Search and Matching in the Labor Market

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the economy

- the preferences of the typical consumer are

$$\mathcal{U}(C) = \mathrm{E}\left[\int_0^\infty e^{-\rho t} U(C_t)\mathrm{d}t\right]$$

- the stock of matches n_t evolves according to

$$\mathrm{D}n_t = -\delta n_t + M(u_t, v_t)$$

  where $u_t = 1 - n_t$ is unemployment and v_t is vacancies

  - the function M is a production function
  - constant returns to scale
  - with $M(1, 0) = 0$ and $M(0, 1) = 0$

- the output of consumption is

$$C_t = u_t x + (1 - u_t)y - av_t$$

  where $y > x > 0$ and $a > 0$

the urn-ball matching function

- suppose there are $[uN]$ unemployed workers
- and $[vN]$ vacancies
- every unemployed worker sends one application,
  - to one randomly selected vacancy
  - in case of multiple applicants, a random applicant is hired
- for each vacancy, the hiring probability is

$$
\begin{aligned}
1 - \Pr[\text{no hire}] &= 1 - \left(1 - \frac{1}{[vN]}\right)^{[uN]} \\
&= 1 - \left(1 - \frac{[uN]/[vN]}{[uN]}\right)^{[uN]} \rightarrow 1 - e^{-u/v}
\end{aligned}
$$

as N becomes large

- so the limiting flow of matches is

$$M(u, v) = \left(1 - e^{-u/v}\right) v$$

the planner

the planner

- the Hamiltonian is

$$\mathcal{H}(n,\mu) = \max_v \{U\,(x + (y-x)n - av) : \mu\,(-\delta n + M(1-n,v))\}$$

- therefore

$$\begin{aligned} \mathrm{D}n_t &= -\delta n_t + M(1-n_t, v_t) \\ \mathrm{D}\mu_t &= (\rho + \delta + \mathrm{D}_1 M(1-n_t, v_t))\mu_t - (y-x)\lambda_t \end{aligned}$$

where

$$\begin{aligned} \lambda_t &= \mathrm{D}U\,(x + (y-x)n_t - av_t) \\ a\lambda_t &\geq \mu_t \mathrm{D}_2 M(1-n_t, v_t), \quad \text{w.e. if } v_t > 0 \end{aligned}$$

the conditions for a steady state

- the condition $\mathrm{D}n_t = 0$ implies

$$u = \frac{\delta}{\delta + M\left(1, v/u\right)}$$

with

$$\mu = \frac{\mathrm{D}U\left(y - \left[y - x + a \times v/u\right] u\right) a}{\mathrm{D}_2 M\left(1, v/u\right)}$$

- the condition $\mathrm{D}\mu_t = 0$ implies

$$\mu = \frac{\mathrm{D}U\left(y - \left[y - x + a \times v/u\right] u\right) a}{\rho + \delta + \mathrm{D}_1 M\left(1, v/u\right)} \times \frac{y - x}{a}$$

with

$$\mu = \frac{\mathrm{D}U\left(y - \left[y - x + a \times v/u\right] u\right) a}{\mathrm{D}_2 M\left(1, v/u\right)}$$

  – this implies

$$\frac{y - x}{a} = \frac{\rho + \delta + \mathrm{D}_1 M\left(1, v/u\right)}{\mathrm{D}_2 M\left(1, v/u\right)}$$

steady state

- an efficiency condition for v/u

$$\frac{y-x}{a} = \frac{\rho + \delta + \mathrm{D}_1 M\left(1, v/u\right)}{\mathrm{D}_2 M\left(1, v/u\right)} \tag{1}$$

- the isoquant determines u

$$u = \frac{\delta}{\delta + M\left(1, v/u\right)} \tag{2}$$

- consumption is

$$c = y - [(y-x)u + av] \tag{3}$$

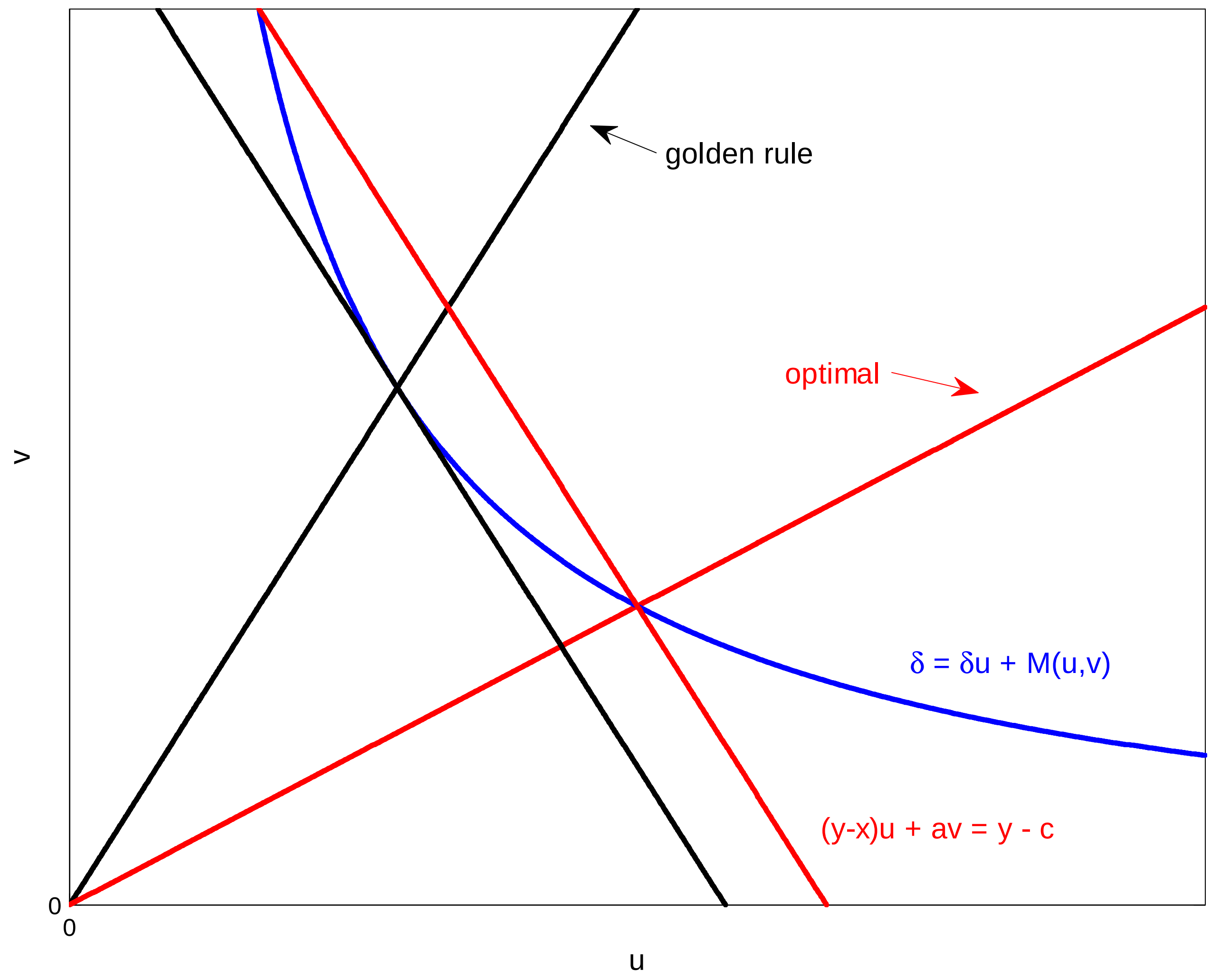
- alternatively, the golden rule is determined by

