Search and Matching Models of Unemployment

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[1]For 8208 only. Please do not circulate.

Abstract

These lecture notes describe a basic version of the Diamond-Mortensen-Pissarides model of unemployment.

Contents

1 Search and Matching

Consider an economy in which consumption is produced by workers who are given a task—a set of instructions for how an individual worker can produce consumption. Every single worker who produces consumption has to have his or her own set of instructions. It takes resources to create these instructions. Unfortunately, the technology is such that nobody can create instructions for themselves. Instead, anyone who uses resources to create a task must find another worker to perform that task.

The first key assumption will be that the process of matching a task with a worker involves delay. The economy has a unit measure of workers. Workers not matched with a task are called unemployed workers. Tasks not matched with workers are called vacancies. Let u_t be the number of unemployed workers and v_t the number of vacancies. It the absence of any delay, this would immediately result in $\min\{u_t, v_t\}$ jobs —matched worker-task pairs— and these jobs would immediately result in the production of consumption. Instead, it will be assumed that there is a "matching function" $M(u_t, v_t)$ that describes the flow of new jobs when there are u_t unemployed workers and v_t vacancies. Jobs are also destroyed, randomly, at the rate δ. When this happens, the worker becomes unemployed, and (the set of instructions for) the task that was performed by the worker disappears. Over a small interval of time Δ, the number of unemployed workers then evolves according to

$$u_{t+\Delta} - u_t \approx -M(u_t, v_t)\Delta + (1 - u_t)\delta\Delta.$$

Assuming that u_t ends up being a differentiable function of time, this yields

$$\mathrm{D}u_t = -M(u_t, v_t) + (1 - u_t)\delta. \tag{1}$$

If we know u_0 and the path for v_t, then this equation allows one to compute the path for u_t.

The second key assumption will be that vacancies can be created instantaneously. There is delay in matching vacancies to unemployed workers, but not in creating vacancies. Specifically, during a small interval of time $[t, t + \Delta)$, anyone can use $a\Delta$ units of consumption to create a vacancy. The vacancy lasts only during that small interval of time, and it will take another $a\Delta$ units of consumption to supply a vacancy during the next small interval of time $[t + \Delta, t + 2\Delta)$, and so on. We will say that the flow cost of "maintaining a vacancy" is a per unit of time.

Clearly, there is a trade-off: maintaining vacancies is costly in terms of current consumption, but it also leads to more employed workers in the future, who can then produce more consumption. Maintaining vacancies is a form of investment, and the stock of matched worker-task pairs is a form of capital. An easy problem to solve is to determine the number of vacancies and the allocation of consumption across consumers that is optimal from the perspective of a planner who cares about the welfare of consumers in the economy. The more tricky question is: what will happen when vacancies are created by individuals (say, entrepreneurs or firms) and newly matched workers and vacancy suppliers have to agree on how much of the output to be produced will go to either party in the match. The most common assumption is that they bargain over this. The resulting equilibrium will typically not correspond to what a planner would do.

2 The Planner's Problem

There is a unit measure of infinitely lived households whose preferences over consumption paths are described by the utility function

$$\mathcal{U}(c) = \int_0^\infty e^{-\rho t} U(c_t) \mathrm{d}t.$$

The discount rate ρ is strictly positive and U is strictly increasing, strictly concave, sufficiently smooth, and its derivative goes to infinity as consumption goes to zero.

There are workers who produce $y > 0$ units of consumption and unemployed who produce $x \in [0, y)$. The fraction of the population who are workers is denoted by $n_t \in [0, 1]$. Aggregate output is thus $n_t y + (1 - n_t)x$. Some of this output can be used to maintain vacancies. It takes $a > 0$ units of output to maintain a vacancy, and the measure of vacancies at time t is denoted by $v_t \in [0, \infty)$. The aggregate resource constraint is thus

$$c_t + v_t a \leq n_t y + (1 - n_t)x, \tag{2}$$

where c_t and v_t are both non-negative. A natural alternative interpretation is that workers can be used not only to produce consumption goods but also to maintain vacancies.

The technology for converting unemployed workers and vacancies into filled jobs is described by a matching function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ that is increasing, concave, and exhibits constant returns to scale. The stock of workers who hold jobs evolves according to

$$\mathrm{D}n_t = -\delta n_t + M(1 - n_t, v_t). \tag{3}$$

Because an employed worker produces more output than an unemployed worker, and since there is no utility cost to producing this extra output, there are incentives to push n_t as close to 1 as possible. But jobs are destroyed at the rate δ, and maintaining the vacancies that can lead to new jobs is a costly investment. What is the optimal allocation of output to consumption and investment in new jobs?

2.1 The Hamiltonian

The constraint (2) is linear in c_t, v_t and n_t, and the concavity of M implies that the set of $\mathrm{D}n_t$, n_t and v_t that satisfy (3) is convex. The Hamiltonian $\mathcal{H}$ for the planner is

$$\mathcal{H}(n,\mu) = \max_{c,v\geq 0} \left\{ U(c) + \mu \left(M(1-n,v) - \delta n \right) : c + va \leq ny + (1-n)x \right\}.$$

This is concave in n and convex in μ. The properties of the utility function ensure that we can focus on interior solutions for consumption. Let λ_t be the Lagrange multiplier for the resource constraint at (n_t, μ_t). The first-order conditions are then

$$\begin{aligned} \mathrm{D}U(c_t) &= \lambda_t, \\ \mu_t \mathrm{D}_2 M(1-n_t, v_t) &\leq a\lambda_t, \quad \text{w.e. if } v_t > 0. \end{aligned}$$

The Hamiltonian dynamics $\mathrm{D}n_t = \mathrm{D}_2\mathcal{H}(n_t,\mu_t)$ and $\mathrm{D}\mu_t = \rho\mu_t - \mathrm{D}_1\mathcal{H}(n_t,\mu_t)$ implies (3) and

$$\mathrm{D}\mu_t = \left[\rho + \delta + \mathrm{D}_1 M(1-n_t, v_t)\right]\mu_t - (y-x)\lambda_t.$$

In addition, there will be a transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\mu_t n_t = 0. \tag{4}$$

Note that $\mathrm{D}\mu_t = (\rho+\delta)\mu_t - (y\lambda_t - [x\lambda_t + \mathrm{D}_1 M(1-n_t,v_t)\mu_t])$. So the equation for μ_t is like the asset pricing equation for an asset that depreciates at the rate δ and generates revenues $y\lambda_t$ that are produced at the cost $x\lambda_t + \mathrm{D}_1 M(1-n_t,v_t)\mu_t$. The cost $x\lambda_t$ arises because unemployed individual do produce $x < y$. The cost $\mathrm{D}_1 M(1-n_t,v_t)\mu_t$ reflects the fact that new jobs "empty the tank" of unemployed individuals who are an input in creating even more new jobs.

Eliminating the Lagrange multiplier λ_t yields

$$\begin{aligned} a\mathrm{D}U(c_t) &\geq \mu_t \mathrm{D}_2 M(1-n_t,v_t), \quad \text{w.e. if } v_t > 0 \qquad (5) \\ \mathrm{D}\mu_t &= \left[\rho+\delta+\mathrm{D}_1 M(1-n_t,v_t)\right]\mu_t - \mathrm{D}U(c_t)(y-x). \qquad (6) \end{aligned}$$

The optimal allocation is determined by the resource constraints (2)-(3) and the optimality conditions (4)-(6), with n_0 given.

The resource constraint (2) can be used to eliminate c_t from (5)-(6). The first-order condition (5) then determines v_t as a function of (n_t, μ_t). And then (3) and (6) pin down $(\mathrm{D}n_t, \mathrm{D}\mu_t)$ as a function of (n_t, μ_t). Starting from a given n_0, one can guess μ_0 and then solve the differential equation forward in time. If the resulting path is such that the transversality condition (4) is satisfied, then the initial guess for μ_0 and the associated path $\{(n_t, \mu_t)\}_{t\geq 0}$ is optimal.

2.2 The Steady State

Conjecture that the optimal combination of unemployment and vacancies converges to a steady state $(1 - n_t, v_t) = (u, v)$. In any steady state, the fact that jobs are destroyed at a positive rate means that the number of vacancies will have to be positive. So (5) will hold with equality. A constant number of unemployed workers and vacancies also implies that consumption is constant, and so the left-hand side of (5) will be constant. It follows that μ_t is constant in a steady state. Imposing $\mathrm{D}\mu_t = 0$ in (6) and using (5) to eliminate μ_t yields

$$\frac{y - x}{a} = \frac{\rho + \delta + \mathrm{D}_1 M(1, v/u)}{\mathrm{D}_2 M(1, v/u)}. \tag{7}$$

Since the matching function is just like a production function, the right-hand side is strictly increasing in v/u. Any solution to this condition will have to be unique. If we assume that the marginal products of M range throughout $(0, \infty)$, then a solution for v/u is guaranteed. Imposing $\mathrm{D}n_t = 0$ in (3) gives

$$\delta = \delta u + M(u, v) \tag{8}$$

The right-hand side can be interpreted as a production function in (u, v), and so this steady-state condition is an isoquant in (u, v) space. We already have v/u, and the steady state level of unemployment is then simply

$$u = \frac{1}{1 + M(1, v/u)/\delta}$$

So there will be a unique steady state.

