Unique Implementation of Permanent Primary Deficits in an Overlapping Generations Economy

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background and our result

- classic overlapping generations economies
  - balanced budget indeterminacy of the price level
  - if autarky is Pareto dominated
- Bassetto and Cui [2018]
  - permanent primary deficits lead to price level indeterminacy
- Cochrane [2005, 2023]
  - we follow his "Money as Stock" interpretation of the FTPL
- here, for the classic OLG economy, we show that
  - with the right continuous Markov strategies,
  - the government can run a permanent primary deficit,
  - and the price level is determinate,
  - even though $r - g < 0$ and there is no risk
  - everything hinges on fiscal policy off the equilibrium path

closely related work

- idiosyncratic capital accumulation risk and incomplete markets
  - infinitely lived agents
  - can give rise to $r - g < 0$ if consumers are sufficiently risk averse
  - makes permanent primary deficits possible

In such a setting:

- ▶ Brunnermeier, Merkel, and Sannikov [2023] (BMS)
  - price level determinacy using grim trigger strategies
  - even when there is a private sector bubble asset
- ▶ Amol and Luttmer [2024]
  - price level determinacy using continuous Markov strategies
  - but *not* if there is a private sector bubble asset

the economy

preferences, technology, and government

- two-period lived overlapping generations
- endowments of the young and old

$$Y_y > 0, \quad Y_o > 0$$

- utility function $U : \mathbb{R}^2_+ \to \mathbb{R}$
  - strictly increasing, smooth, with convex indifference curves
- the government imposes taxes on endowments
  - at rates $\theta_y \in (0, 1)$ and $\theta_o \in (0, 1)$ for young and old consumers
  - so government tax revenues are

$$T = \theta_y Y_y + \theta_o Y_o > 0$$

- the government targets expenditures equal to

$$\Gamma \in [T, Y_y + Y_o)$$

in units of consumption

the trading and settlement subperiods

- dates $t \in \{0, 1, \ldots\}$
- the price of *cum-dividend* government stock is $s_t \geq 0$,
  - in units of consumption
- the government dividend is $d_t \geq 0$ per share, in units of consumption
  - this dividend could be a function of s_t (more later)
- during the date-t *trading subperiod*
  - a market-clearing price s_t is determined,
  - and all associated trades are agreed upon,
  - among consumers,
  - and between the government and consumers
- during the date-t subsequent *settlement subperiod*
  - the government pays the dividend
  - and all agreed upon trades are settled

a more familiar numeraire

- suppose $s_t > 0$ in units of date-t consumption
  - express prices in units of cum-dividend stock
  - then the price level (price of consumption) is

$$P_t = \frac{1}{s_t}$$

- one unit of stock in the $t+1$ trading period
  - date-t cost in units of consumption

$$s_t - d_t$$

  - date-t cost in units of cum-dividend stock

$$Q_t = P_t(s_t - d_t) = 1 - \frac{d_t}{s_t} \leq 1$$

  - this is the price of a nominal discount bond

▶ but we do not want to assume $s_t > 0$

  - this should be a result

free disposal and no arbitrage

- dividends required to be non-negative
  - owning stock is never a liability
- no arbitrage requirements
  1. because stock at date $t+1$ can be abandoned at no cost
  $$s_t \geq d_t$$
  2. cannot buy something at date $t+1$ for nothing at date t
  $$s_{t+1} > 0 \Rightarrow s_t > d_t$$
- if $s_t > 0$ this implies
$$Q_t = 1 - \frac{d_t}{s_t} \in [0, 1]$$
- if $s_{t+1} > 0$ this implies
$$Q_t \in (0, 1]$$

a bit of corporate finance

- a *regular dividend*
  - is a transfer of consumption goods to existing shareholders
- a *stock dividend*
  - is a transfer of new shares to existing shareholders
- Brealey, Myers, and Allen (2008) write
  - "For example, if the firm pays a stock dividend of 5%, it sends each shareholder 5 extra shares for every 100 shares currently owned"
  - page 445 of *Principles of Corporate Finance, Ninth Edition*
- we will do standard general equilibrium
  - there is no means of payment
  - can decompose a stock dividend into
    - a transfer of consumption goods,
    - together with a sale of new stock

nominal interest rates

- a *stock dividend* is a dividend determined by a number $\delta_t \in [0,\infty)$ via

$$d_t = \frac{\delta_t}{1+\delta_t} \times s_t$$

  - one unit of stock receives $\delta_t \in [0,\infty)$ units of ex-dividend stock,
  - during the settlement subperiod
  - the implied nominal interest rate is then $i_t = \delta_t$, since

$$\frac{1}{1+i_t} = Q_t = 1 - \frac{d_t}{s_t} = 1 - \frac{\delta_t}{1+\delta_t} = \frac{1}{1+\delta_t}$$