$$\max_{v} \left\{y - [(y-x)u + av] : \delta u + M(u, v) \geq \delta\right\}$$

which, instead of (1), gives

$$\frac{y-x}{a} = \frac{\delta + \mathrm{D}_1 M\left(1, v/u\right)}{\mathrm{D}_2 M\left(1, v/u\right)}$$

the optimal and golden rule allocations



Nash bargaining

a symmetric equilibrium

- complete markets and no aggregate risk
  - everyone only cares about expected present values,
  - discounted at the risk-free rates r_t
- the firm and worker use a particular Nash bargaining rule
  - to share the joint surplus of a match
  - anticipating that the same is true in all other current and future matches
- the job finding and filling rates are

$$\phi_t = M\left(\frac{v_t}{u_t}, 1\right), \quad \psi_t = M\left(1, \frac{v_t}{u_t}\right)$$

  - note that, by construction, $\phi_t/\psi_t = v_t/u_t$

three connected asset pricing equations

- unemployed worker

$$r_t U_t = x + \mathrm{D}U_t + \phi_t(V_t - U_t)$$

  – the effective flow earnings are $x + \phi_t(V_t - U_t) \geq x$

- employed worker earning w_t

$$r_t V_t = w_t + \mathrm{D}V_t - \delta(V_t - U_t)$$

- a matched firm paying w_t

$$r_t F_t = y - w_t + \mathrm{D}F_t - \delta F_t$$

▶ employed worker surplus

$$r_t\left(V_t - U_t\right) = w_t - [x + \phi_t(V_t - U_t)] + \mathrm{D}\left(V_t - U_t\right) - \delta(V_t - U_t)$$

▶ joint surplus of a match

$$r_t\left(F_t + V_t - U_t\right) = y - [x + \phi_t(V_t - U_t)] + \mathrm{D}\left(F_t + V_t - U_t\right) - \delta\left(F_t + V_t - U_t\right)$$

sharing the surplus

- the joint surplus of the match satisfies

$$(r_t + \delta)(F_t + V_t - U_t) = y - [x + \phi_t(V_t - U_t)] + \mathrm{D}(F_t + V_t - U_t)$$

- the time-t surplus is shared according to

$$V_t - U_t = \beta (F_t + V_t - U_t)$$

  for some $\beta \in (0,1)$

  – everyone expects the same sharing rule to be used everywhere and at all times

- eliminate $V_t - U_t$ from the asset pricing equation for the joint surplus

$$(r_t + \delta + \beta\phi_t)(F_t + V_t - U_t) = y - x + \mathrm{D}(F_t + V_t - U_t)$$

  – now use $\lambda_t = \mathrm{D}U(C_t)$ to define

$$s_t = \lambda_t (F_t + V_t - U_t)$$

- together with $r_t = \rho - \mathrm{D}\lambda_t/\lambda_t$, this implies

$$(\rho + \delta + \beta\phi_t)\, s_t = (y - x)\,\lambda_t + \mathrm{D}s_t$$

free entry

- consider a steady state with
  - risk-free rate $r_t = r$, job-filling rate $\psi_t = \psi$, firm value $F_t = F$
- consider paying the flow $a > 0$ until matched
  - the waiting time has a density $\psi e^{-\psi T}$
- the present value of this is

$$\int_0^\infty \psi e^{-\psi T} \left(- \int_0^T e^{-rt} a \mathrm{d}t + e^{-rT} F \right) \mathrm{d}T$$

$$= - \int_0^\infty \psi e^{-\psi T} \left(\int_0^T e^{-rt} a \mathrm{d}t \right) \mathrm{d}T + \int_0^\infty e^{-(r+\psi)T} \psi F \mathrm{d}T$$

$$= \int_0^\infty e^{-(r+\psi)s} \left(-a + \psi F \right) \mathrm{d}s = \frac{-a + \psi F}{r + \psi}$$

- free entry forces $\psi F \leq a$
- vacancies must be positive in a steady state
  - this requires $\psi F = a$

more generally

- let G_t be the present value of maintaining a vacancy
  - until a match occurs, or until $G_t \geq 0$ hits zero
  - whichever comes first
- this present value must satisfy

$$r_t G_t = -a + \mathrm{D}G_t + \psi_t(F_t - G_t), \quad G_t \geq 0$$

- if $G_s > 0$ for $s \in [t, T)$, then this yields

$$G_t = \int_0^T \exp\left(-\int_0^t (r_s + \psi_s)\mathrm{d}s\right)(\psi_t F_t - a)\mathrm{d}t + \exp\left(-\int_0^T (r_s + \psi_s)\mathrm{d}s\right) G_T$$

- given a trajectory $\{\psi_s F_s - a\}_{s \geq t}$, define $\tau = \inf\{T \geq t : G_T = 0\}$,
  - and conclude that

$$G_t = \int_0^\tau \exp\left(-\int_0^t (r_s + \psi_s)\mathrm{d}s\right)(\psi_t F_t - a)\mathrm{d}t$$

- cannot have this be strictly positive at any time t
  - therefore, must have $\psi_t F_t \leq a$, and thus $(1 - \beta)\psi_t s_t \leq \lambda_t a$

the equilibrium conditions

- the state $[u_t, s_t]$ evolves according to

$$\mathrm{D}u_t = (1-u_t)\delta - M(u_t, v_t) \tag{1}$$

$$\mathrm{D}s_t = \left(\rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right)\right) s_t - (y - x)\,\lambda_t, \tag{2}$$

where v_t and λ_t are jointly determined by

$$\lambda_t a = (1-\beta) M\left(\frac{u_t}{v_t}, 1\right) s_t \tag{3}$$

$$\lambda_t = \mathrm{D}U(u_t x + (1-u_t)y - a v_t) \tag{4}$$

- eliminating λ_t gives the equilibrium condition for v_t

$$s_t = \frac{\mathrm{D}U(u_t x + (1-u_t)y - a v_t)a}{(1-\beta)M\,(u_t/v_t, 1)}$$