2.2.1 The Golden Rule

Filled jobs are the capital stock of this economy. As in the one- and two-sector models of physical capital we have studied before, we can construct a golden rule. It is determined by maximizing consumption subject to the steady-state condition (8),

$$\max_{u,v} \{y - [(y-x)u + av] : \delta \leq \delta u + M(u,v)\} \tag{9}$$

The inequality is a relaxed version of (3) in any steady state. One can interpret $(y-x)u + av$ as the cost of unemployed workers and maintaining vacancies, and so (9) requires minimization of this cost subject to the isoquant (8). The first-order conditions for the golden rule are simply

$$\frac{y-x}{a} = \frac{\delta + \mathrm{D}_1 M(1, v/u)}{\mathrm{D}_2 M(1, v/u)}, \tag{10}$$

which says that the line $(y-x)u + av = \texttt{constant}$ and the isoquant (8) are tangent. As in the steady-state efficiency condition (7), the right-hand side of (10) only depends on v/u, and it is strictly increasing. The same assumption about the marginal products of M will guarantee the existence of a unique golden rule.

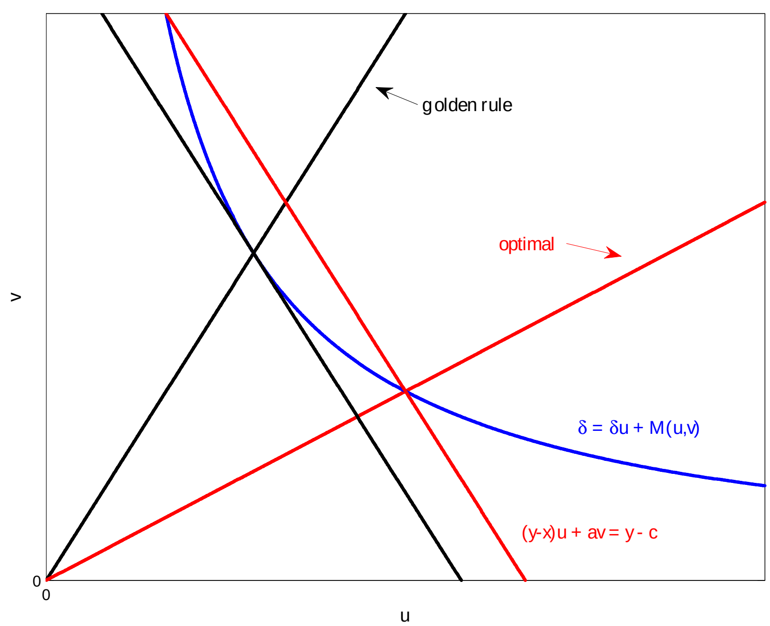

FIGURE 1 Golden Rule and Optimal Steady State

A comparison of (7) and (10) shows that the v/u will be higher in under the golden rule than in the optimal steady state. The isoquant (8) then implies that the number of employed workers will be higher (unemployment will be lower) under the golden rule than in the efficient steady state. This is familiar from models of physical capital accumulation. An illustration is given in Figure 1. A corollary is that increasing steady state unemployment from the optimal steady state (and thus lowering v/u along the isoquant (8)) will move the economy further away from the golden rule and lower consumption.

2.3 The Shadow Present Value Budget Constraint

The number of matched workers n_t is the capital stock in this economy, and μ_t is the shadow price for the constraint that limits how fast this stock can be augmented. One can interpret $\mu_t n_t$ as the shadow value of the capital stock at time t. In anticipation of decentralizations to come, the following constructs a shadow present value budget constraint, with $\mu_0 n_0$ as one of the components of wealth.

Observe that (3) and (6) with $\lambda_t = \mathrm{D}U(c_t)$ imply

$$\begin{aligned}
\mathrm{D}\left[\mu_t n_t\right] &= n_t \mathrm{D}\mu_t + \mu_t \mathrm{D} n_t \\
&= \left[\rho + \delta + \mathrm{D}_1 M(1-n_t, v_t)\right] \mu_t n_t - (y-x)\lambda_t n_t \\
&\quad + \mu_t \left[-\delta n_t + M(1-n_t, v_t)\right] \\
&= \left[\rho + \mathrm{D}_1 M(1-n_t, v_t)\right] \mu_t n_t - (y-x)\lambda_t n_t + \mu_t M(1-n_t, v_t).
\end{aligned}$$

Since the matching function M exhibits constant returns to scale,

$$\begin{aligned}
\mathrm{D}_1 M(1-n_t, v_t)\mu_t n_t + \mu_t M(1-n_t, v_t) &= \mu_t \mathrm{D}_1 M(1-n_t, v_t) + \mu_t \mathrm{D}_2 M(1-n_t, v_t) v_t \\
&= \mu_t \mathrm{D}_1 M(1-n_t, v_t) + a\lambda_t v_t,
\end{aligned}$$

where the second equality follows from the first-order condition (5). Combining these two observations gives

$$\mathrm{D}\left[\mu_t n_t\right] = \rho \mu_t n_t - \left[\lambda_t \left[(y-x)n_t - a v_t\right] - \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right]$$

and thence

$$\mathrm{D}\left[e^{-\rho t}\mu_t n_t\right] = -e^{-\rho t}\left[\lambda_t \left[(y-x)n_t - a v_t\right] - \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right].$$

Integrating this differential equation from $t = 0$ to $t = T$ yields

$$\mu_0 n_0 = \int_0^T e^{-\rho t}\left[\lambda_t \left[(y-x)n_t - a v_t\right] - \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right] \mathrm{d}t + e^{-\rho T}\mu_T n_T.$$

Using the transversality condition (4) and the resource constraint (2) gives

$$\begin{aligned}\mu_0 n_0 &= \int_0^\infty e^{-\rho t}\left[\lambda_t\left[(y-x)n_t - av_t\right] - \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right] \mathrm{d}t \\ &= \int_0^\infty e^{-\rho t}\left[\lambda_t(c_t - x) - \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right] \mathrm{d}t.\end{aligned}$$

Another way to write this is

$$\int_0^\infty e^{-\rho t}\lambda_t c_t \mathrm{d}t = \mu_0 n_0 + \int_0^\infty e^{-\rho t}\left[\lambda_t x + \mu_t \mathrm{D}_1 M(1-n_t, v_t)\right] \mathrm{d}t.$$

The left-hand side is the shadow present value of consumption. The second term on the right-hand side is the shadow present value of what households can do on their own: simultaneously produce x units of consumption and contribute to the creation of new matches. The shadow value $\mu_0 n_0$ of the initial stock of matches reflects the contribution of already having n_0 matches at the initial date.

3 Wage Bargaining

Continue with the same technology for producing consumption, maintaining vacancies, and creating matches, as described so far, and for which we have characterized the allocation a planner would choose. This section describes a decentralized model of what can happen in this environment.

3.1 Complete Markets for Consumption

Assume there is a complete set of markets that allows households to trade consumption subject to a present-value budget constraint. There is no aggregate risk and idiosyncratic risk (such as the loss of employment, or success in finding a job) is shared across households, along the lines described in an earlier lecture note. There are no risk premia and consumption flows are valued using expected present discounted values. Let r_t be the risk-free interest rate.

The typical household maximizes utility subject to

$$\int_0^\infty \exp\left(-\int_0^t r_s \mathrm{d}s\right) c_t \mathrm{d}t \leq \texttt{wealth},$$

where wealth remains to be specified. The first-order condition for consumption is

$$e^{-\rho t}\mathrm{D}U(c_t) = \lambda_0 \exp\left(-\int_0^t r_s \mathrm{d}s\right), \tag{11}$$

where λ_0 is the Lagrange multiplier on the present-value budget constraint of the household. Differentiating this first-order condition with respect to t gives

$$r_t = \rho - \frac{1}{\mathrm{D}U(c_t)} \frac{\mathrm{d}\left[\mathrm{D}U(c_t)\right]}{\mathrm{d}t}.$$

This leads to the familiar Euler equations

$$r_t = \rho + \gamma(c_t) \times \frac{\mathrm{D}c_t}{c_t} \tag{12}$$

where $\gamma(c) = -c\mathrm{D}^2U(c)/\mathrm{D}U(c)$ is the coefficient of relative risk aversion.

3.2 The Labor "Market"

Write $u_t = 1 - n_t$ for the measure of unemployed workers. As before, the number of vacancies at time t is v_t. Existing matches are destroyed exogenously at an average rate $\delta > 0$. Either party in a match can also choose to break it up at any time. Destroyed matches are gone forever. The flow of new matches will be $M(u_t, v_t)$.

