4$ and $\varepsilon = 4$



$$\frac{\mathcal{E}/\rho}{\mathcal{L}/\phi} = \frac{\texttt{discounted per-capita flow of new firms created}}{\texttt{employment in multiples of fixed labor}}$$

some smell tests

- recall

$$\frac{\mathcal{E}/\rho}{\mathcal{L}/\phi} = \frac{\texttt{discounted per-capita flow of new firms created}}{\texttt{employment in multiples of fixed labor}}$$

- if $\mathcal{L}/\phi = 20$ then this gives

$$\mathcal{E} = \rho \times \mathcal{L}/\phi \times 0.004 = 0.05 \times 20 \times 0.004 = 0.004$$

- the steady state number of firms is then

$$\frac{N}{H} = \frac{\mathcal{E}}{\eta + \frac{1}{2}\sigma^2\psi^2} = \frac{0.004}{0.01 + \frac{1}{2}(0.4 \times 1.183)^2} = 0.033$$

  - the underlying exit rate is

$$\frac{1}{2}\sigma^2\psi^2 = \frac{1}{2}(0.4 \times 1.183)^2 = 0.112$$

- US private sector employment is about 135 million people

$$N = 0.033 \times 135 = 4.155 \texttt{ million}$$

  - the actual number of employer firms is about 6.1 million

- the market value of the average firm relative to fixed labor is

$$\frac{q}{w\phi} = \int_0^\infty U\left(a+u\right) f(u)\mathrm{d}u$$

where

$$f(u) = \psi^2 u e^{-\psi u}$$

and

$$U(a+u) = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^u - 1 - \frac{1-e^{-\xi u}}{\xi}\right), \;\; \xi = -\psi + \sqrt{\psi^2 + \frac{\rho}{\sigma^2/2}}$$

- this yields

$$\frac{q}{w\phi} = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(\left(\frac{\psi}{\psi-1}\right)^2 - 1 - \frac{1}{\xi}\left(1 - \left(\frac{\psi}{\psi+\xi}\right)^2\right)\right)$$

  – our parameters imply $q/(w\phi) = 153$

- entrepreneurial income relative to labor income is then

$$\frac{q\mathcal{E}}{w\mathcal{L}} = \rho \times \frac{q}{w\phi} \times \frac{\mathcal{E}/\rho}{\mathcal{L}/\phi} = 0.05 \times 153 \times 0.004 = 0.0306$$

  – note well that these are serial entrepreneurs

- the aggregate value of firms relative to labor income is now

$$\frac{qN}{w\mathcal{L}H} = \frac{q}{\phi w}\frac{N/H}{\mathcal{L}/\phi} = 153 \times \frac{0.033}{20} = 0.251$$

- the market cap of US stock markets is around 1.5 times US GDP,
  - which does not include the value of privately held businesses
  - so aggregate firm values seem off by an order of magnitude…
- but the model economy
  - has no physical capital,
  - only "organization capital"

▶ contribution of selection to consumption growth

$$\kappa - \frac{\eta + \theta}{\varepsilon - 1} = \frac{\mathrm{D}Z_t - \theta}{\varepsilon - 1} = -\frac{\mu}{\varepsilon - 1} = \frac{\sigma^2 \psi}{\varepsilon - 1} = \frac{(0.4)^2 \times 1.183}{4 - 1} = 0.063$$

- but a reasonable number for κ is about 2% per annum
- so θ will be have to be significantly negative
- tail index and high exit rates force large $-\mu$ and σ^2

concluding remark

- the equilibrium will satisfy

$$\mu + \frac{1}{2}\sigma^2 < \eta$$

- this implies a per-capita consumption growth rate

$$\kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1} > \frac{1}{\varepsilon - 1}\left(\theta + \frac{1}{2}\sigma^2\right)$$

- for individual firms

$$\mathrm{d}\left[e^{z_t}\right] = \left[e^{z_t}\right]\left(\left(\theta + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t\right)$$

- the scenario $\theta + \frac{1}{2}\sigma^2 = 0$ shows that
  - even if e^{z_t} is “only” a martingale for individual firms…
  - …the overall economy will grow

adopting the latest science

exogenous growth

- new firms at time t can enter at the exogenous entry point

$$x_t = x + (\theta - \mu)t$$

  for some exogenous x and μ

- conjecture that along a BGP $[L_t, N_t]/H_t = [L, N]/H$ and

$$Z_t = Z + (\theta - \mu)t$$

- as before, the optimal exit decision implies

$$b_t = b + (\theta - \mu)t, \quad e^a = \frac{1/\varepsilon}{1 - 1/\varepsilon} \frac{e^{b-Z} L}{\phi N}$$

  – now, the exogenous μ fixes a via the optimal stopping problem
  – BGP equilibrium conditions must deliver $x - b$, $Z - x$, and L/N