- using δ_t as the policy instrument *pegs* the nominal interest rate
- more generally,

$$i_t = \frac{1-Q_t}{Q_t} = \frac{1-\left(1-\frac{d_t}{s_t}\right)}{1-\frac{d_t}{s_t}} = \frac{d_t}{s_t - d_t}$$

  - the government could use d_t as its policy instrument, and then
  - the equilibrium value of s_t determines the nominal interest rate

the decision problem of the date-t young

- given $s_t \geq d_t$ and $s_{t+1} \geq 0$, date-t consumers solve

$$\max_{\left(C_y, C_o\right) \in \mathbb{R}_+^2} U(C_y, C_o)$$

subject to

$$\begin{aligned} 0 &\leq D, \\ C_y + (s_t - d_t)\, D &\leq (1-\theta_y) Y_y, \\ C_o &\leq (1-\theta_o) Y_o + s_{t+1} D. \end{aligned}$$

- this is well defined if $\{s_t, d_t, s_{t+1}\}$ is arbitrage-free
- in case there are arbitrage opportunities,
  - if $s_t - d_t < 0$ and $s_{t+1} \geq 0$ then choose $C_{y,t} = 10^{100} \times (Y_y + Y_o)$
  - if $s_t - d_t = 0$ and $s_{t+1} > 0$ then choose $C_{o,t+1} = 10^{100} \times (Y_y + Y_o)$

the decision problem of the date-t young

- if $s_t > 0$ and $s_{t+1} > 0$, then the problem is the familiar

$$\max_{(C_y, C_o) \in \mathbb{R}^2_+} U(C_y, C_o)$$

subject to

$$\begin{aligned} 0 &\leq D, \\ P_t C_y + Q_t D &\leq (1 - \theta_y) P_t Y_y, \\ P_{t+1} C_o &\leq (1 - \theta_o) P_{t+1} Y_o + D. \end{aligned}$$

- this is as if the Federal Reserve is paying the same rate of interest $i_t = (1 - Q_t)/Q_t$ as the US Treasury

tracing out the offer curve

- as long as $\{s_t, d_t, s_{t+1}\}$ is arbitrage-free
- the consumption choices $C_{o,t}$ and $C_{y,t+1}$ imply $D = D_t$, where

$$(s_t - d_t)D_t \;=\; (1-\theta_y)Y_y - C_{y,t},$$

$$s_{t+1}D_t \;=\; C_{o,t+1} - (1-\theta_o)Y_o.$$

- by varying $s_t > d_t$ and $s_{t+1} > 0$ one can trace out an offer curve
  - assume it is an increasing function $f(\cdot)$,

$$s_{t+1}D_t = f\left((s_t - d_t)D_t\right) \in [0, \infty)$$

  - only defined for

$$(s_t - d_t)D_t \in [0, (1-\theta_y)Y_y]$$

  - we will keep θ_y and θ_o fixed throughout perfect foresight equilibrium trajectories

perfect foresight equilibrium trajectories

- government trajectories for purchases and dividends

$$\{G_t\}_{t=0}^{\infty} \subset [0, Y_y + Y_o), \quad \{d_t\}_{t=0}^{\infty} \subset [0, \infty)$$

- trajectories for the prices $\{s_t\}_{t=0}^{\infty} \subset [0, \infty)$ and the supplies of government stock $\{D_t\}_{t=0}^{\infty} \subset (0, \infty)$ that satisfy

$$(s_t - d_t)D_t + T = s_t D_{t-1} + G_t \tag{1}$$

  – for all $t \in \{0, 1, \ldots\}$, given some initial $D_{-1} > 0$

- so that $\{s_t, d_t\}_{t=0}^{\infty}$ is arbitrage-free and

$$s_{t+1} D_t = f\left((s_t - d_t)D_t\right) \tag{2}$$

  for all $t \in \{0, 1, \ldots\}$

- this is a standard definition: Radner equilibrium with a large agent

▶ what is not to like?