  – given the state $[u_t, s_t]$

the condition for v_t given $[u_t, s_t]$

- recall that

$$s_t = \frac{\mathrm{D}U(y - [y - x + av_t/u_t] \times u_t)a}{(1-\beta)M(u_t/v_t, 1)}$$

- holding fixed u_t, this implies

$$\frac{\partial v_t}{\partial s_t} > 0$$

  - a higher marginal utility weighted surplus attracts vacancies

- holding fixed s_t, this implies

$$\frac{\partial v_t/u_t}{\partial u_t} < 0$$

  - this effect is close to zero if the curvature of $U(\cdot)$ is small
  - so then high unemployment implies high vacancies…
  - if s_t barely responds, and more so if s_t increases with u_t

the steady state

- the condition $\mathrm{D}s_t = 0$ implies

$$\left(\rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right)\right) \times s_t = (y - x)\,\lambda_t, \quad s_t = \frac{a\lambda_t}{(1-\beta)M\left(u_t/v_t, 1\right)}$$

which gives

$$\frac{y - x}{a} = \frac{\rho + \delta + \beta M\left(1, v/u\right)}{(1-\beta)M\left(1/(v/u), 1\right)} \qquad (v/u)$$

  – the right-hand side is increasing in v/u
  – an increase in $(y - x)/a$ will increase v/u
  – an increase in β or $\rho + \delta$ lowers v/u

- the condition $\mathrm{D}u_t = 0$ implies

$$u = \frac{\delta}{\delta + \phi}, \quad \phi = M\left(1, \frac{v}{u}\right) \qquad (u)$$

  – the right-hand side of u is decreasing in v/u

▶ an increase in δ will lower v/u and therefore raise unemployment

properties of the phase diagram

- the state $[u_t, s_t]$ evolves according to

$$\mathrm{D}u_t = (1-u_t)\delta - M(u_t, v_t)$$

$$\mathrm{D}s_t = \left(\rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right)\right) s_t - (y-x)\,\lambda_t$$

where v_t and λ_t are determined by

$$\lambda_t = \mathrm{D}U(u_t x + (1-u_t)y - a v_t) \tag{1}$$

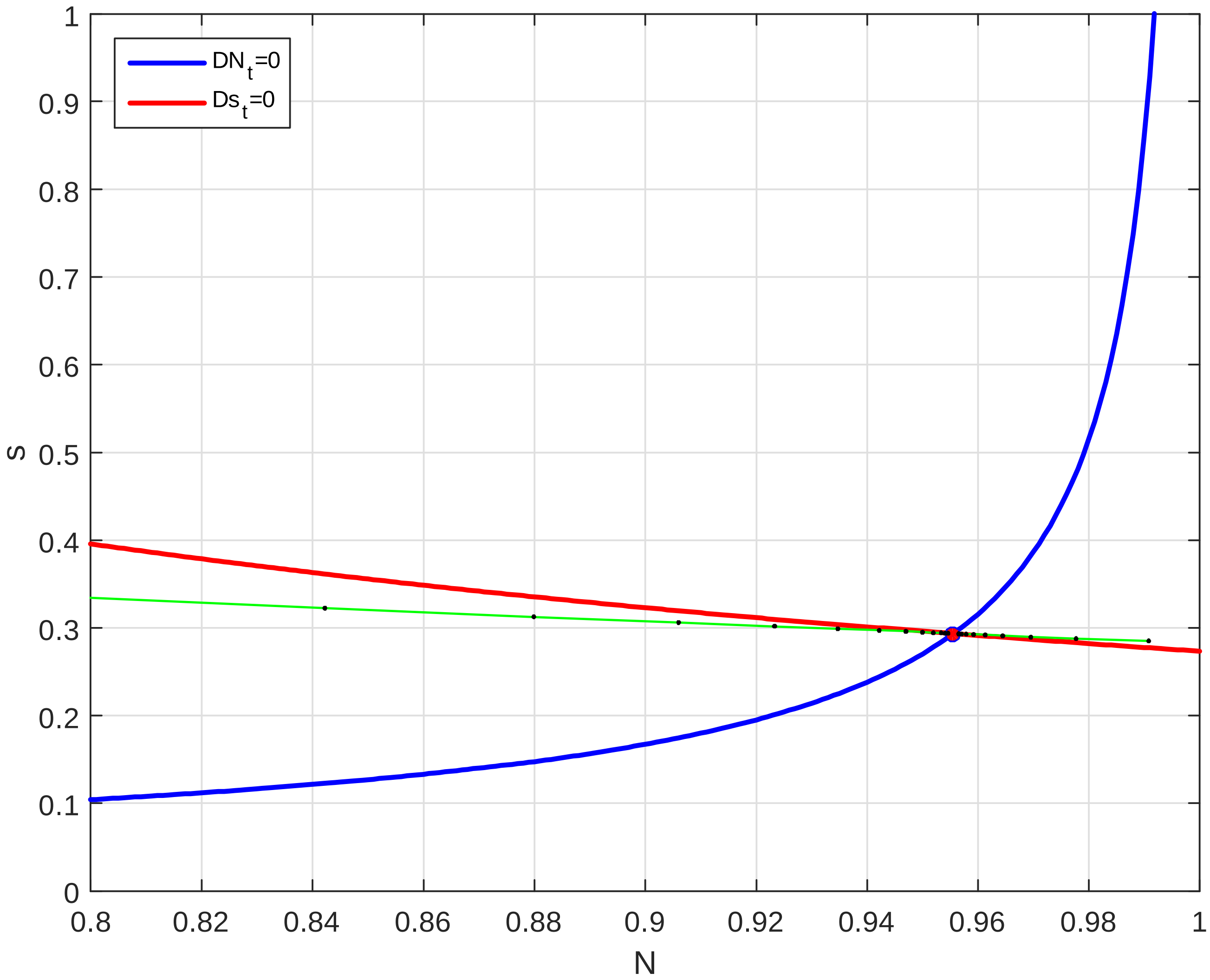
$$s_t = \frac{\mathrm{D}U(u_t x + (1-u_t)y - a v_t)a}{(1-\beta)M\left(u_t/v_t, 1\right)} \tag{2}$$

- holding fixed u_t, v_t increases with s_t via (2), and that reduces $\mathrm{D}u_t$
- holding fixed s_t, an increase in u_t
  - must lower v_t/u_t and increase λ_t via

$$s_t = \frac{\mathrm{D}U(y - [y - x + a v_t/u_t]\, u_t)a}{(1-\beta)M\left(1/(v_t/u_t), 1\right)} \Rightarrow \frac{\partial\left(v_t/u_t\right)}{\partial u_t} > 0,$$