- it is convenient to again write $u = z - b_t$, and

$$\Delta = x - b$$

  – with μ exogenous, this endogenous variable will now determine the shape of the stationary distribution of productivities

the KFE

- the Kolmogorov forward equation is

$$\mathrm{D}_1 m(t,u) = -\mu \mathrm{D}_2 m(t,u) + \frac{1}{2}\sigma^2 \mathrm{D}_{22} m(t,u), \quad z \in (0,\Delta) \cup (\Delta,\infty)$$

  – the boundary conditions at 0 and Δ are

$$m(t,0) = 0, \quad m(t,\Delta_-) = m(t,\Delta_+),$$

  – and $m(t,u)$, $\mathrm{D}_2 m(t,u)$, and $\mathrm{D}_{22} m(t,u)$ must vanish for large u

- the Brownian motion implies that the flow of exiting firms is $\frac{1}{2}\sigma^2 \mathrm{D}_2 m(t,0)$
- there will be an entry flow at Δ

the number of firms

- integrating the KFE over u gives

$$\begin{aligned}\mathrm{D}N_t \;=\; & -\mu\left[m(t,\Delta_-) - m(t,0)\right] + \frac{1}{2}\sigma^2\left[\mathrm{D}_2 m(t,\Delta_-) - \mathrm{D}_2 m(t,0)\right] \\ & -\mu\left[m(t,\infty) - m(t,\Delta_+)\right] + \frac{1}{2}\sigma^2\left[\mathrm{D}_2 m(t,\infty) - \mathrm{D}_2 m(t,\Delta_+)\right]\end{aligned}$$

  – the boundary conditions imply that this reduces to

$$\mathrm{D}N_t = -\frac{1}{2}\sigma^2 \mathrm{D}_2 m(t,0) + \frac{1}{2}\sigma^2\left[\mathrm{D}_2 m(t,\Delta_-) - \mathrm{D}_2 m(t,\Delta_+)\right]$$

  – the first term on the RHS represents the exit flow

  – the second term must therefore measure the entry flow

- we will use this to back out the required entry rate

the stationary version of the KFE

- conjecture that there is a stationary density

$$N_t = Ne^{\eta t}, \;\; m(t,u) = f(u)N_t$$

- this implies

$$\mathrm{D}_t m(t,u) = f(u)\eta N_t$$

and

$$\left[\, \mathrm{D}_2 m(t,u) \;\; \mathrm{D}_{22} m(t,u) \,\right] = \left[\, \mathrm{D}f(u) \;\; \mathrm{D}^2 f(u) \,\right] N_t$$

- the resulting KFE is

$$\eta f(u) = -\mu \mathrm{D}f(u) + \frac{1}{2}\sigma^2 \mathrm{D}f(u), \quad u \in (0,\Delta) \cup (\Delta,\infty)$$

- the boundary conditions are

$$f(0) = 0, \;\; \lim_{u\uparrow\Delta} f(u) = \lim_{u\downarrow\Delta} f(u), \;\; \lim_{u\to\infty} f(u) = 0$$

- the required entry rate is

$$\alpha = \eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0) = \frac{1}{2}\sigma^2 \left[\mathrm{D}f(\Delta_-) - \mathrm{D}f(\Delta_+)\right]$$

the stationary density

- recall

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D} f(u), \quad u \in (0, \Delta) \cup (\Delta, \infty)$$

- solutions of the form $e^{-\zeta u}$ require

$$\eta = \mu\zeta + \frac{1}{2}\sigma^2\zeta^2$$

and hence $\zeta \in \{\zeta_-, \zeta_+\}$, where

$$\zeta_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- now we can consider solutions of the form

$$\begin{aligned} f(u) &= A_- e^{-\zeta_- u} + A_+ e^{-\zeta_+ u}, \quad u \in (0, \Delta) \\ f(u) &= B_- e^{-\zeta_- u} + B_+ e^{-\zeta_+ u}, \quad u \in (\Delta, \infty) \end{aligned}$$

  - the boundary conditions pin down $[A_-, A_+, B_-, B_+]$ up to scale
  - requiring $f(u)$ to integrate to 1 gives the scale

the solution

- the stationary density is

$$f(u) \;=\; \frac{\zeta e^{-\zeta u}}{(e^{\zeta_* \Delta} - 1)/\zeta_*} \times \min\left\{\frac{e^{(\zeta+\zeta_*)u} - 1}{\zeta + \zeta_*}, \frac{e^{(\zeta+\zeta_*)\Delta} - 1}{\zeta + \zeta_*}\right\},$$

  where

$$\zeta \;=\; -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \quad \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- note that the right tail behaves like $e^{-\zeta u}$

- the implied exit rate is

$$\frac{1}{2}\sigma^2 \mathrm{D} f(0) = \frac{1}{2}\sigma^2 \zeta \times \left(\frac{e^{\zeta_* \Delta} - 1}{\zeta_*}\right)^{-1}$$

the condition for $\mathrm{E}[e^u] < \infty$

- a model with a continuum of firms must have $\mathrm{E}[e^u] < \infty$
- this requires $\zeta > 1$, where