  – this says that $G_t - T > 0$ causes $s_t - d_t > 0$

implied savings trajectories

- given an equilibrium trajectory $\{G_t, d_t, s_t, D_t\}_{t=0}^{\infty}$, define

$$x_t = (s_t - d_t)D_t$$

  for all $t \in \{0, 1, \ldots\}$

  – this is how much young consumers save at date t

  – the absence of arbitrage opportunities implies $x_t \geq 0$

- recall that for all $t \in \{0, 1, \ldots\}$

$$\begin{aligned} (s_t - d_t)D_t + T &= s_t D_{t-1} + G_t && (1) \\ s_{t+1} D_t &= f\left((s_t - d_t)D_t\right) && (2) \end{aligned}$$

▶ hence $x_0 + T = s_0 D_{-1} + G_0$, and

$$x_{t+1} + T = f\left(x_t\right) + G_{t+1} \qquad \text{(recursion)}$$

for all $t \in \{0, 1, \ldots\}$

  – this recursion only depends on $\{G_t\}_{t=0}^{\infty}$, and not on $\{d_t\}_{t=0}^{\infty}$

▶ the minimal requirement is that $\{x_t\}_{t=0}^{\infty} \subset [0, \infty)$

  – also need to construct arbitrage-free $\{s_t, d_t\}_{t=0}^{\infty}$ and $\{D_t\}_{t=0}^{\infty} \subset (0, \infty)$

steady states and other equilibrium trajectories

- suppose the government targets $G_t = \Gamma > T$ for all $t \in \{0, 1, \ldots\}$
- the steady state condition for savings is

$$x + T = f(x) + \Gamma$$

- suppose $\mathrm{D}f(0) < 1$ and $\Gamma > T$ is not too large
  - then there are two steady states $\left\{\underline{\xi}, \overline{\xi}\right\} \subset (0, (1 - \theta_y)Y_y)$,

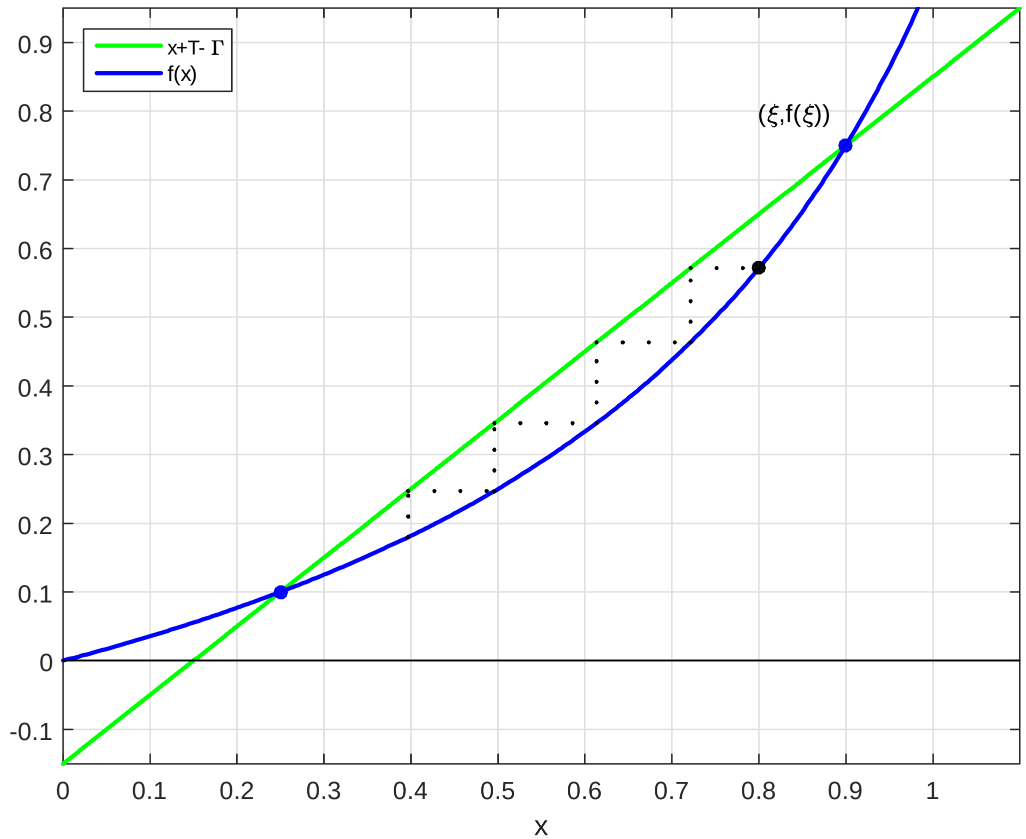
$$x + T = f(x) + \Gamma \Leftrightarrow x \in \left\{\underline{\xi}, \overline{\xi}\right\},$$