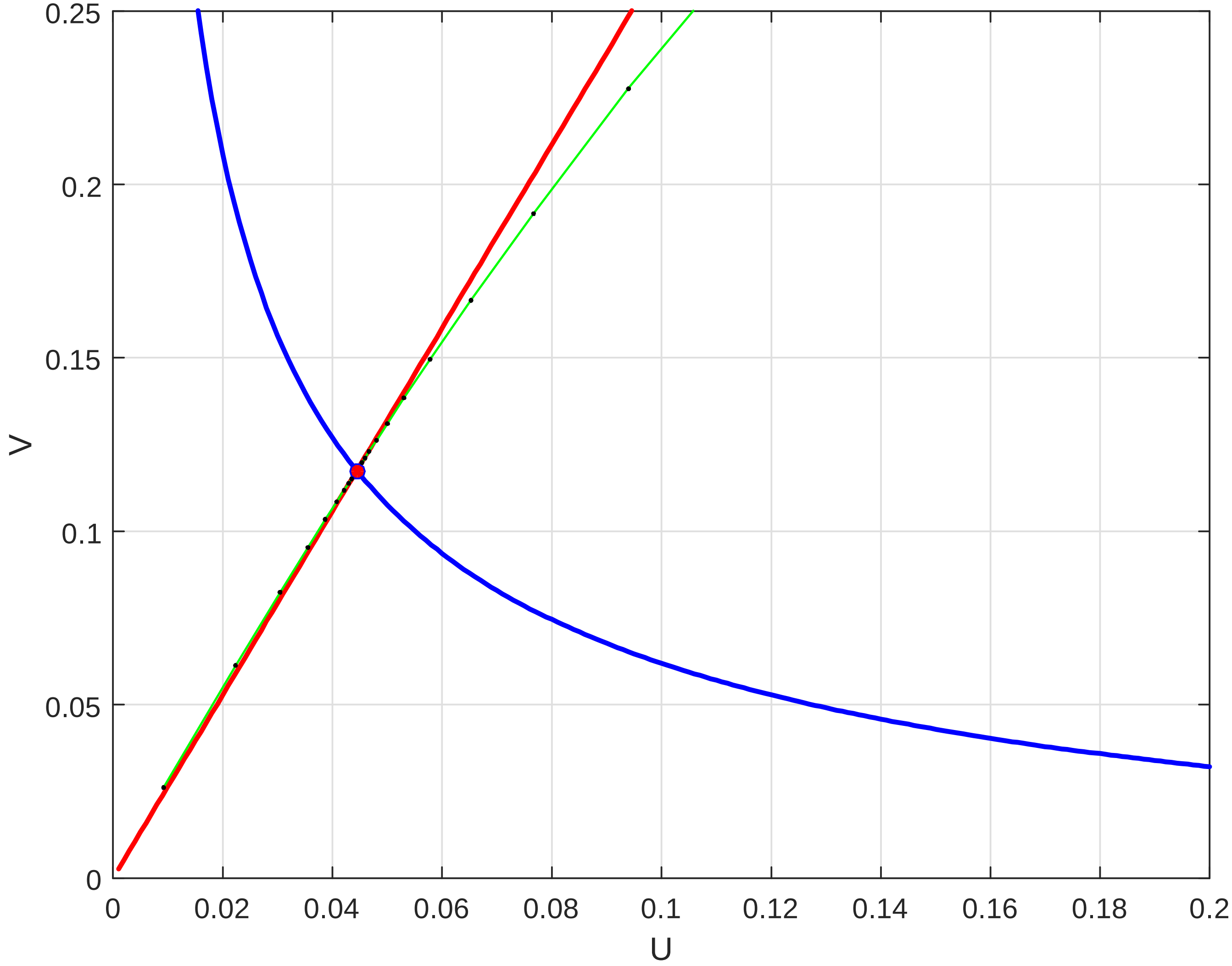
  - which then lowers $\mathrm{D}s_t$

the phase diagram



- black dots at quarterly intervals

the UV-diagram



- black dots at quarterly intervals

the Hosios condition

- common elements,

$$\mathrm{D}u_t \;=\; (1-u_t)\delta - M(u, v_t)$$

$$\lambda_t \;=\; \mathrm{D}U\left(x + (1-u_t)(y-x) - av_t\right)$$

▶ planner

$$\mathrm{D}\mu_t = (\rho + \delta + \mathrm{D}_1 M(u_t, v_t))\mu_t - (y-x)\lambda_t$$

where

$$a\lambda_t = \mu_t \mathrm{D}_2 M(u_t, v_t)$$

▶ bargaining

$$\mathrm{D}s_t = \left(\rho + \delta + \beta M\left(1, \frac{v_t}{u_t}\right)\right) s_t - (y-x)\,\lambda_t$$

where

$$a\lambda_t = (1-\beta) M\left(u_t/v_t, 1\right) s_t$$

- same thing if $M(u,v) \propto u^{\beta} v^{1-\beta}$

the underlying wages

- recall
$$\begin{bmatrix} F_t \\ V_t - U_t \end{bmatrix} = \begin{bmatrix} 1-\beta \\ \beta \end{bmatrix} (F_t + V_t - U_t)$$
and
$$\begin{aligned} \mathrm{D}F_t &= (r_t+\delta)F_t - (y - w_t) \\ \mathrm{D}\,(V_t - U_t) &= (r_t+\delta+\phi_t)\,(V_t-U_t) - (w_t - x) \end{aligned}$$
and
$$a = \psi_t F_t$$
- write the differential equation as
$$\begin{aligned} \mathrm{D}\,[\beta F_t] &= (r_t+\delta)\beta F_t - \beta\,(y - w_t) \\ \mathrm{D}\,[(1-\beta)\,(V_t - U_t)] &= (r_t+\delta+\phi_t)\,(1-\beta)\,(V_t-U_t) - (1-\beta)\,(w_t - x) \end{aligned}$$
- since $\beta F_t = (1-\beta)\,(V_t - U_t)$ *identically*, this gives
$$(r_t+\delta)\beta F_t - \beta\,(y-w_t) = (r_t+\delta+\phi_t)\,\beta F_t - (1-\beta)\,(w_t - x)$$

- recall

$$(r_t + \delta)\beta F_t - \beta\left(y - w_t\right) = \left(r_t + \delta + \phi_t\right)\beta F_t - (1-\beta)\left(w_t - x\right)$$