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- note that

$$\begin{aligned} \frac{\partial}{\partial x}\left(-x + \sqrt{x^2+1}\right) &= -1 + \frac{x}{\sqrt{x^2+1}} < 0 \\ 1 &= -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} \Rightarrow \eta = \mu + \frac{1}{2}\sigma^2 \end{aligned}$$

- so μ cannot be too large,

$$\texttt{drift of firm employment =}\ \mu + \frac{1}{2}\sigma^2 < \eta$$

  – with exogenous growth, this has to be assumed
- at $\eta = 0.01$, $\mu = 0$ would require $\sigma < 0.14$
  – there is evidence to suggest that σ is larger than that, forcing $\mu < 0$

given Δ, rapid firm growth implies less churn

- recall that the exit rate equals

$$\frac{1}{2}\sigma^2 \mathrm{D}f(0) = \frac{1}{2}\sigma^2 \zeta \left(\frac{e^{\zeta_* \Delta} - 1}{\zeta_*}\right)^{-1}$$

where

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \quad \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- note that $(e^x - 1)/x$ is increasing in x
- and we have

$$\frac{\partial \zeta}{\partial \mu} < 0, \quad \frac{\partial \zeta_*}{\partial \mu} > 0$$

- hence

$$\frac{\partial}{\partial \mu}\left(\frac{1}{2}\sigma^2 \mathrm{D}f(0)\right) < 0$$

  - this holds Δ fixed
  - an implication of stationarity, not of the exit policy

the $\eta \downarrow 0$ limit

- suppose $\mu < 0$, and consider

$$\begin{aligned}\zeta_* &= \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} \\ &= -\frac{\mu}{\sigma^2} \times \left(-1 + \sqrt{1 + \frac{\eta}{\sigma^2/2}\left(\frac{\mu}{\sigma^2}\right)^{-2}}\right) \qquad \text{(signs!)}\end{aligned}$$

- then

$$\frac{\zeta_*}{\eta} = -\frac{2}{\mu} \times \frac{1}{\frac{\eta}{\sigma^2/2}\left(\frac{\mu}{\sigma^2}\right)^{-2}} \left(-1 + \sqrt{1 + \frac{\eta}{\sigma^2/2}\left(\frac{\mu}{\sigma^2}\right)^{-2}}\right)$$

- by l'Hôpital

$$\lim_{x \to 0} \frac{1}{x}\left(-1 + \sqrt{1+x}\right) = \left(\frac{1}{2}\frac{1}{\sqrt{1+x}}\right)_{x=0} = \frac{1}{2}$$

- hence

$$\lim_{\eta \downarrow 0} \frac{\zeta_*}{\eta} = -\frac{1}{\mu}$$

the special case $\eta = 0$

- this requires $\mu < 0$
  - selection must contribute to aggregate productivity growth
- then

$$\zeta = -\frac{\mu}{\sigma^2/2}, \quad \zeta_* = 0$$

  - contrast with the tail index $\psi = -\mu/\sigma^2$ obtained in the economy with learning from incumbents
- the density reduces to

$$f(u) = e^{-\zeta u} \times \min\left\{\frac{e^{\zeta u} - 1}{\Delta}, \frac{e^{\zeta \Delta} - 1}{\Delta}\right\}$$

- the implied exit rate becomes

$$\frac{1}{2}\sigma^2\zeta \times \frac{1}{\Delta} = -\frac{\mu}{\Delta} \qquad (!)$$

components of the mean

- on $(0, \Delta)$

$$\int_0^\Delta e^u f(u)\mathrm{d}u = \frac{\zeta\zeta_*}{e^{\zeta_*\Delta}-1}\int_0^\Delta \left(\frac{e^{(1+\zeta_*)u} - e^{-(\zeta-1)u}}{\zeta+\zeta_*}\right)\mathrm{d}u$$
$$= \frac{\zeta\zeta_*}{e^{\zeta_*\Delta}-1}\frac{1}{\zeta+\zeta_*}\left(\frac{e^{(1+\zeta_*)\Delta}-1}{1+\zeta_*} - \frac{1-e^{-(\zeta-1)\Delta}}{\zeta-1}\right)$$

- on (Δ, ∞)

$$\int_\Delta^\infty e^u f(u)\mathrm{d}u = \frac{e^{(\zeta+\zeta_*)\Delta}-1}{\zeta+\zeta_*}\frac{e^{-(\zeta-1)\Delta}}{(e^{\zeta_*\Delta}-1)/\zeta_*}\int_\Delta^\infty \zeta e^{-(\zeta-1)(u-\Delta)}\mathrm{d}u$$
$$= \frac{e^{(\zeta+\zeta_*)\Delta}-1}{\zeta+\zeta_*}\frac{e^{-(\zeta-1)\Delta}}{(e^{\zeta_*\Delta}-1)/\zeta_*}\frac{\zeta}{\zeta-1}$$