  - and a continuum of equilibrium trajectories $\{x_t\}_{t=0}^{\infty} \subset \left(\underline{\xi}, \overline{\xi}\right)$
  - see diagram on the next page
- since $\overline{\xi}$ is furthest up the offer curve
  - the steady state $x_t = \overline{\xi}$ Pareto dominates $\{x_t\}_{t=0}^{\infty} \subset [\underline{\xi}, \overline{\xi})$
  - from hereon write

$$\xi = \overline{\xi}$$

  - and focus on implementing $x_t = \xi$

two steady states, a continuum of equilibrium trajectories



- can set $d_t = 0$ for all $t \in \{0, 1, \ldots\}$ and back out

$$\frac{D_t}{D_{t-1}} = \frac{x_t}{x_t + T - \Gamma} > 1, \quad s_t = \frac{x_t + T - \Gamma}{D_{t-1}} > 0$$

  – arbitrage-free by construction

perfect foresight equilibrium

admissible government policies

- the government selects $\{G_t(\cdot), d_t(\cdot), D_t(\cdot)\}_{t=0}^{\infty}$
  - the argument is $s \in [0, \infty)$
  - history is represented by t
- given $D_{t-1} > 0$, government policy must satisfy

$$G_t(s) \geq 0, \quad d_t(s) \geq 0, \quad D_t(s) \geq 0,$$

  together with

$$(s - d_t(s))\, D_t(s) + T = sD_{t-1} + G_t(s),$$

  identically in $s \in [0, \infty)$
- we restrict attention to policies that satisfy $D_t(s) \geq D_{t-1}$
  - avoid buying back government stock, so that $D_t > 0$ for all t
- note that $s = 0$ gives

$$T = G_t(0) + d_t(0)D_t(0)$$

  - could take $G_t(0) \in [0, T)$, $D_t(0) = D_{t-1}$, and $d_t(0) > 0$
  - so $d_t(0) > 0$ forces $s > 0$, or else there is an arbitrage

definition of equilibrium

- fix some $D_{-1} > 0$
- the sequence $\{G_t, d_t, s_t, D_t\}_{t=0}^{\infty}$ is
  - a *perfect foresight equilibrium trajectory* if
  - it admits no short-term arbitrage opportunities and satisfies

$$(s_t - d_t)D_t + T = s_t D_{t-1} + G_t, \tag{1}$$
$$s_{t+1}D_t = f\left((s_t - d_t)D_t\right), \tag{2}$$

  for all $t \in \{0, 1, \ldots\}$

- given an admissible policy $\{G_t(\cdot), d_t(\cdot), D_t(\cdot)\}_{t=0}^{\infty}$,
  - a *perfect foresight equilibrium* is
  - a perfect foresight equilibrium trajectory that satisfies

$$d_t = d_t(s_t), \quad G_t = G_t(s_t), \quad D_t = D_t(s_t), \tag{3}$$

  for all $t \in \{0, 1, \ldots\}$

constructing equilibria for monotone policies

- for an admissible policy, define

$$x_t(s) = (s - d_t(s))D_t(s)$$

  - assume that this $x_t : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, and then define