- this implies

$$w_t = (1-\beta)x + \beta\left(y + \phi_t F_t\right)$$

and hence

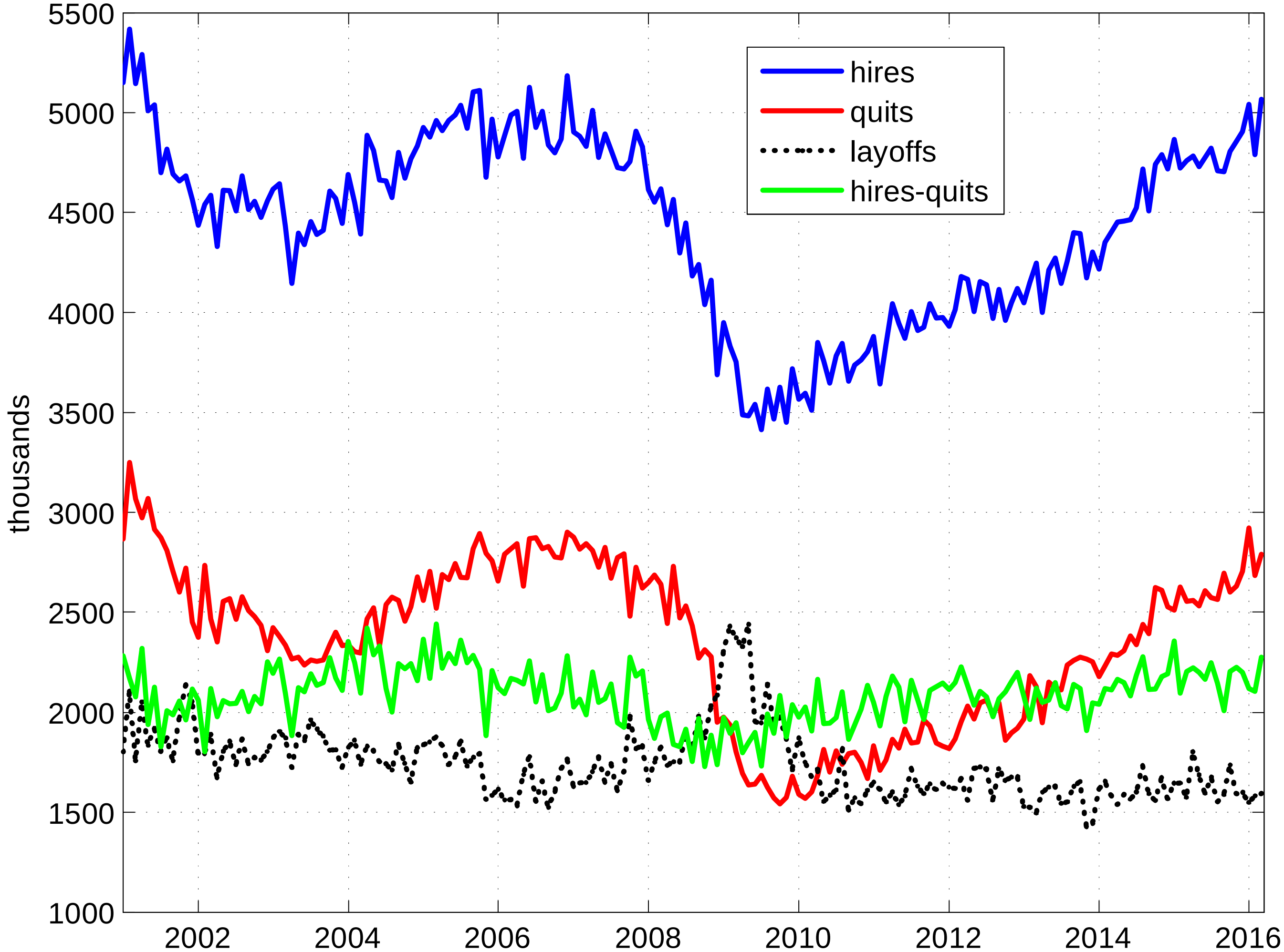
$$w_t = (1-\beta)x + \beta\left(y + a \times \frac{\phi_t}{\psi_t}\right)$$

or

$$w_t = (1-\beta)x + \beta\left(y + a \times \frac{v_t}{u_t}\right)$$

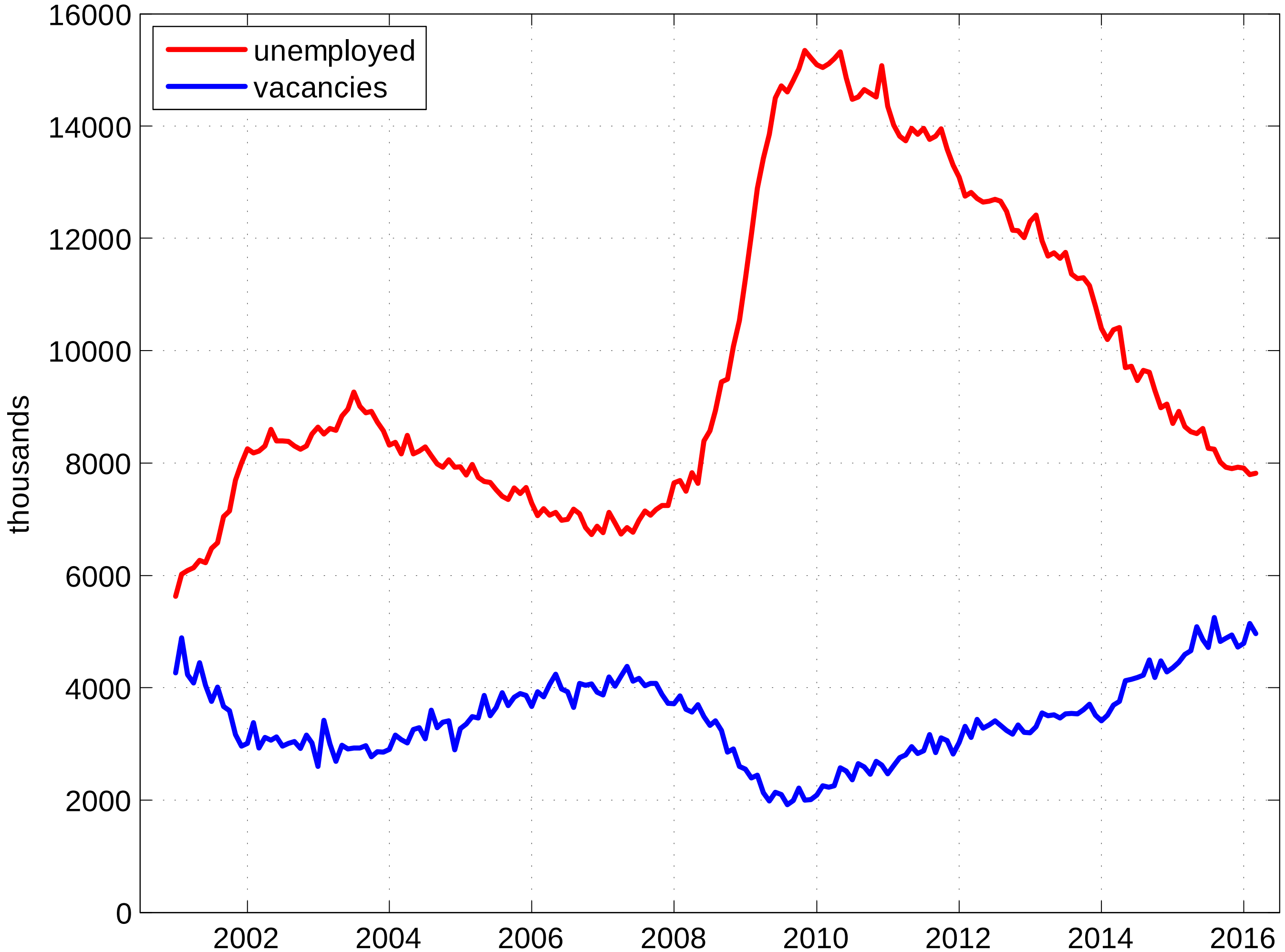
- so wages will be high when v_t/u_t is high