3.2.1 Match Creation

Define

$$\phi_t = M\left(1, \frac{v_t}{u_t}\right), \quad \psi_t = M\left(\frac{u_t}{v_t}, 1\right). \tag{13}$$

These are, respectively, the job finding and the vacancy filling rates. Assuming that all unemployed workers have an equal shot at finding a match, an unemployed worker searching for a match will be successful at the rate $M(u_t, v_t)/u_t = M(1, v_t/u_t) = \phi_t$. Similarly, vacancies are filled at the rate $M(u_t, v_t)/v_t = M(u_t/v_t, 1) = \psi_t$. The variable v_t/u_t is called labor market tightness, traditionally written as $\theta_t = v_t/u_t$. Note that ϕ_t is increasing in θ_t while ψ_t is decreasing in θ_t. A tight labor market is a labor market in which it is easy to find a job and hard to fill vacancies.

3.2.2 Inside a Match

The matched pair of a firm and a worker are in joint possession of an asset (the match) that can continue and bear fruit only if both parties cooperate and agree on the terms of employment. In a frictionless market, immediate competition determines the terms of trade. Unhappy workers can switch employers without delay, and unhappy employers can hire other workers without delay. In such a frictionless environment, matches are free: they can be reproduced at no cost. Here instead, creating a new match is costly: there is delay, during which potential employers have to pay to maintain vacancies, and unemployed workers only produce low output.

An important assumption will be that employers and workers cannot sign binding agreements. At any point in time while a match is still viable, both parties simply have to agree to continue the match. Consider a particular match and suppose the firm and its worker agree to continue the match as long as the job is not destroyed exogenously. Abstract from the possibility of signing bonuses, severance payments, or any other lumpy transfers between the worker and the firm. We will not need them here. Let $\{w_s\}_{s\geq t}$ be the path of wages paid to the worker from date t on, conditional on the continued viability of the match.

Continuation Values Given this path of wage payments, define F_t to be the value of the match to the firm: the expected present value of $\{y - w_s\}_{s\geq t}$. This present value satisfies the asset pricing or Bellman equation

$$r_t F_t = y - w_t + \mathrm{D}F_t - \delta F_t \tag{14}$$

and a transversality condition. Define V_t to be the expected present value of current and future earnings of the worker in this match. While the match lasts, these earnings are simply the wages $\{w_s\}_{s\geq t}$. When the match ends, the worker will be unemployed, earn x, and search for a new match. Let U_t be the expected present value of all current and future earnings of someone who is unemployed at time t. Then V_t must satisfy

$$r_t V_t = w_t + \mathrm{D}V_t - \delta(V_t - U_t) \tag{15}$$

and a transversality condition. The firm and the worker inside this match will never meet again after the match is broken up. And they know what will happen to a worker once their match has broken down. They can both take U_t as given. Destruction of the match leaves the firm with nothing.

Adding up the two Bellman equations (14) and (15) equations gives

$$r_t(F_t + V_t) = y + \delta U_t + \mathrm{D}(F_t + V_t) - \delta\left(F_t + V_t\right).$$

This differential equation is solved by

$$\begin{aligned} F_t + V_t &= \int_t^\infty \delta e^{-\delta(\tau - t)} \left(\int_t^\tau \exp\left(-\int_t^s r_a \mathrm{d}a \right) y \mathrm{d}s + \exp\left(-\int_t^\tau r_a \mathrm{d}a \right) U_\tau \right) \mathrm{d}\tau \\ &= \mathrm{E}_t \left[\int_t^\tau \exp\left(-\int_t^s r_a \mathrm{d}a \right) y \mathrm{d}s + \exp\left(-\int_t^\tau r_a \mathrm{d}a \right) U_\tau \right] \end{aligned} \tag{16}$$

where $\mathrm{E}_t[\cdot]$ is the expectation over the random time τ at which the match is destroyed exogenously. To confirm that this is the solution, note that integration by parts implies

$$\int_t^\infty \delta e^{-\delta(\tau - t)} \left(\int_t^\tau \exp\left(-\int_t^s r_a \mathrm{d}a \right) y \mathrm{d}s \right) \mathrm{d}\tau = \int_t^\infty \exp\left(-\int_t^s (r_a + \delta) \mathrm{d}a \right) y \mathrm{d}s,$$

as long as we know that the present value of y forever is finite. So the proposed solution (16) can also be written as

$$\exp\left(-\int_0^t (r_a + \delta) \mathrm{d}a \right) (F_t + V_t) = \int_t^\infty \exp\left(-\int_0^s (r_a + \delta) \mathrm{d}a \right) (y + \delta U_s) \, \mathrm{d}s.$$

Note that the right-hand side depends on t only through the lower limit of integration. Differentiating with respect to t proves the result. Equation (16) says that $F_t + V_t$ is the expected present value of the output y earned while the match lasts, plus the present value of all earnings of the worker following the destruction of the match. Clearly, (16) does not depend on the wages earned by the worker while in the match.

Wage Determination If the match were broken up at date t, the continuation value of the worker would be U_t, and that of the firm would be zero. So the surplus generated by the match, in present value, is equal to $F_t + V_t - U_t$. This surplus does not depend on the wages paid by the firm to the worker while the match lasts. The firm gains from continuing the match as long as $F_t \geq 0$ and the worker gains as long as $V_t \geq U_t$. If the surplus $F_t + V_t - U_t$ is strictly positive, there will be many paths of wages $\{w_s\}_{s \geq t}$ that are consistent with both the firm and the worker consenting to a continuation of the match at date t.

It will be assumed here that the agreed upon wages are such that the worker receives a constant share of the surplus,

$$V_t - U_t = \beta\left(F_t + V_t - U_t\right) \tag{17}$$

for some bargaining parameter $\beta \in [0, 1)$. This division of the surplus is imposed for all t while the match lasts.[1] To see that this does indeed pin down wages, note that $F_t + V_t$ is determined by (16), with U_τ taken as given by the firm and the worker in this match. The division of surplus (17) then determines $V_t - U_t$, and hence also F_t. The fact that (17) is imposed continuously while the match continues allows one to infer $\mathrm{D}F_t$, by taking a derivative of the path of F_t. And then the Bellman equation (14) pins down w_t via $w_t = y - (r_t + \delta)F_t + \mathrm{D}F_t$.

The division of surplus (17) can be viewed as the outcome of a dynamic game in which both the firm and the worker can break up that match at any time, and in which the worker is supposed to produce y and the firm is supposed to pay wages. Such a dynamic game will have many subgame perfect equilibria, and (17) describes one of them as long as the overall surplus $F_t + V_t - U_t$ is positive.

3.2.3 The Value of Search and Incentives to Create Vacancies

Because of (16), the value of $F_t + V_t$ is the same in any continuing match. Consider equilibria in which the same bargaining rule (17) is used in all continuing matches. Then wages are the same for all employed workers, and so is the value V_t of being an employed worker. Moreover, searching unemployed workers can all anticipate to receive those wages in future matches. Since unemployed workers find matches at the rate ϕ_t, the present value of all current and future earnings of an unemployed worker must satisfy

$$r_t U_t = x + \phi_t (V_t - U_t) + \mathrm{D}U_t \tag{18}$$

and a transversality condition. Subtracting (18) from (15) gives

$$r_t\left(V_t - U_t\right) = w_t - \left[x + \phi_t (V_t - U_t)\right] + \mathrm{D}\left(V_t - U_t\right) - \delta (V_t - U_t) \tag{19}$$

So the value of being an employed worker rather than an unemployed worker is going to be the present value of $w_t - [x + \phi_t (V_t - U_t)]$ for the duration of the match. The earnings of an unemployed worker are the output flow x and the expected capital gains from finding a job. The gain from being employed is the amount by which the wage exceeds these earnings.

[1] See Hall and Milgrom [2008] for an alternative division of the match surplus.

There is free entry into vacancy creation, and the rate at which any vacancy is filled is equal to ψ_t. In any equilibrium, it will have to be the case that

$$a \geq \psi_t F_t, \quad \text{w.e. if } v_t > 0. \tag{20}$$

Consumption would be zero if $\psi_t F_t > a$, and preferences are such that this will not happen in equilibrium. Vacancies will not be supplied if the cost of doing so exceeds the expected gain.

Note that (17) implies $F_t + V_t - U_t = (V_t - U_t)/\beta$ and thus $F_t = (1-\beta)(V_t - U_t)/\beta$. So (20) can be stated more explicitly as

$$\frac{\beta a}{1-\beta} \geq M\left(\frac{u_t}{v_t}, 1\right)(V_t - U_t), \quad \text{w.e. if } v_t > 0.$$

If $M(\infty, 1) = \infty$ then this forces $v_t > 0$ and we obtain an increasing relationship between $V_t - U_t$ and v_t/u_t. The labor market will be tight if and only if the surplus of being an employed worker over being an unemployed worker is high.