- adding up and cancelling some terms (…) yields

$$\mathrm{E}\left[e^u\right] = \frac{\zeta\zeta_*}{(\zeta-1)(\zeta_*+1)}\frac{e^{(1+\zeta_*)\Delta}-1}{e^{\zeta_*\Delta}-1} \qquad (!)$$

  – the $\Delta \downarrow 0$ limit is $\zeta/(\zeta-1)$ (as if Pareto)

the mean of this distribution

- note that

$$\frac{1}{2}\sigma^2\zeta\zeta_* \;=\; \eta$$

$$\frac{1}{2}\sigma^2(\zeta-1)(\zeta_*+1) \;=\; \eta - \left(\mu + \frac{1}{2}\sigma^2\right)$$

- hence

$$\mathrm{E}\left[e^u\right] = \frac{\zeta\zeta_*}{(\zeta-1)(\zeta_*+1)}\frac{e^{(1+\zeta_*)\Delta}-1}{e^{\zeta_*\Delta}-1}$$

  becomes

$$\mathrm{E}\left[e^u\right] = \frac{\eta}{\eta-\left(\mu+\frac{1}{2}\sigma^2\right)}\frac{e^{(1+\zeta_*)\Delta}-1}{e^{\zeta_*\Delta}-1}$$

  – increasing in μ, and explodes when $\mu + \frac{1}{2}\sigma^2 \uparrow \eta$

- assuming $\mu + \frac{1}{2}\sigma^2 < 0$, the $\eta \downarrow 0$ limit is

$$\mathrm{E}\left[e^u\right] = \frac{\zeta}{\zeta-1}\frac{e^{\Delta}-1}{\Delta}, \quad \zeta = -\frac{\mu}{\sigma^2/2}$$

equilibrium conditions for the BGP

- the steady state measure of firms must satisfy

$$\frac{N}{H} = \frac{\mathcal{E}\left(q/w\right)}{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)}, \quad \frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}\left(q/w\right)}{1 + (\varepsilon - 1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u}$$

  – and the stopping problem implies $q/w = \phi U\left(a + \Delta\right)$

- clearing the steady state market for firms gives

$$\frac{q}{w} = \phi U\left(a + \Delta\right), \quad \frac{\mathcal{E}\left(q/w\right)}{\mathcal{L}\left(q/w\right)} = \frac{1}{\phi} \frac{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)}{1 + (\varepsilon - 1)\int_0^\infty e^{a+u} f(u)\mathrm{d}u}$$

  – together with the mapping $\Delta \mapsto f(\cdot)$

  – to be solved for $\Delta = x - b$; non-existence of BGP if $\mu + \frac{1}{2}\sigma^2 \geq \eta$

- at date $t = 0$ we have x_0 and $n(0, z)$ (and therefore N_0)

  – if it is true that

$$n(0, z) = f(z - x_0 + \Delta)N_0, \quad z \in (-\infty, \infty)$$

  then the economy is in a steady state equilibrium at $t = 0$

improving abandoned technologies

a knowledge diffusion mechanism

- active firms can protect their trade secrets for free
  - every active firm has a monopoly
  - product qualities change randomly over time
  - firms have to pay a continuation cost to remain active
  - exit when product quality falls below some threshold
- trade secrets of exiting firms become public knowledge
  - quality stops changing upon exit
  - falls further behind the exit threshold of active firms
  - no point paying a cost just to protect the trade secret
- entrants can improve the quality of abandoned technologies
  - cost: entrepreneurs could be workers instead
  - best to build on the most recently abandoned technologies

endogenous growth again

- now suppose entrants can use

$$x_t = b_t + \Delta$$

  for some exogenously specified $\Delta > 0$

- the equilibrium conditions are

$$\frac{q}{\phi w} = U\left(a + \Delta\right), \quad \frac{\mathcal{E}\left(q/w\right)}{\mathcal{L}\left(q/w\right)} = \frac{1}{\phi}\frac{\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)}{1 + (\varepsilon - 1)\int_0^\infty e^{a+u}f(u)\mathrm{d}u}$$

  – together with the mappings $\mu \mapsto [a, U(\cdot)]$ and $\mu \mapsto f(\cdot)$
  – to be solved for μ
  – steady state equilibrium automatically implies $\mu + \frac{1}{2}\sigma^2 < \eta$

- at date $t = 0$ we have x_0 and $n(0, z)$ (and therefore N_0)

  – we have an equilibrium if it is true that

$$n(0, z) = f(z - x_0 + \Delta)N_0, \quad z \in (-\infty, \infty)$$

the mapping $\mu \mapsto [a, U(\cdot), f(\cdot)]$

- the value function is

$$U(y) = \frac{1}{\rho}\frac{\xi}{1+\xi}\left(e^{y-a} - 1 - \frac{1 - e^{-\xi(y-a)}}{\xi}\right)$$

and

$$e^{a} \;=\; \frac{\xi}{1+\xi}\left(1 - \frac{\mu + \sigma^2/2}{\rho}\right), \;\; \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\rho}{\sigma^2/2}}$$