JOLTS *monthly* flows, seasonally adjusted



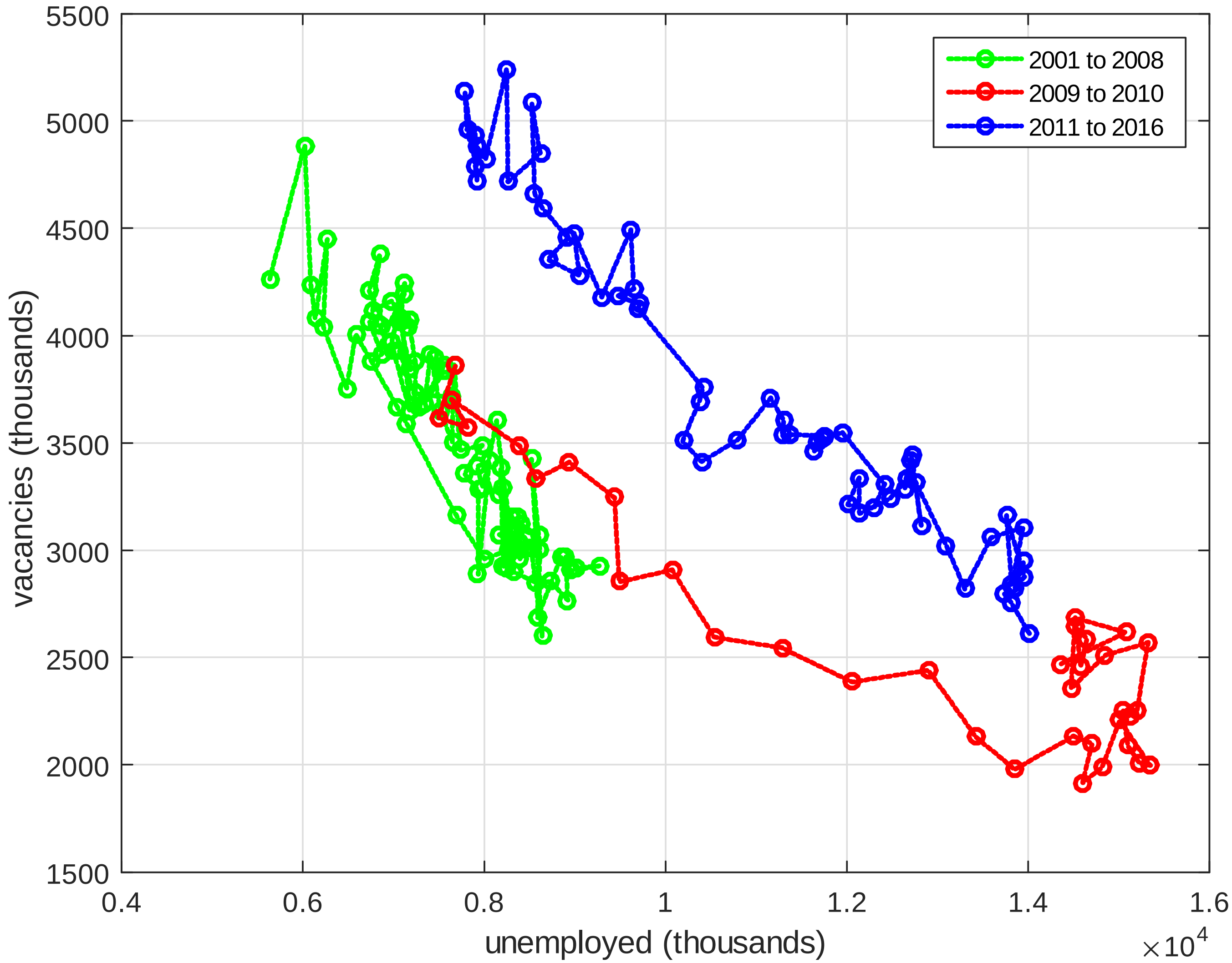
- this graph is correctly labeled…

stocks of unemployed and vacancies



- recession: unemployment doubles and vacancies are cut in half

the JOLTS Beveridge curve



a back-of-the-envelope calculation

- over the period 2010 to 2016,

	monthly flow		end of period stock
hire-quits	2 million	employed	150 million
layoffs	1.5 million	unemployed	8 million

  - so the 2016 unemployment rate was $8/158$, or about 5.1%

- suppose we take δ to reflect layoffs

$$\delta = \frac{12 \times 1.5}{150} = 0.12$$

- suppose everyone in the labor force and all quits were job-to-job
  - then hires minus quits reflects hiring from unemployment
  - can interpret ϕ to be the job-finding rate out of unemployment

$$\phi = \frac{12 \times 2}{8} = 3$$

- the steady state unemployment rate is

$$u = \frac{\delta}{\delta + \phi} = \frac{0.12}{0.12 + 3} \approx 3.8\%$$

the model speed of convergence

- suppose $v_t/u_t = v/u$ (not a bad approximation in the model), then

$$\mathrm{D}u_t \;=\; (1-u_t)\delta - M\left(1, \frac{v_t}{u_t}\right)u_t$$

$$=\; \delta - (\delta+\phi)\,u_t$$

where $\phi = M(1, v/u)$ is the steady state job finding rate

- the steady-state unemployment rate is

$$u = \frac{\delta}{\delta+\phi}$$

and the speed of convergence is $\delta+\phi$

- suppose $\phi = 3$ (find a job after 4 months) and $u = 0.04$,

$$0.04 = \frac{\delta}{\delta+3} \Rightarrow \delta + 3 = \frac{3}{0.96} = 3.125$$