3.2.4 The Implied Wages

Adding (14) to (19) gives

$$r_t (F_t + V_t - U_t) = y - [x + \phi_t(V_t - U_t)] + \mathrm{D}(F_t + V_t - U_t) - \delta(F_t + V_t - U_t). \tag{21}$$

This is the asset pricing equation for the joint surplus of a match. The match produces y, and the worker could "produce" $x + \phi_t(V_t - U_t)$ while unemployed. The surplus sharing rule (17) says that we can transform this asset pricing equation into an asset pricing equation for the worker surplus $V_t - U_t$ by multiplying (21) by β. This yields

$$r_t (V_t - U_t) = \beta (y - [x + \phi_t(V_t - U_t)]) + \mathrm{D}(V_t - U_t) - \delta(V_t - U_t).$$

But we already have the asset pricing equation (19) for the surplus of a worker. The two equations should be the same. This will be true if and only if

$$w_t - [x + \phi_t(V_t - U_t)] = \beta (y - [x + \phi_t(V_t - U_t)]). \tag{22}$$

In other words, the sharing rule for the flow surplus $y - [x + \phi_t(V_t - U_t)]$ is the same as the sharing rule (17) for the present values.

Suppose now that $v_t > 0$, so that (20) implies $a = \psi_t F_t$. The sharing rule (17) says that this is the same as $a/(1-\beta) = \psi_t(V_t - U_t)/\beta$. Using this to eliminate $V_t - U_t$ from (22) gives the flow surplus for workers

$$w_t - \left[x + \frac{\beta a}{1-\beta}\frac{\phi_t}{\psi_t}\right] = \beta\left(y - \left[x + \frac{\beta a}{1-\beta}\frac{\phi_t}{\psi_t}\right]\right).$$

The flow surplus $y - w_t$ for firms is then simply $1-\beta$ times the joint flow surplus. Using $\phi_t = M(u_t, v_t)/u_t$ and $\psi_t = M(u_t, v_t)/v_t$, this can be written as

$$\begin{pmatrix} w_t - \left[x + \frac{\beta a}{1-\beta}\frac{v_t}{u_t}\right] \\ y - w_t \end{pmatrix} = \begin{pmatrix} \beta \\ 1-\beta \end{pmatrix}\left(y - \left[x + \frac{\beta a}{1-\beta}\frac{v_t}{u_t}\right]\right).$$

The outside option for an employed worker, $x + (\beta a/(1-\beta))v_t/u_t$ is high when v_t/u_t is high. That is, when the labor market is tight. This means that the joint surplus of any particular job is low: the match produces y, as always, but it easy for workers to find alternative employment. The sharing rule (17) means that the flow surplus from a match is low for both firms and workers when the labor market is tight. For firms, this simply means low flow profits $y - w_t$. But workers also benefit from their improved outside option,

$$w_t = (1-\beta)x + \beta\left(y + a \times \frac{v_t}{u_t}\right). \tag{23}$$

Wages are increasing in labor market tightness.

3.3 The Marginal Utility Weighted Surplus

As usual, it is convenient to eliminate the risk-free interest rate by considering marginal utility weighted values instead of values measured in units of consumption. To this end, define

$$s_t = \mathrm{D}U(c_t)\left(F_t + V_t - U_t\right).$$

Differentiating this with respect to t and using the Euler equation together with the asset pricing equation (21) for the joint surplus of a match gives

$$\rho s_t = \mathrm{D}U(c_t)\left(y - [x + \phi_t(V_t - U_t)]\right) + \mathrm{D}s_t - \delta s_t.$$

The sharing rule (17) says that $\mathrm{D}U(c_t)(V_t - U_t) = \beta s_t$, and so this becomes $\rho s_t = \mathrm{D}U(c_t)(y - x) - \beta\phi_t s_t + \mathrm{D}s_t - \delta s_t$, or

$$\mathrm{D}s_t = (\rho + \delta + \beta\phi_t)\, s_t - \mathrm{D}U(c_t)(y - x).$$

The free-entry condition (20) for vacancy creation can also be restated in terms of s_t, as

$$\mathrm{D}U(c_t)a \geq (1-\beta)\psi_t s_t \quad \text{w.e. if } v_t > 0.$$

Taking into account the resource constraint (2) and the definition (13) of ψ_t, this condition determines v_t as a function of the state (u_t, s_t).

3.4 Summary of the Equilibrium Conditions

Given the state (u_t, s_t), consumption and the supply of vacancies are determined by

$$c_t + v_t a = (1-u_t)y + u_t x, \tag{24}$$

$$\mathrm{D}U(c_t)a \geq (1-\beta)M\left(\frac{u_t}{v_t}, 1\right)s_t, \quad \text{w.e. if } v_t > 0, \tag{25}$$

and then (u_t, s_t) evolves according to

$$\mathrm{D}u_t = \delta(1-u_t) - M(u_t, v_t), \tag{26}$$

$$\mathrm{D}s_t = \left[\rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right)\right]s_t - \mathrm{D}U(c_t)(y-x). \tag{27}$$

Solving (24)-(25) for c_t and v_t as a function of the state (u_t, s_t) and then using this solution to eliminate c_t and v_t from (26)-(27) produces a differential equation for (u_t, s_t). The initial value u_0 is given and there is a transversality condition that requires $e^{-\rho t}\mathrm{D}U(c_t)$ times wealth to converge to zero.

3.4.1 The Hosios Condition for Efficiency

Recall from (5)-(6) that efficiency requires that

$$\begin{aligned}\mathrm{D}U(c_t)a &\geq \mu_t \mathrm{D}_2 M(u_t, v_t), \quad \text{w.e. if } v_t > 0 \\ \mathrm{D}\mu_t &= [\rho + \delta + \mathrm{D}_1 M(u_t, v_t)]\,\mu_t - \mathrm{D}U(c_t)(y-x),\end{aligned}$$

where μ_t is the Lagrange multiplier for the upper bound on $\mathrm{D}[1-u_t]$. These two conditions resemble (25) and (27), except that marginal products of $M(u_t, v_t)$ matter for efficiency while the equilibrium conditions depend on the average products $M(u_t, v_t)/v_t$ and $M(u_t, v_t)/u_t$. In the special case of Cobb-Douglas matching functions that are proportional to $u_t^\beta v_t^{1-\beta}$, these marginal and average products are related via

$$\left[\frac{\mathrm{D}_1 M(u_t, v_t)u_t}{M(u_t, v_t)}, \frac{\mathrm{D}_2 M(u_t, v_t)v_t}{M(u_t, v_t)}\right] = [\beta, 1-\beta]\,.$$

If we now take $s_t = \mu_t$, then the efficiency conditions (5)-(6) and the equilibrium conditions (25) and (27) coincide. The resource constraints are the same, and so the efficient and equilibrium allocations coincide.

3.4.2 Vacancies and Consumption as a Function of (u_t, s_t)

Write $c_t = c(u_t, s_t)$ and $v_t = v(u_t, s_t)$ for the c_t and v_t that solve (24)-(25). To focus on the most interesting case, assume that $M(\infty, 1) = \infty$ so that we can rule out $v_t = 0$. Combining (24)-(25) then gives

$$\mathrm{D}U((1-u_t)y + u_t x - av_t)a = (1-\beta)M\left(\frac{u_t}{v_t}, 1\right)s_t. \tag{28}$$

The left-hand side is strictly increasing in v_t and the right-hand side is strictly decreasing as long as u_t and s_t are both strictly positive. Suppose that utility satisfies the Inada condition $\mathrm{D}U(0) = \infty$. Then the left-hand side will go off to ∞ as av_t approaches total output $(1-u_t)y + u_t x$ from below. On the other hand, the right-hand side diverges as v_t approaches 0 from above. So this equilibrium condition will have a unique solution for v_t whenever u_t and s_t are strictly positive.

It is easy to see from (28) that $v(u_t, s_t)$ is strictly increasing in s_t: the higher the surplus $s_t = \mathrm{D}U(c_t)(F_t + V_t - U_t)$, the more vacancies will be maintained. Since maintaining vacancies is costly, this immediately implies that $c(u_t, s_t)$ is decreasing in s_t.

It is also immediate from (28) that $v(u, \cdot)$ is invertible. The fact that given any $u_t \in (0,1)$, (28) implies one-to-one relationship between the surplus value s_t and the supply of vacancies v_t means that one can restate the differential equation in terms of (u_t, v_t) only, using $\mathrm{D}v_t = \mathrm{D}_1 v(u_t, s_t)\mathrm{D}u_t + \mathrm{D}_2 v(u_t, s_t)\mathrm{D}s_t$.

To examine the dependence of the supply of vacancies $v(u_t, s_t)$ on u_t, write (28) as

$$s_t = \frac{\mathrm{D}U(y - (y - x + av_t/u_t)u_t)a}{(1-\beta)M(u_t/v_t, 1)}. \tag{29}$$

The right-hand side of this equation is increasing in labor market tightness v_t/u_t, and increasing in u_t when holding v_t/u_t fixed. For given s_t, an increase in u_t therefore requires a reduction in v_t/u_t. This says that $v(u_t, s_t)/u_t$ is decreasing in u_t. Holding fixed s_t, labor market tightness will be low when unemployment is high. But, as we will see, the equilibrium value of the surplus s_t will be high when unemployment is high, and this is an important force in the other direction, towards raising labor market tightness.