$$\Gamma_t(u) = G_t\left(x_t^{-1}(u)\right)$$

- the equilibrium conditions

$$(s_t - d_t(s_t))D_t(s_t) + T = s_t D_{t-1} + G_t(s_t) \quad (1)$$

$$s_{t+1} D_t(s_t) = f\left((s_t - d_t(s_t))D_t(s_t)\right) \quad (2)$$

for $t \in \{0, 1, \ldots\}$ imply

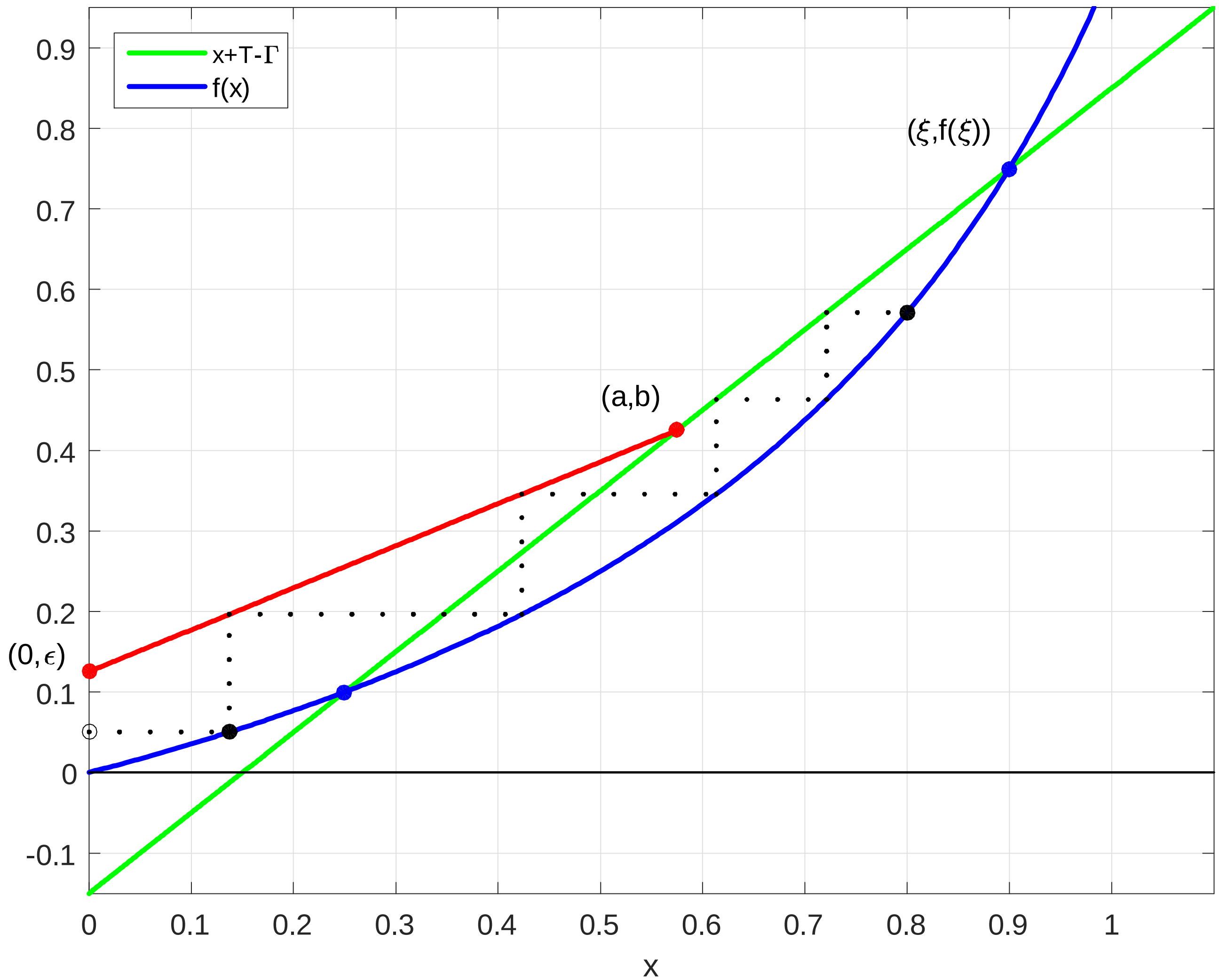
$$x_{t+1} + T = f\left(x_t\right) + \Gamma_{t+1}(x_{t+1}) \quad \text{(recursion)}$$

for $t \in \{0, 1, \ldots\}$

- a perfect foresight equilibrium *implies* a trajectory $\{x_t\}_{t=0}^{\infty} \subset [0, \infty)$
  - can establish *uniqueness* by showing uniqueness of $\{x_t\}_{t=0}^{\infty}$