- the stationary density is

$$f(y) = \frac{\zeta e^{-\zeta(y-b)}}{(e^{\zeta_* \Delta} - 1)/\zeta_*} \times \min\left\{\frac{e^{(\zeta+\zeta_*)(y-b)} - 1}{\zeta + \zeta_*}, \frac{e^{(\zeta+\zeta_*)\Delta} - 1}{\zeta + \zeta_*}\right\}$$

and

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}, \;\; \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

key properties of $\mu \mapsto [a, U(\cdot), f(\cdot)]$

- the exit threshold and value of an entrant

$$\frac{\mathrm{d}a}{\mathrm{d}\mu} < 0, \quad \frac{\mathrm{d}(q/w)}{\mathrm{d}\mu} = \phi \times \frac{\mathrm{d}U(a+\Delta)}{\mathrm{d}\mu} > 0$$

- the exit rate

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1}{2}\sigma^2 \mathrm{D}f(0)\right) < 0$$

- the tail index

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\mu} = \frac{\mathrm{d}}{\mathrm{d}\mu}\left(-\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}\right) < 0$$

- the mean size (tricky since $\mathrm{d}a/\mathrm{d}\mu$)

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\int_0^\infty e^{a+u} f(u)\mathrm{d}u > 0$$

  – importantly,

$$\mu + \frac{1}{2}\sigma^2 \uparrow \eta \text{ implies } \int_0^\infty e^{a+u} f(u)\mathrm{d}u \to \infty$$

$\Rightarrow$ demand for N/H is decreasing in q/w, supply is increasing

the Fréchet version of Roy

- suppose the talent distribution is independent Fréchet

$$\mathcal{P}(x, y) = \exp\left(-T_E x^{-\theta} - T_L y^{-\theta}\right)$$

for some positive T_E and T_L, and $\theta > 1$, and

  - x is new entry opportunities per unit of time
  - y is units of labor per unit of time

- the occupational choice probabilities are

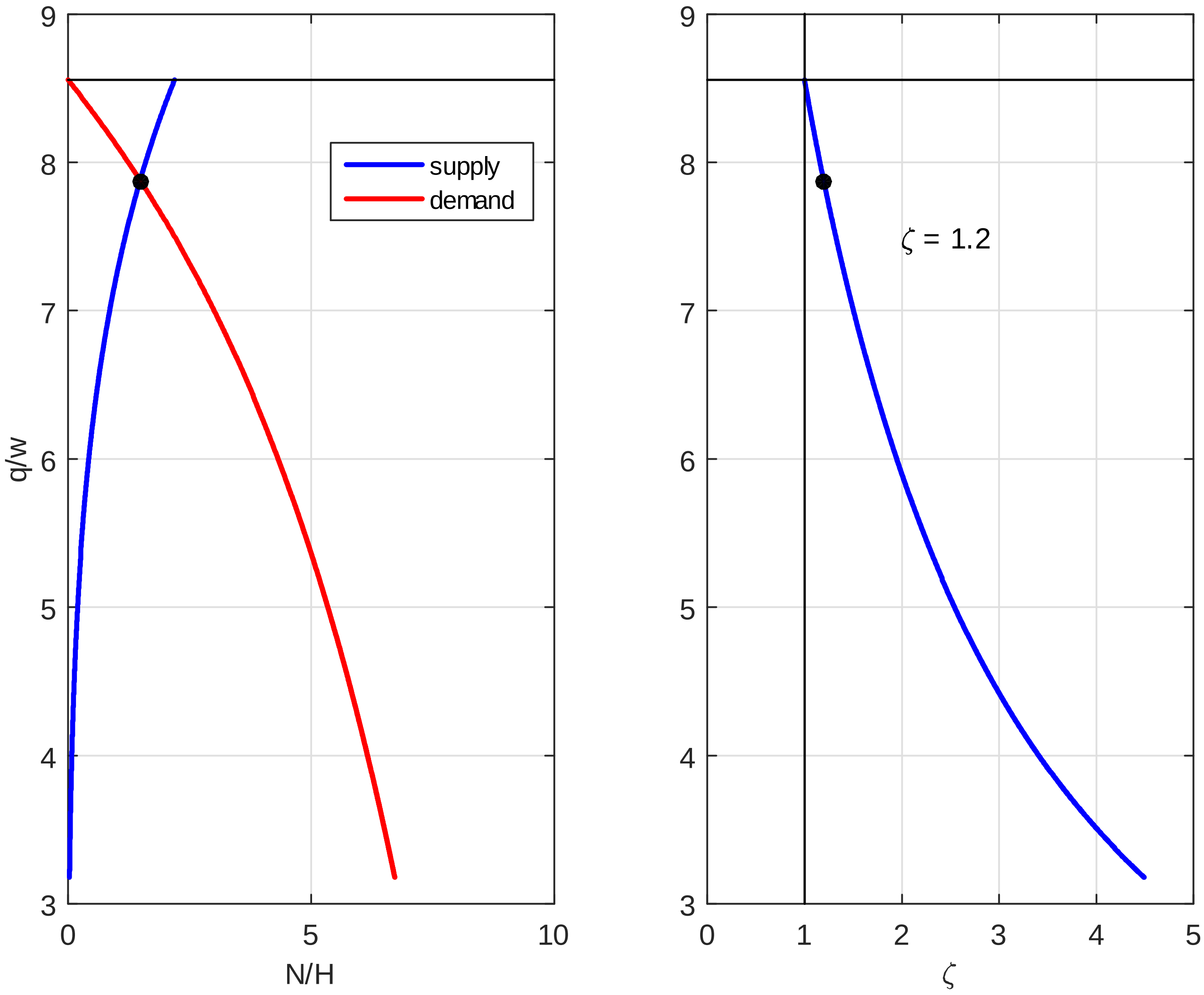
$$\begin{bmatrix} \mathcal{P}_E(s) \\ \mathcal{P}_L(s) \end{bmatrix} = \frac{1}{T_E s^\theta + T_L} \begin{bmatrix} T_E s^\theta \\ T_L \end{bmatrix}$$