- the half-life T of a deviation from steady state is

$$\frac{1}{2} = e^{-3.125\times T} \Rightarrow T = \frac{\ln(2)}{3.125} \approx 0.22$$

or about $0.22 \times 12 = 2.64$ months…

adjustment costs

it takes a job to create a job

- an employed worker can produce $1 - a_t \in [0, 1]$ units of consumption
  - and maintain $v_t = G(1, a_t)$ vacancies
  - the production function G exhibits constant returns to scale
- the supply of potential workers is $\mathcal{L}$
  - and $N_t \in (0, \mathcal{L})$ have a job at time t
- the aggregate technology is then

$$C_t = N_t - A_t$$

together with

$$\mathrm{D}N_t = -\delta N_t + M(\mathcal{L} - N_t, V_t), \quad V_t = G(N_t, A_t)$$

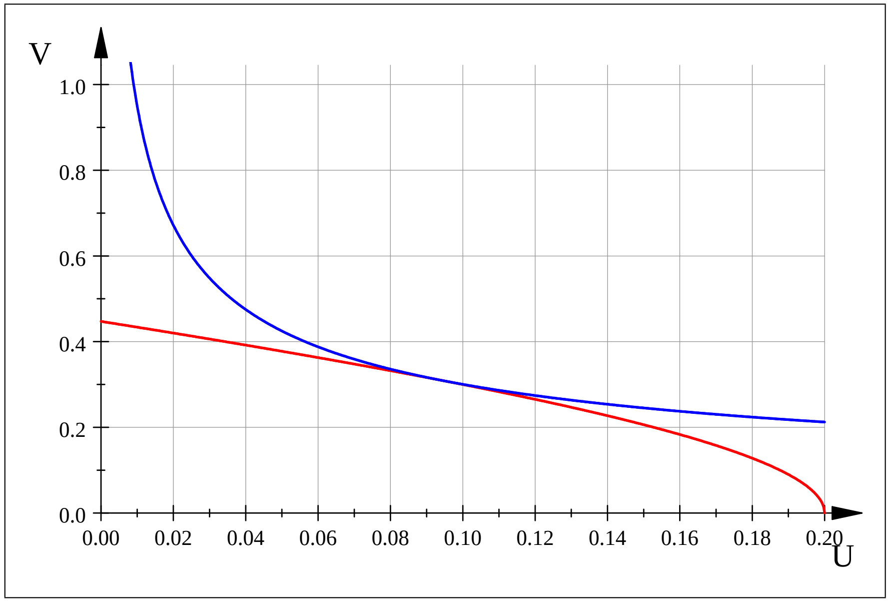
  - if $G(1, a)$ is linear in a, we have the standard model
  - curvature in $G(1, \cdot)$ makes vacancies above steady state expensive

this will not be enough

- suppose adjustment costs cause $A_t/N_t \approx a$, its steady state value
- the supply of vacancies will be low when unemployment is high
- but the effect is small
  - suppose U_t *doubles* from 0.05 to 0.10
  - then N_t goes from 0.95 to 0.90
  - since $V_t \approx G(1, a)N_t$,
  - this implies V_t down by a bit more than 5%
  - better than the sharp increase of the standard model
- but the Beveridge curve suggests vacancies down by 50%

adjustment cost formulation

- let $\mathcal{A}(N,V)$ solve
$$V = G\left(N, \mathcal{A}(N,V)\right)$$
  - this makes $\mathcal{A}(\mathcal{L}-U,V)$ increasing and convex in (U,V)
- the golden rule is determined by
$$\max_{U\in[0,\mathcal{L}],V\geq 0} \{\mathcal{L}-U-\mathcal{A}(\mathcal{L}-U,V) : \delta\mathcal{L} = \delta U + M(U,V)\}$$
  - two isoquants that must be tangent



two state variables: projects and matches

projects and matches

- the population of possible workers is $\mathcal{L}$
- there are also serial entrepreneurs who generate a flow $\mathcal{E}$ of projects
- a project can be used to produce consumption
  - and to create new projects,
  - but neither if not matched with a worker
- the technology is $C_t = N_t$ and

$$\begin{aligned} \mathrm{D}K_t &= -\delta K_t + \gamma N_t + \mathcal{E} \\ \mathrm{D}N_t &= -(\delta + \lambda)N_t + M\left(\mathcal{L} - N_t, K_t - N_t\right) \end{aligned}$$

  where δ, λ, and γ are positive, and
  - the measure of projects is $K_t \in (0, \infty)$
  - the measure of projects matched with workers is $N_t \in [0, \min\{\mathcal{L}, K_t\}]$
  - and $M(U_t, V_t)$ is the flow of new matches,

$$U_t = \mathcal{L} - N_t, \quad V_t = K_t - N_t$$

- with $\delta = \delta_{\mathrm{F}} + \delta_{\mathrm{K}}$, the firm size distribution will be approximately Pareto... (along the lines of Luttmer [*ReStud*, 2011])

the steady state conditions

- imposing $\mathrm{D}[K_t, N_t] = 0$ gives

$$\mathrm{D}K_t = 0 \Rightarrow \delta K = \gamma N + \mathcal{E}$$

$$\mathrm{D}N_t = 0 \Rightarrow (\delta + \lambda)\mathcal{L} = (\delta + \lambda)(\mathcal{L} - N) + M(\mathcal{L} - N, K - N)$$