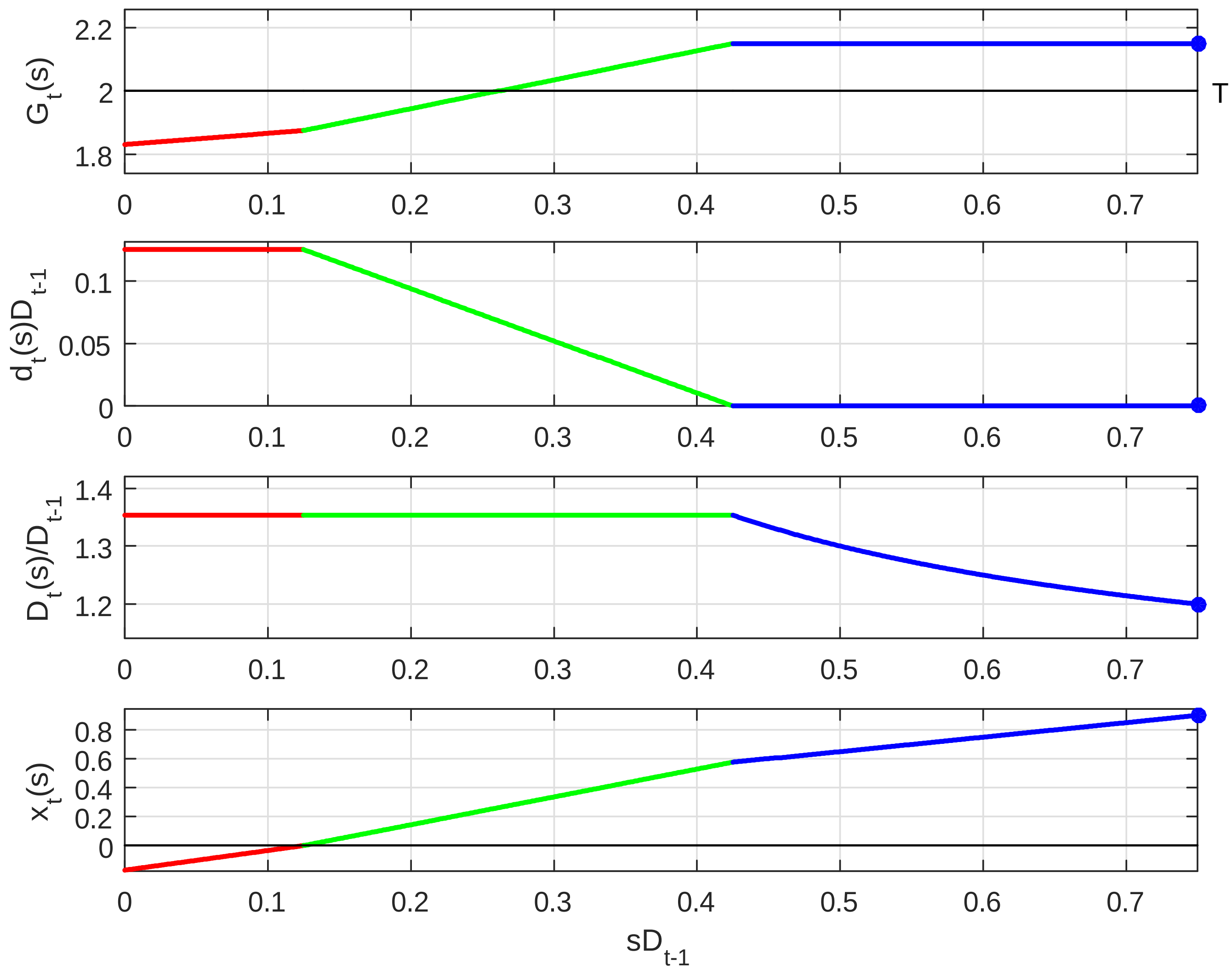
example 1

the unique equilibrium $x_0 = \xi$



- the upper envelope of the green and red lines is $x + T - \Gamma_t(x)$
- cannot start with $x_0 = 0$ because $0 = f(x_0) = x_1 + T - \Gamma_1(x_1) > 0$

the admissible policy, and $x_t(\cdot)$, for $\Gamma/T = 1.075$



- the blue dots represent the targeted steady state
  - zero nominal interest rate while $\sqrt[25]{D_t/D_{t-1}} \approx \sqrt[25]{1.2} \approx 1.007$

three zones for policy off the equilibrium path

- business as usual

$$s_t D_{t-1} \in [b, \infty)$$

  - simply sell some more stock if the price is low
  - no dividends

- the danger zone

$$s_t D_{t-1} \in [\varepsilon, b]$$

  - cap the sales of new stock
  - reduce spending and start to pay some dividends
  - run a strict primary surplus at the lower bound of this zone