3.4.3 The Time Derivatives $(\mathrm{D}u_t, \mathrm{D}s_t)$ as a Function of s_t

It follows from (26) that a high s_t implies a low $\mathrm{D}u_t$. To see how $\mathrm{D}s_t$ depends on s_t, combine (25) and (27) and use the fact that $v_t > 0$ to conclude that

$$\frac{\mathrm{D}s_t}{s_t} = \rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right) - \left(\frac{y-x}{a}\right)(1-\beta)M\left(\frac{u_t}{v_t}, 1\right).$$

The right-hand side is increasing in v_t, and so a high s_t implies a high $\mathrm{D}s_t/s_t$.

It follows that $\mathrm{D}u_t < 0$ above the $\mathrm{D}u_t = 0$ curve and $\mathrm{D}s_t > 0$ above the $\mathrm{D}s_t = 0$ curve, in a (u_t, s_t) diagram with u_t on the horizontal axis and s_t on the vertical axis. Of course, the inequalities flip when s_t is below these curves.

3.5 The Steady State

The steady state is defined by $\mathrm{D}[u_t, s_t] = 0$. This yields two curves in $[u, s]$ space,

$$\mathrm{D}u_t = 0 \Rightarrow \delta = \delta u + M(u, v), \tag{30}$$

$$\mathrm{D}s_t = 0 \Rightarrow \rho + \delta + \beta M\left(1, \frac{v}{u}\right) = (1-\beta)M\left(\frac{u}{v}, 1\right)\left(\frac{y-x}{a}\right), \tag{31}$$

where $v = v(u, s)$ is defined by (28). More conveniently, as in (29),

$$s = \frac{\mathrm{D}U(x + (1-u)(y-x) - av)a}{(1-\beta)M(u/v, 1)}. \tag{32}$$

To construct the $\mathrm{D}u_t = 0$ and $\mathrm{D}s_t = 0$ curves in $[u, s]$ space, one can vary u and v along the two curves (30)-(31) and then use (32) to trace out the implicit s. Alternatively, one can construct the steady-state directly in $[u, v]$ space using (30)-(31) only, subject to the constraint that the implied consumption is strictly positive. And then (32) will pin down the implied surplus of an employed worker.

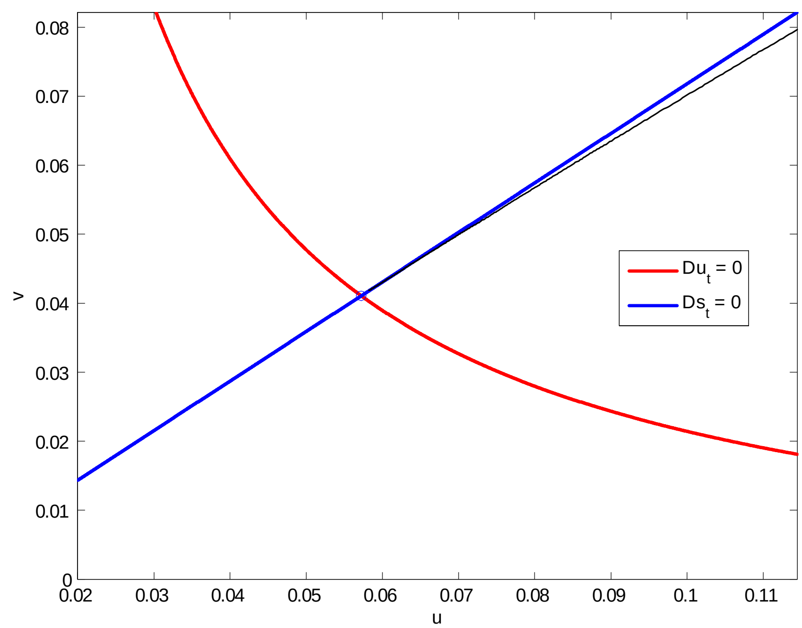

FIGURE 2 Steady State and Transition Dynamics From High u_0

First, observe that the $\mathrm{D}u_t = 0$ curve (30) is an isoquant for the increasing and concave production function $\delta u + M(u, v)$. So this defines v as a decreasing and convex function of u. Second, the $\mathrm{D}s_t = 0$ curve (31) only depends on the ratio v/u. The left-hand side is increasing in v/u and the right-hand side is decreasing in v/u. Any solution will be unique. A solution is guaranteed by assuming $M(\infty, 1) = M(1, \infty) = \infty$. So the steady state condition $\mathrm{D}s_t = 0$ pins down labor market tightness, and hence an upward sloping line in $[u, v]$ space. The unemployment rate can then be inferred from (30), or

$$u = \frac{1}{1 + M\left(1, v/u\right)/\delta}. \tag{33}$$

This is well defined and in $(0, 1)$, and so the two curves (30) and (31) intersect in $[u, v]$ space. They do so only once. The two curves are shown in Figure 2, together with the equilibrium trajectories for initial conditions above and below the steady state level of unemployment.

3.5.1 Labor Market Tightness

To further interpret, write the $\mathrm{D}s_t = 0$ condition (31) as

$$a = M\left(\frac{u}{v}, 1\right) \times \frac{(1-\beta)(y-x)}{\rho + \delta + \beta M(1, v/u)} \tag{34}$$

The first factor on the right-hand side is the rate at which vacancies are filled, and the second factor is the share $1-\beta$ of the surplus $F+V-U$ that goes to the vacancy suppliers. In this economy, the cost of maintaining vacancies is always a units of consumption. Both factors on the right-hand side of (34) are decreasing in the labor market tightness variable v/u. In a tight labor market, vacancies are filled slowly, and they are less valuable because the unemployed quickly find jobs and so the effective discount rate at which the flow surplus $(1-\beta)(y-x)$ is discounted will be high.

To better understand the role of the job-finding rate $M(1, v/u)$ in (34), recall from (18) that the value of an unemployed worker has to satisfy

$$\rho U = x + M\left(1, \frac{v}{u}\right)(V - U)$$

in a steady state. Solving for U gives

$$U = \frac{x + M(1, v/u)V}{\rho + M(1, v/u)}.$$

This is an increasing function of v/u, and taking $M(1, v/u) \to \infty$ gives $U \to V$, holding fixed V. That is, a very high job finding rate means that the value of employed workers cannot be much above the value of unemployed workers. Unemployed workers will soon be employed workers if the job-finding rate $M(1, v/u)$ is very high. The difference between the earnings of unemployed and employed workers can then not be large in present value. The bargaining solution says that $F = (1-\beta)(V-U)/\beta$, and so the value of a filled vacancy can then also not be very large. This is exactly what the effective discount rate $\rho + \delta + \beta M(1, v/u)$ entails.

Comparative Statics The steady-state condition (34) for v/u only depends on y, x and a via the "surplus ratio" $(y-x)/a$. The labor market tightness variable v/u is increasing in this surplus ratio. If the output gain from employment is high relative to the cost of maintaining vacancies, then there will be a lot of vacancies per unemployed worker in the steady state.

The right-hand side of (34) is decreasing in β, and so the steady state u/v ratio will rise with an increase in β. Improving the bargaining power of employed workers will unambiguously raise unemployment and lower vacancies because the $\mathrm{D}u_t = 0$ curve does not depend on β. The effect on welfare must necessarily be ambiguous: we know that efficiency requires that β satisfies the Hosios condition. So there is an optimal amount of bargaining power.

3.5.2 Wages

We already have wages from (23). To confirm that these wages will be in (x, y) in the steady state, note that setting $\mathrm{D}s_t$ in (27) together with the definition $s_t = \mathrm{D}U(c_t)(F_t + V_t - U_t)$ gives

$$F + V - U = \frac{y - x}{\rho + \delta + \beta M(1, v/u)}.$$

Setting $\mathrm{D}F_t = 0$ in the asset pricing equation for F_t yields $F = (y - w)/(\rho + \delta)$, and we have the sharing rule $\beta F = (1 - \beta)(V - U)$. Eliminating F and $V - U$ from these three conditions gives

$$w = x + \frac{\rho + \delta + M(1, v/u)}{\rho + \delta + \beta M(1, v/u)} \times \beta(y - x).$$

This is a convex combination of x and y, as expected. The weight on $y - x$ is larger than β, as we can already infer from (23).

3.5.3 The Fast-Matching Limit

Suppose the matching function can be parameterized as $M(u, v) = A\widehat{M}(u, v)$ for some baseline matching function $\widehat{M}(u, v)$ and a matching productivity parameter A. Use this to write the $\mathrm{D}s_t = 0$ condition (34) as

$$a = \widehat{M}\left(\frac{u}{v}, 1\right) \times \frac{(1 - \beta)(y - x)}{\frac{\rho+\delta}{A} + \beta\widehat{M}(1, v/u)}.$$

The right-hand side is decreasing in v/u and so an increase in A will raise v/u. Taking the $A \to \infty$ eliminates $(\rho+\delta)/A$ from the equilibrium condition, and then $M(1, v/u)/M(u/v, 1) = v/u$ implies

$$\frac{v}{u} = \frac{1 - \beta}{\beta}\frac{y - x}{a}.$$

That is, increasing the productivity of the matching function without bound raises labor market tightness to an upper bound that is proportional to $(y - x)/a$ and increasing in the share of the surplus in a match that goes to the vacancy suppliers. It follows from (23) that wages converge to y.