and the factor supplies are implied by

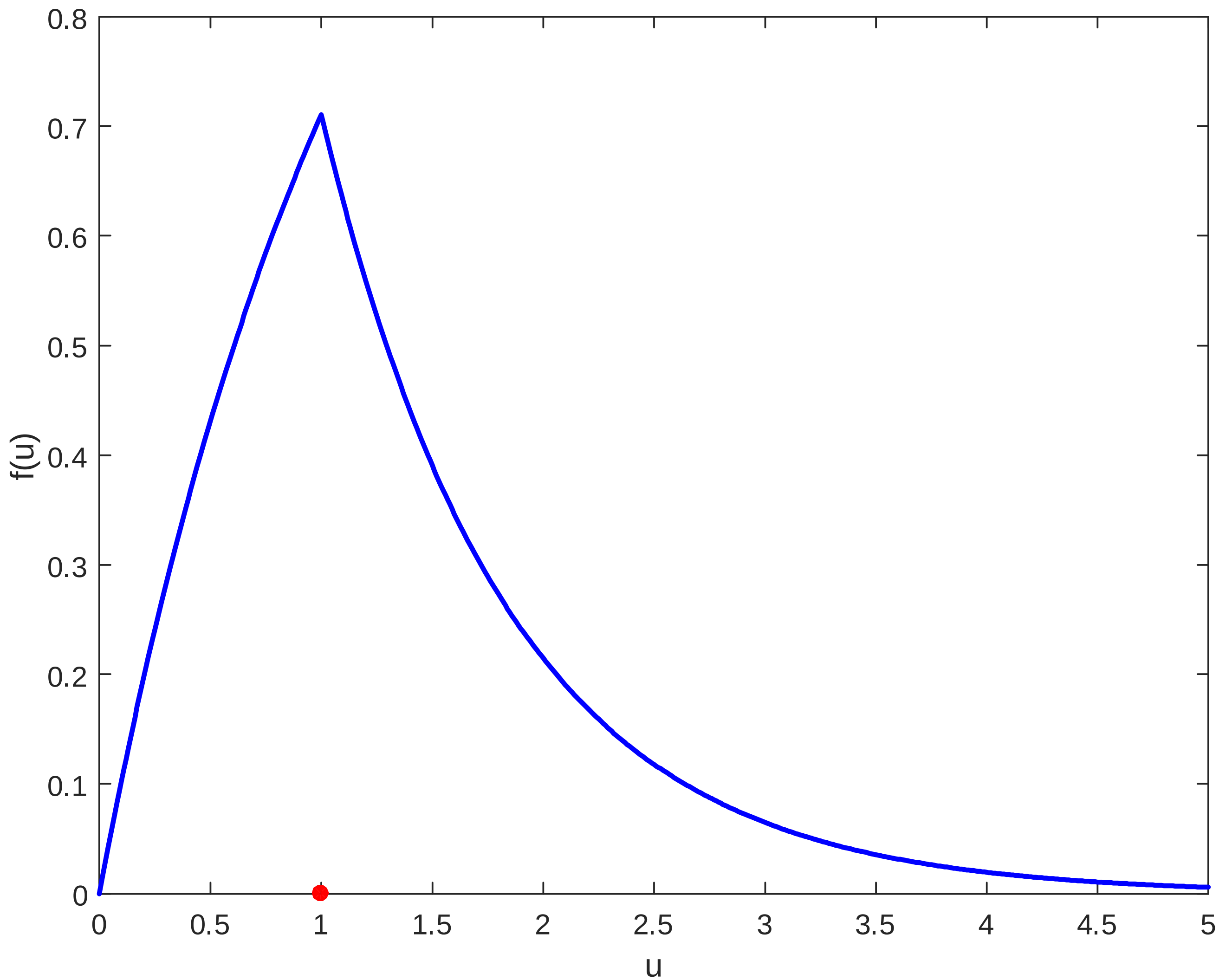
$$\begin{bmatrix} s\mathcal{E}(s) \\ \mathcal{L}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_E(s) \\ \mathcal{P}_L(s) \end{bmatrix} \times \left(T_E s^\theta + T_L\right)^{1/\theta} \Gamma\left(1 - 1/\theta\right)$$