- the region $\mathrm{D}K_t \geq 0$ is defined by

$$\delta K_t \leq \gamma N_t + \mathcal{E}$$

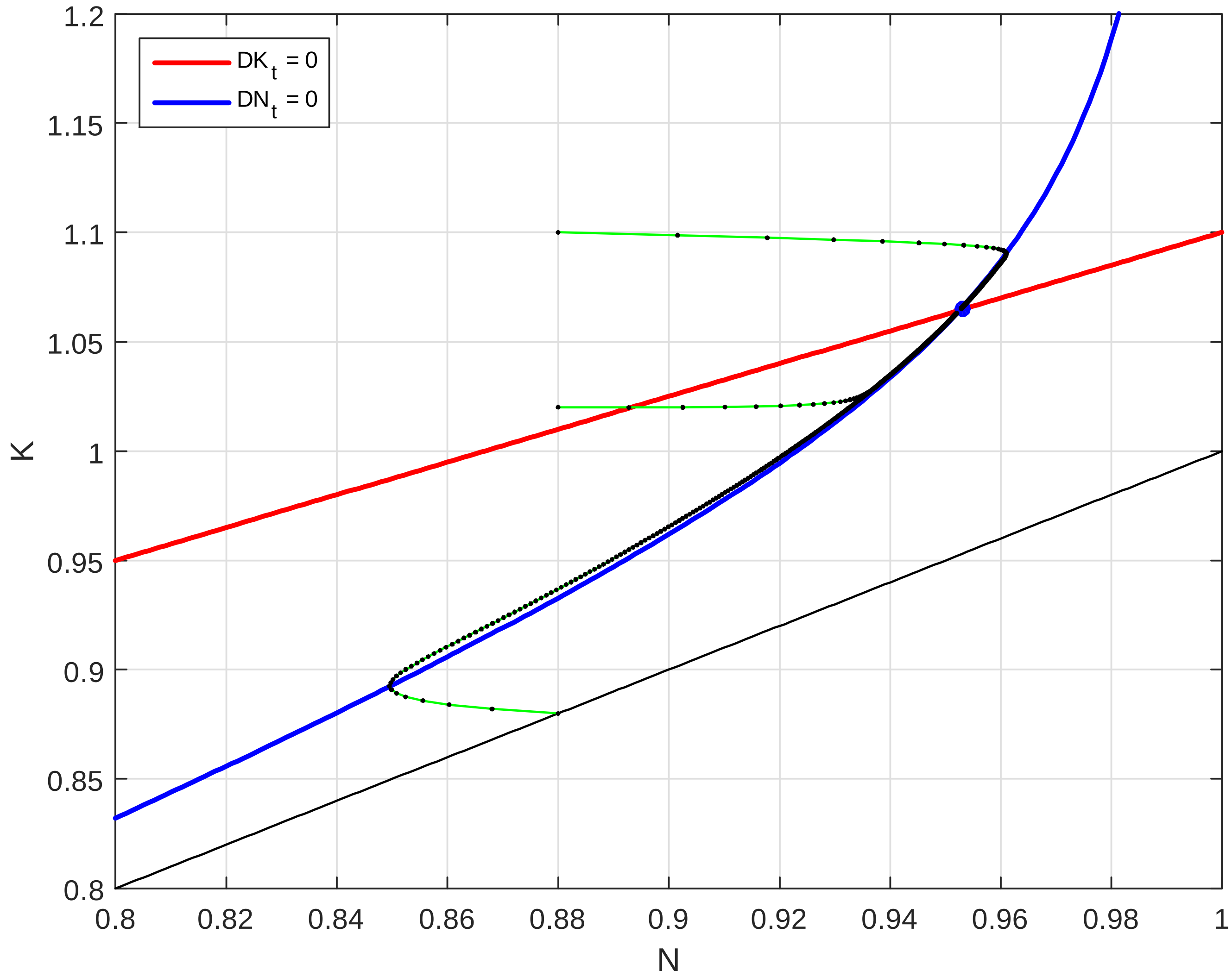
- the region $\mathrm{D}N_t \geq 0$ is defined by

$$(\delta + \lambda)\mathcal{L} \leq (\delta + \lambda)(\mathcal{L} - N_t) + M(\mathcal{L} - N_t, K_t - N_t)$$

  – the concavity of $M(\cdot, \cdot)$ implies that this set is convex
  – the boundary defines a strictly increasing function $N \mapsto K$,
  – that starts at $[N, K] = [0, 0]$ and asymptotes at $N = \mathcal{L}$

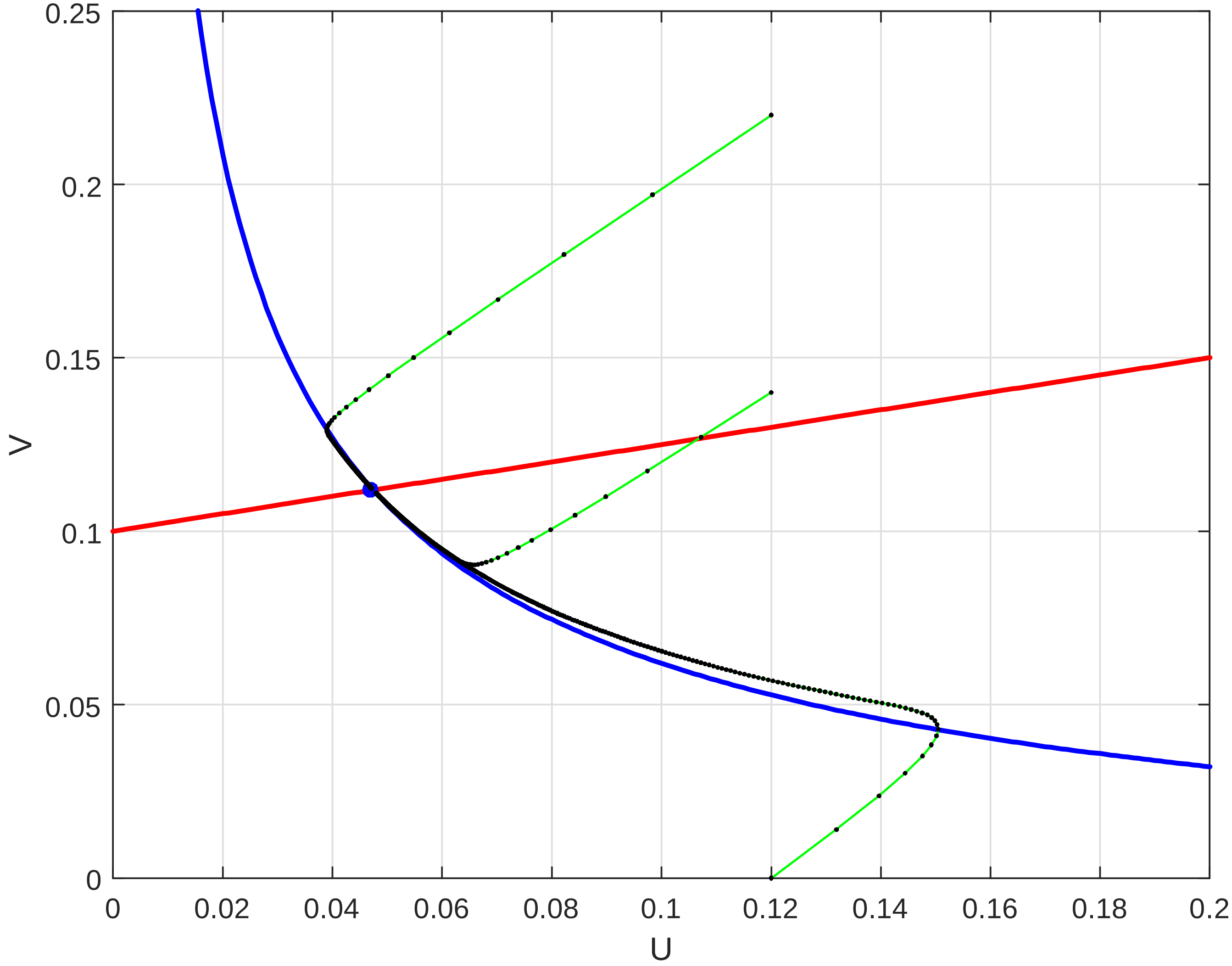
- two state variables, no forward-looking prices to consider
  – this implies a unique steady state, and it is stable
  – vacancies are a stock!

the phase diagram for projects and matches



- trajectories of 25 years, with black dots at quarterly intervals

unemployment and vacancies



- trajectories of 25 years, with black dots at quarterly intervals