Of course, (33) implies that there is no unemployment in this limit. Since v/u converges to a finite and positive limit, this means that v converges to zero as well. The resources used to maintain full employment are negligible in the fast-matching limit, because it takes essentially no time to maintain the vacancies required to make up for job destruction. In this economy, it is not really costly to design the jobs that allow workers to produce y instead of x. The only thing that is costly is searching for the workers to do those jobs, and those costs disappear in the fast-matching limit.

3.6 The Phase Diagram

Figure 3 shows the equilibrium in $[u_t, s_t]$ space. The flow utility function is logarithmic and the matching function is Cobb-Douglas.

3.6.1 The Curve $\mathrm{D}s_t = 0$

Recall that $\mathrm{D}s_t = 0$ pins down v_t/u_t at its steady state level. From (29) we know that, holding fixed v_t/u_t, the zero-profit condition for maintaining vacancies implies that s_t is increasing in u_t. So $\mathrm{D}s_t = 0$ implies an upward-sloping curve in $[u_t, s_t]$ space.

3.6.2 The Curve $\mathrm{D}u_t = 0$

The $\mathrm{D}u_t = 0$ curve is an isoquant in $[u, v]$ space, determined by $\delta = \delta u + M(u, v)$. So u pins down v from this isoquant, and then we can use the zero-profit condition (28) (or (29)) to calculate

$$s = \frac{\mathrm{D}U(y - [(y - x)u + av])a}{(1 - \beta)M(u/v, 1)}.$$

Increasing u lowers v along the $\mathrm{D}u_t = 0$ isoquant, and so vacancy filling rate $M(u/v, 1)$ is increasing in u. Holding fixed the marginal utility of consumption, the requirement that the profits from maintaining vacancies are zero forces a lower s. But the effect on consumption of an increase in u along the $\mathrm{D}u_t = 0$ isoquant is ambiguous. Recall the golden rule displayed in Figure 1. If u is below the golden rule, then increasing u will raise consumption and the marginal utility of consumption will fall, further lowering s. So the $\mathrm{D}u_t = 0$ curve is guaranteed to be downward sloping when u is below the golden rule. But the effect of u on s is ambiguous when u is above the golden rule (as it is in the steady state of the planner's problem, and hence in the steady state equilibrium if the Hosios condition holds), and then increasing u will lower consumption and increase the marginal utility of consumption.

But because of Figure 2, we can be sure there is only one steady state, and this means that the $\mathrm{D}u_t = 0$ and $\mathrm{D}s_t = 0$ curves cross precisely once. Along the $\mathrm{D}u_t = 0$ isoquant, $u = 1/(1 + M(1, v/u)/\delta)$ implies that $v/u \uparrow \infty$ as u goes to zero. So this means $u/v \downarrow 0$ and hence $1/M(u/v, 1) \uparrow \infty$. At the same time, we know from Figure 1 that consumption declines and so the marginal utility of consumption must rise as u becomes small. So $s \uparrow \infty$ as $u \downarrow 0$. Since the $\mathrm{D}s_t = 0$ curve is upward sloping, this tells us that the $\mathrm{D}u_t = 0$ curve cuts the $\mathrm{D}s_t = 0$ from above.

This is enough to argue that the stable manifold will be upward sloping. The case in which s is decreasing in u along the $\mathrm{D}u_t = 0$ curve, anywhere in the neighborhood of the steady state, is shown in Figure 3.

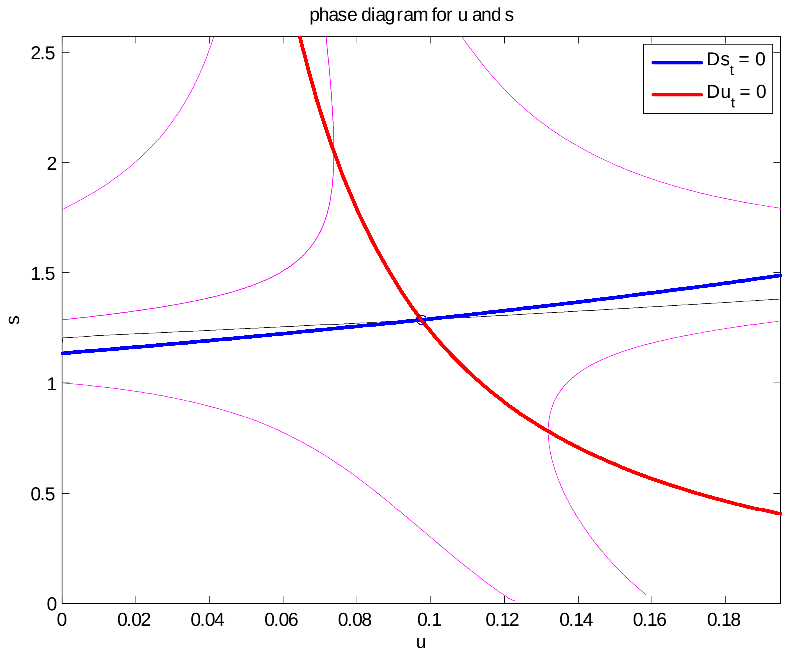

FIGURE 3 The Phase Diagram

When unemployment is high, the marginal utility weighted value of a vacancy is high, and the effect of this is to increase the supply of vacancies. A high number of unemployed workers and a high supply of vacancies both help bringing unemployment back down to its steady state.

Figure 4 shows the equilibrium trajectories for parameters taken from Pissarides [2009]. The steady state employment rate is 5.7% in this calibration and steady-state wages are 0.983. The initial level of unemployment in Figure 4 is 10%. This implies initial wages of about 0.980, only a fraction below the steady state. The most striking fact about these trajectories is the speed of convergence: unemployment is essentially back to the steady state in about half a year. The next section explains why this happens.

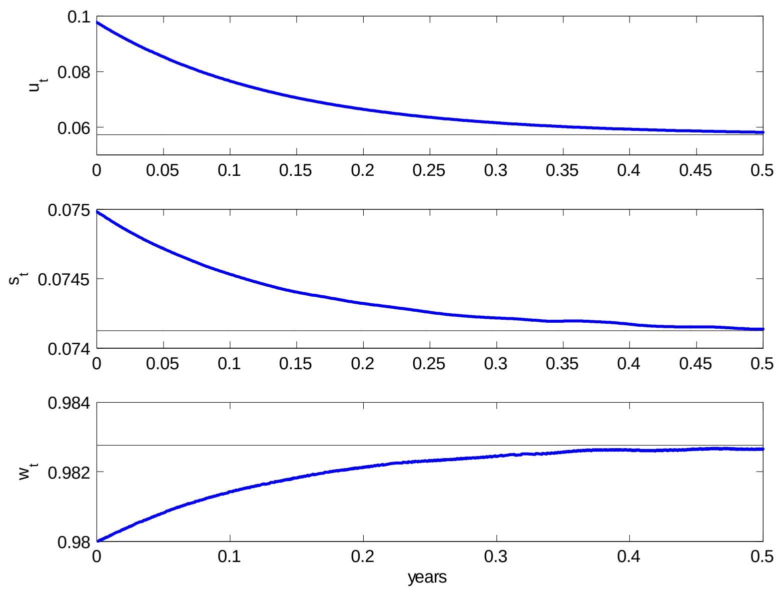

FIGURE 4 Equilibrium Trajectories for Pissarides [2009]

4 A Calibration

Monthly data on hires, quits, layoffs, and vacancies are available from the Job Openings and Labor Turnover Survey (JOLTS) conducted by the Bureau of Labor Statistics (BLS). The BLS also publishes data on the numbers of employed and unemployed workers, based on their Current Population Survey (CPS.) The important flows are shown in Figure 5.