- but the income distributions are identical across occupations...

an example with $\rho = 0.05$, $\sigma = 0.3$ and $\varepsilon = 4$



the implied stationary density



the Zipf asymptote

- demand and supply for firms

$$\frac{N}{H} = \frac{1}{\phi}\frac{\mathcal{L}(s)}{1+(\varepsilon-1)\int_0^\infty e^{a+u}f(u)\mathrm{d}u}$$

$$\frac{N}{H} = \frac{\mathcal{E}(s)}{\eta+\frac{1}{2}\sigma^2\mathrm{D}f(0)}$$

  where $s = V(a+\Delta)$

  - average firm size explodes as s increases and $\mu+\frac{1}{2}\sigma^2 \uparrow \eta$
  - hence, the demand for firms goes to zero

- to approach Zipf
  - shift the supply curve in
  - along the downward sloping demand curve
  - can use shifts in $\mathcal{E}(\cdot)$ and $\mathcal{L}(\cdot)$ (from the Roy model)

so what determines growth?

- per-capita consumption equals

$$\frac{C_t}{H_t} = \left(e^{Z_t} N_t\right)^{1/(\varepsilon-1)} \times \frac{L}{H}, \quad Z_t = Z + (\theta - \mu)t, \quad N_t = \frac{N}{H} \times H_t$$

- two components

1. improvements in a productivity index

2. gains from variety

growth with a constant population

- this implies $\zeta = -\mu/(\sigma^2/2)$ and $\zeta_* = 0$
  - the density near 0 is then

$$f(u) = \frac{1 - e^{-\zeta u}}{\Delta}, \quad u \in [0, \Delta]$$

- implied entry and exit rates

$$\frac{1}{2}\sigma^2 \mathrm{D} f(0) = \frac{1}{2}\sigma^2 \times \frac{\zeta}{\Delta} = -\frac{\mu}{\Delta}$$

  - this can be written as

$$\theta - \mu = \theta + \frac{1}{2}\sigma^2 \mathrm{D} f(0)\Delta$$

▶ so growth follows from

1. incumbent firms improving their own productivities at the rate θ
2. replacing firms, selectively, with firms that are better

$$z_t[entry] = z_t[exit] + \Delta$$

▶ the entry and exit rate (the amount of churning) is *endogenous*

- could enrich the model by making θ and Δ endogenous as well

notes on calibration

employment growth accounting

- in a steady state, the employment flows must satisfy

$$\eta \mathrm{E}\left[e^{z}\right] = \underbrace{\left(\mu + \frac{1}{2}\sigma^2\right)\mathrm{E}\left[e^{z}\right]}_{\text{surviving incumbents}} + \underbrace{\left(\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)\right)e^{x}}_{\text{entrants}} - \underbrace{\frac{1}{2}\sigma^2 \mathrm{D}f(0)e^{b}}_{\text{exit}}$$

and thus

$$\eta = \mu + \frac{1}{2}\sigma^2 + \frac{1}{\mathrm{E}\left[e^{z-b}\right]}\left\{\left(\eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0)\right)e^{\Delta} - \frac{1}{2}\sigma^2 \mathrm{D}f(0)\right\}$$

- to verify this, recall that

$$\frac{1}{2}\sigma^2 \mathrm{D}f(0) = \frac{1}{2}\sigma^2 \times \frac{\zeta\zeta_*}{e^{\zeta_*\Delta} - 1}$$

and

$$\mathrm{E}\left[e^{z-b}\right] = \frac{\zeta\zeta_*}{(\zeta-1)(\zeta_*+1)}\frac{e^{(1+\zeta_*)\Delta} - 1}{e^{\zeta_*\Delta} - 1}$$

- therefore

$$\begin{aligned} &\mu + \frac{1}{2}\sigma^2 + \frac{\eta e^{\Delta} + \frac{1}{2}\sigma^2 \mathrm{D}f(0)\left(e^{\Delta} - 1\right)}{\mathrm{E}\left[e^{z-b}\right]} \\ =\ &\mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2(\zeta - 1)(\zeta_* + 1) = \eta \end{aligned}$$

a measurement issue

- employment growth accounting gives

$$\eta \mathrm{E}\left[e^{z}\right] = \underbrace{\left(\mu + \frac{1}{2}\sigma^{2}\right)\mathrm{E}\left[e^{z}\right]}_{\text{surviving incumbents}} + \underbrace{\left(\eta + \frac{1}{2}\sigma^{2}\mathrm{D}f(0)\right)e^{x}}_{\text{entrants}} - \underbrace{\frac{1}{2}\sigma^{2}\mathrm{D}f(0)e^{b}}_{\text{exit}}$$

- the entry term

$$\left(\eta + \frac{1}{2}\sigma^{2}\mathrm{D}f(0)\right) \times e^{x}$$

  reflects employment created by new firms *at the precise time they enter*

  - this is a flow of new employees per unit of time

- in practice, measure employment created from t_0 to $t_1 > t_0$
  - includes incumbent employment growth from t_0 to t_1, for firms that survive that long
  - typically, $t_1 - t_0$ is at least 1 year, sometimes 5 years, or even 10

- this model does not capture learning (Jovanovic) or the idea that entry may be a process that takes some time after the firm first appears on our radar screens

some obvious targets

- the growth rate is

$$\kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1}$$

  - at $\varepsilon = 5$, an annual population growth rate $\eta = 0.01$ adds only 0.25% to the growth rate

- the tail index ζ does not depend on Δ

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- the exit rate very much depends on Δ

$$\frac{1}{2}\sigma^2 \mathrm{D}f(0) = \frac{\eta}{e^{\zeta_* \Delta} - 1}, \quad \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}$$

- average relative to minimum employment is

$$\mathrm{E}\left[e^{z-b}\right] = \frac{\eta}{\eta - \left(\mu + \frac{1}{2}\sigma^2\right)} \frac{e^{(1+\zeta_*)\Delta} - 1}{e^{\zeta_* \Delta} - 1}$$

without population growth

- this gives $\kappa = (\theta - \mu)/(\varepsilon - 1)$ and

$$\zeta = -\frac{\mu}{\sigma^2/2}$$

  - the condition $\zeta > 1$ is the same as $\mu + \frac{1}{2}\sigma^2 < 0$
  - at $\zeta = 1.1$, $\sigma = 0.4$, and $\varepsilon = 5$ this yields

$$\kappa = \frac{\theta - \mu}{\varepsilon - 1} = \frac{\theta}{\varepsilon - 1} + \frac{1}{2}(0.4)^2 \times \frac{1.1}{5 - 1} = \frac{\theta}{\varepsilon - 1} + 0.022$$

- the exit and entry rates are

$$\frac{1}{2}\sigma^2 \mathrm{D}f(0) = -\frac{\mu}{\Delta} = \frac{1}{2}\sigma^2\zeta \times \frac{1}{\Delta}$$

  - an exit rate of 0.1 per annum this then implies

$$\Delta = \frac{0.1}{\frac{1}{2} \times (0.4)^2 \times 1.1} = 1.14$$

  which gives $e^{\Delta} = 3.12$

some round numbers

- suppose

$$\eta = 0.01, \quad \mu = -0.125, \quad \sigma = 0.50$$

so that

$$\eta = 0.01 > 0 = \mu + \frac{1}{2}\sigma^2$$

  – implies that employment at surviving incumbents is a martingale
  – on average, the contribution to employment of incumbents is zero

▶ the tail index is

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} = 1.074456$$

and

$$\zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} = 0.074456$$

- the US exit rate is about 0.10 per annum
  - then
$$0.10 = \frac{1}{2}\sigma^2 \mathrm{D} f(0) = \frac{\eta}{e^{\zeta_* \Delta} - 1}$$
so that $\eta = 0.01$ and $\zeta_* = 0.074456$ gives
$$\Delta = \frac{\ln(1.1)}{0.074456} = 1.2801$$
- so the employment of entrants must be $e^{\Delta} \approx 3.597$ times the employment of firms that exit
- the mean employment size relative to exit size is
$$\mathrm{E}\left[e^{z-b}\right] = \frac{\eta}{\eta - \left(\mu + \frac{1}{2}\sigma^2\right)} \frac{e^{(1+\zeta_*)\Delta} - 1}{e^{\zeta_* \Delta} - 1} \approx 29.567$$
- the employment dynamics accounting is
$$\eta = \mu + \frac{1}{2}\sigma^2 + \frac{1}{\mathrm{E}\left[e^{z-b}\right]} \left\{ \left(\eta + \frac{1}{2}\sigma^2 \mathrm{D} f(0) \right) e^{\Delta} - \frac{1}{2}\sigma^2 \mathrm{D} f(0) \right\}$$
or
$$0.01 = 0 + \frac{0.11 \times 3.597 - 0.10}{29.567}$$

my papers on this topic

References

[1] Selection, Growth, and the Size Distribution of Firms, *Quarterly Journal of Economics*, 2007.

[2] Models of Growth and Firm Heterogeneity, *Annual Review of Economics*, 2010.

[3] Technology Diffusion and Growth, *Journal Economic Theory*, 2012.

[4] Four Models of Knowledge Diffusion and Growth, *MPLS Fed working paper 724*, 2015.

[5] Bounded Learning from Incumbent Firms, *MPLS Fed working paper 771*, 2020.

Model B is in the second half of [1], and C is in the first half of [1]. Model D is in [3]. The multiplicity of stationary distributions in model B was briefly addressed in [1] and in its online appendix. A more presentable version of that appendix appears in Section 5 of [4]. A complete argument for the distribution used here is given in [5], based on the assumption that improvements by imitation are bounded.