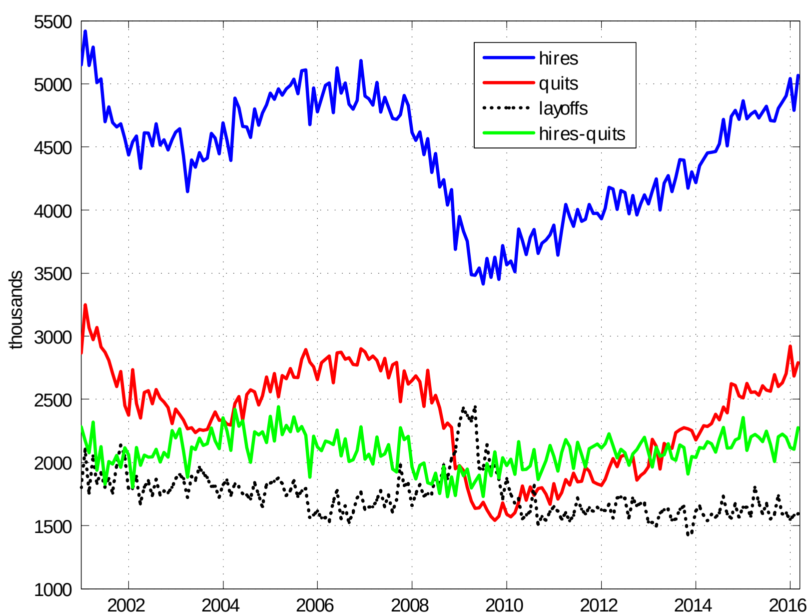

FIGURE 5 JOLTS Flows

See the class web site for more of the evidence for the US labor market. It is important to note that these data are highly seasonal. The numbers in Figure 5 are seasonally adjusted. The unadjusted data is available at the JOLTS web site of the BLS.

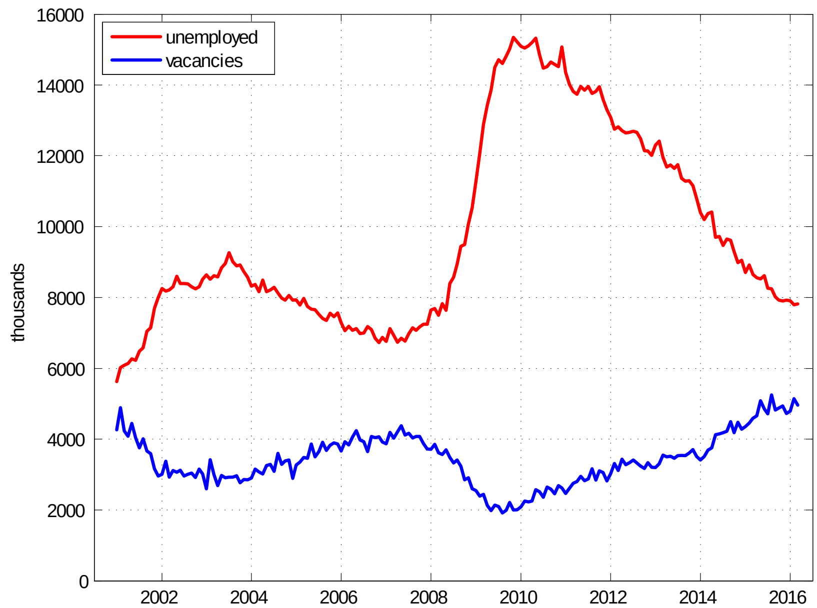

FIGURE 6 Unemployment and Vacancies

4.1 Evidence on Job Finding and Destruction Rates

The unemployment rate in the US has come down very significantly from its most recent peak in 2009, when it was close to 10%. The number of employed in the US is about 150 million, and the number of unemployed is around 8 million. This implies an unemployment rate of $8/158$ or about 5.1%. The number of hires currently is around 5 million per *month*, and the number of quits around 3 million per *month*. Many of these quits are accounted for by job-to-job transitions. The number of layoffs is about 1.75 million per *month*. If we interpret these layoffs (rather than all separations, which includes both layoffs and quits) as resulting from δ shocks, then this says that

$$\delta \approx \frac{1.75 \times 12}{150} = 0.14$$

per annum. Suppose there are no quits into unemployment. That is, every quit is initiated by a hire in a job-to-job transition. Then the net flow of hires that will absorb unemployed workers is about 2 million per month. So the job finding rate for unemployed workers is

$$\phi = \frac{12 \times 2}{8} = 3$$

per annum. In other words, a quarter of the unemployed find a job in every month.

4.1.1 Implied Steady-State Unemployment

Conjecture that ϕ is (close to) the steady state job finding rate for unemployed workers. What will be the steady-state unemployment rate if there are no aggregate shocks? An easy way to add some realism is to take into account that there is population growth.[2] Suppose that the labor force is l_t and the flow of new workers is j_t. Suppose these new workers initially join the population of unemployed workers and then find jobs, just like unemployed workers who have already had jobs in the past. One could imagine alternative assumptions, but if this is the case, then the number of employed workers evolves according to

$$\mathrm{D}n_t = -\delta n_t + M(l_t - n_t, v_t), \quad \mathrm{D}l_t = j_t.$$

Suppose that the population growth rate is η, and has been for a long time. This implies that in a steady state

$$[j_t, l_t, n_t, v_t] = [j, l, n, v]e^{\eta t}.$$

The differential equation for n_t then implies that

$$(\eta + \delta)n = M(l - n, v) = \phi \times (l - n).$$

The number of unemployed workers is $u = l - n$, and this implies an unemployment rate of

$$\frac{u}{l} = \frac{\eta + \delta}{\eta + \delta + \phi}.$$

Notice that the population growth rate simply acts like an addition to the random job destruction rate. In the US, population growth is about 1% per annum, and so the implied steady-state unemployment rate is

$$u = \frac{\eta + \delta}{\eta + \delta + \phi} = \frac{0.15}{0.15 + 3} = \frac{1}{21} \approx 0.048,$$

or 4.8%. This is not much different from the 4.5% that we would have estimated without accounting for population growth. And it is only a fraction below the current unemployment rate of 5.1%. So the current unemployment rate is close to where it will be in a steady

[2]One should really include exit from the labor force as well. A simple and fairly realistic way to do that is to have fixed finite (working) life spans T. But that will make what happens in every match depend on the age of the worker. This itself may be a very useful step towards realism, but it also complicates the analysis. Continue to assume that $T = \infty$.

state, consistent with our conjecture that $\phi = 3$ is close to the steady state. Of course, this assumes there will be no further aggregate shocks.

4.1.2 Using Vacancy Data to Fit a Cobb-Douglas Matching Function

At any point in time, we can learn about the matching function from

$$\phi_t = M\left(1, \frac{v_t}{u_t}\right)$$

together with data on u_t, v_t and ϕ_t, obtained from data on the stock of unemployed workers, the stock of vacancies, and data on the flow of workers hired per unit of time. Suppose that the matching function is Cobb-Douglas,

$$M(u, v) = \mu u^{\alpha} v^{1-\alpha}.$$

This implies the log-linear relation

$$\ln(\phi_t) = \ln(\mu) + (1-\alpha)\ln(v_t/u_t).$$

At the peak of the 2008-2009 recession in the US, the difference between hires and quits was roughly 2 million, just has it is now (the difference between hires and quits is very stable compared to just about everything else in JOLTS.) But the number of unemployed workers was as high as 16 million in the summer of 2009. The resulting job-finding rate for the unemployed would then have been $\phi_t = 12 \times 2/16 = 1.5$, only half of its steady state value. The JOLTS data on vacancies show that the number of vacancies in the summer of 2009 was about $1/7$ of the number of unemployed workers, while now it is up to $2/3$. Taking this to be the steady state, and fitting the matching function to both the steady state and the trough of the most recent recession gives, roughly,

$$\begin{bmatrix} \ln(3) \\ \ln(3/2) \end{bmatrix} \approx \begin{bmatrix} 1 & \ln(2/3) \\ 1 & \ln(1/7) \end{bmatrix} \begin{bmatrix} \ln(\mu) \\ 1-\alpha \end{bmatrix}.$$

Solving for $\ln(\mu)$ and $1-\alpha$ yields

$$\begin{bmatrix} \ln(\mu) \\ 1-\alpha \end{bmatrix} \approx \begin{bmatrix} 1 & \ln(2/3) \\ 1 & \ln(1/7) \end{bmatrix}^{-1} \begin{bmatrix} \ln(3) \\ \ln(3/2) \end{bmatrix} \approx \begin{bmatrix} 1.28 \\ 0.45 \end{bmatrix},$$

which implies $\mu \approx 3.6$.

Figure 7 expands on this back-of-the-envelope estimate. It shows monthly data on ϕ_t (measured in as JOLTS hires minus quits per annum) and v_t/u_t for the period December 2000 through February 2016.

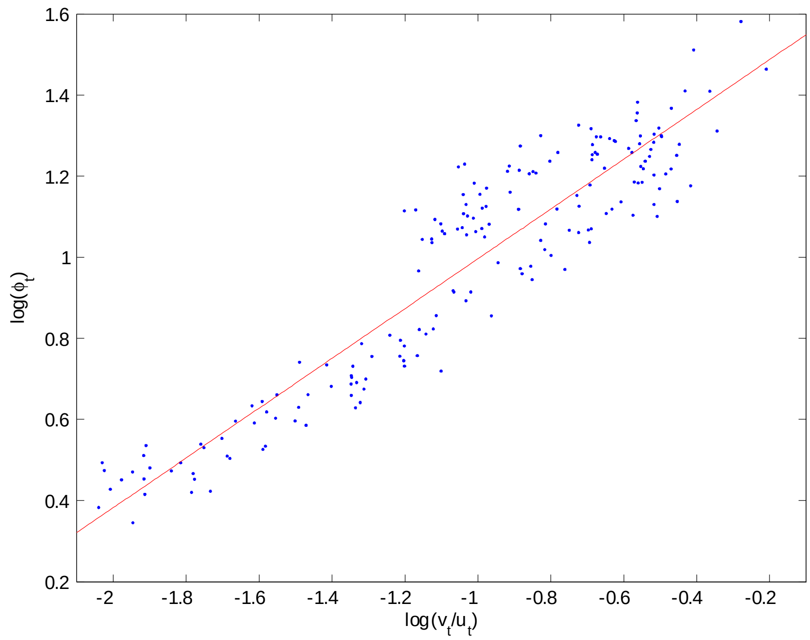

FIGURE 7 Fitting a Matching Function with 2000-2016 Data

The regression line shown fits well and is precisely estimated. It has a slope 0.61 and the intercept is 1.6, which implies $\mu \approx 5$.

4.2 Difficulties

We have not tried to explain why unemployment is not in its steady state. A simple answer is to assume that there was an recent aggregate shock that destroyed matches. There must have been a large aggregate shock that, somehow, destroyed a lot of matches in the US economy, resulting in 10% unemployment. The year 2008 certainly was eventful. But the difficulty with interpreting what happened subsequently through the lens of this model of the labor market is that vacancies and unemployment should co-move, as illustrated by the equilibrium trajectory shown in Figure 2. Vacancies in the summer of 2009 should have been at an all-time high. Instead, they are at an all-time high now. The evidence of this latest recession is not special: there is a strong negative correlation between u_t and v_t in the data, and the resulting downward sloping cloud in u-v space is known as the Beveridge curve.

Another difficulty is that the unemployment rate converges much more slowly back to the steady state than is predicted by this model. To see this, continue to suppose that

$l_t = le^{\eta t}$ and write the dynamics for n_t as

$$\mathrm{D}\left[n_t e^{-\eta t}\right] = -(\eta+\delta)[n_t e^{-\eta t}] + M(l - [n_t e^{-\eta t}], v_t e^{-\eta t})$$

and then

$$\mathrm{D}\left[u_t e^{-\eta t}\right] = -[\eta+\delta+M(1, v_t/u_t)]\left[u_t e^{-\eta t}\right] + (\eta+\delta)l$$

or

$$\mathrm{D}\left[u_t/l_t\right] = -[\eta+\delta+\phi_t]\left[u_t/l_t\right] + \eta+\delta.$$

If the job finding rate $\phi_t = M(1, v_t/u_t)$ is close to its steady state, then this predicts that the unemployment rate should converge to its steady state at the incredible rate of $\eta+\delta+\phi = 3.15$. The implied half life of a deviation from steady state is $\ln(2)/(\eta+\delta+\phi) \approx 0.22$ years, or about 2.64 months. The unemployment rate peaked in the summer of 2009 at about 10%, or about 5.2% above the steady state. A constant job finding rate equal to its steady state value of 3 would say that this should have shrunk to about

$$4.8 + e^{-3.15}(10 - 4.8) \approx 5.02$$

only one year later, in the summer of 2010. In fact, the US economy is only now, in 2016, reaching this level of unemployment. Even a constant job-finding rate at the low level of 2009, about 1.5, would imply fast convergence to the steady state. In the summer of 2010 unemployment would have been 5.8%, a level it only reached in 2015.

A much lower but constant job-finding rate would solve the speed of convergence problem, but only at the cost of predicting a much higher steady-state level of unemployment $(\eta+\delta)/(\eta+\delta+\phi)$. What the data want is a job-finding rate that is low when unemployment is high, for long enough to slow down convergence, but not forever, to avoid steady-state unemployment rates that are too high. This is precisely what the empirical Beveridge curve predicts in the presence of a stable matching function, but the model does not generate a realistic Beveridge curve because the model generates vacancies that are well above their steady state when unemployment is high.

4.3 Outline of a Solution: Vacancies Are a Stock

The following is taken from Luttmer [2011], where a complete model with endogenous entry and replication decisions is presented.

Abstract from population growth. Suppose workers have to be assigned to "projects," one per project. Let k_t be the stock of projects at time t and continue to let n_t denote the number of employed workers. Suppose any project not assigned to a worker is automatically a vacancy. So $n_t \in [0, k_t]$, $u_t = 1 - n_t$, and $v_t = k_t - n_t$. Employed workers can generate new projects (replicate the project they are currently assigned to) randomly at the rate $\gamma > \delta$. So we have

$$\begin{aligned} \mathrm{D}k_t &= -\delta k_t + \gamma n_t, \\ \mathrm{D}n_t &= -\delta n_t + M(1 - n_t, k_t - n_t). \end{aligned}$$

Observe that this system now has two state variables. Vacancies can no longer jump up to quickly soak up high unemployment. The steady state is determined by

$$\delta n = M(1 - n, k - n), \quad \delta k = \gamma n.$$

Figure 8 shows the trajectories for $u_t = 1 - n_t$ and $v_t = k_t - n_t$ together with the $\mathrm{D}k_t = 0$ and $\mathrm{D}n_t = \mathrm{D}u_t = 0$ curves.

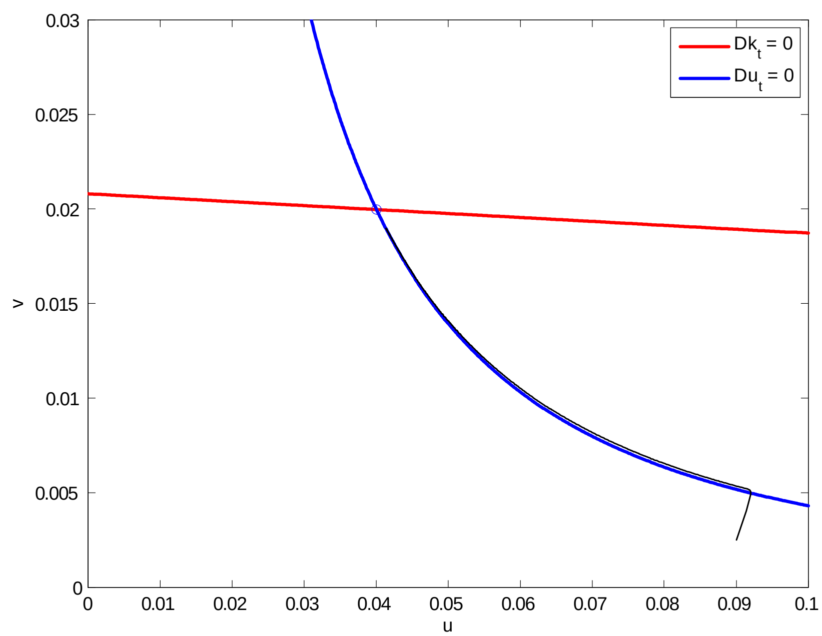

FIGURE 8 Project Replication and Matching

The matching function is very productive in this example, and this very quickly puts the trajectory close to the downward-sloping $\mathrm{D}u_t = 0$ curve. Figure 9 displays the resulting time series for $u_t = 1 - n_t$ and $v_t = k_t - n_t$. Note the similarity to the US evidence following the 2008 recession, shown in Figure 6.

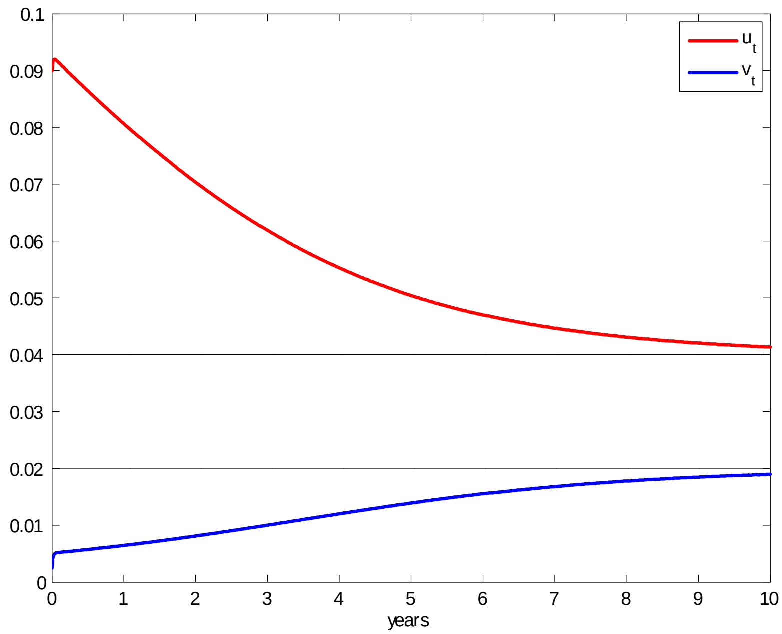

FIGURE 9 Starting From High Unemployment

5 Further Reading

Pissarides [2000] and Shimer [2010] are two monographs on search and matching models of unemployment. A very useful survey can be found in Rogerson, Shimer and Wright [2005].

References

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