- the arbitrage zone

$$s_t D_{t-1} \in [0, \varepsilon]$$

  - aggregate dividends at the $\varepsilon > 0$ cap
  - the ex-dividend price is now negative
  - comes at a cost to the government, continue to reduce spending
  - but this offers consumers an arbitrage opportunity

formulas for the underlying policy

- if $sD_{t-1} \in [0, \varepsilon]$ then

$$G_t(s) = \frac{\Gamma - T}{b} \times (sD_{t-1} - \varepsilon) + T - \varepsilon, \quad d_t(s)D_{t-1} = \varepsilon, \quad \frac{D_t(s)}{D_{t-1}} = 1 + \frac{\Gamma - T}{b}$$

- if $sD_{t-1} \in [\varepsilon, b]$ then

$$\begin{aligned} \begin{bmatrix} G_t(s) \\ d_t(s)D_{t-1} \end{bmatrix} &= \begin{bmatrix} T - \varepsilon \\ \varepsilon \end{bmatrix} \left(1 - \frac{sD_{t-1} - \varepsilon}{b - \varepsilon} \right) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \frac{sD_{t-1} - \varepsilon}{b - \varepsilon} \\ \frac{D_t(s)}{D_{t-1}} &= 1 + \frac{\Gamma - T}{b} \end{aligned}$$

- if $sD_{t-1} \in [b, \infty)$ then

$$G_t(s) = \Gamma, \quad d_t(s) = 0, \quad \frac{D_t(s)}{D_{t-1}} = 1 + \frac{\Gamma - T}{sD_{t-1}}.$$

▶ the implied recursion is

$$\max\left\{ \varepsilon + (b - \varepsilon) \times \frac{x_{t+1}}{a}, x_{t+1} + T - \Gamma \right\} = f(x_t)$$

where $a = b + \Gamma - T$

the implied nominal interest rate policy

- if $s - d_t(s) > 0$, then the implicit nominal interest rate policy is

$$\frac{1}{1+i_t(s)} = 1 - \frac{d_t(s)}{s}, \text{ and thus } \quad i_t(s) = \frac{d_t(s)}{s - d_t(s)}$$

  – at the unique equilibrium price $s = s_t$, this yields $i_t(s) = 0$

- but, in the danger zone, when $sD_{t-1} \in [\varepsilon, b]$,

$$d_t(s)D_{t-1} = \left(1 - \frac{sD_{t-1} - \varepsilon}{b - \varepsilon}\right)\varepsilon$$

  and hence

$$i_t(s) = \frac{\varepsilon}{1+b-\varepsilon}\frac{1 - \frac{sD_{t-1}-\varepsilon}{b-\varepsilon}}{\frac{sD_{t-1}-\varepsilon}{b-\varepsilon}} \to \infty \text{ as } sD_{t-1} \downarrow \varepsilon$$

▶ but the government is *not* issuing new stock at a really high rate

  – the growth factor $D_t(s)/D_{t-1} > 1$ is constant for $sD_{t-1} \in [0, \varepsilon]$

  – instead, the dividend $\varepsilon > 0$ is financed by a surplus,

$$sD_{t-1} \leq \varepsilon \Rightarrow T - G_t(s) \geq \varepsilon > 0$$

example 2

a purely nominal policy (as in BMS)

- take $m \in [0, \infty)$ and $i \in [0, \infty)$, and consider

$$d_t(s) = \frac{i}{1+i} \times s, \quad \frac{D_t(s) - D_{t-1}}{D_{t-1}} = m$$

  for all $t \in \{0, 1, \ldots\}$

- admissibility requires

$$s\left(D_t(s) - D_{t-1}\right) + T = d_t(s) D_t(s) + G_t(s)$$

  – this implies

$$G_t(s) = T + \left(\frac{m}{1+m} - \frac{i}{1+i}\right)(1+m) s D_{t-1}$$

  – taking $m > i$ finances a primary deficit if $s > 0$

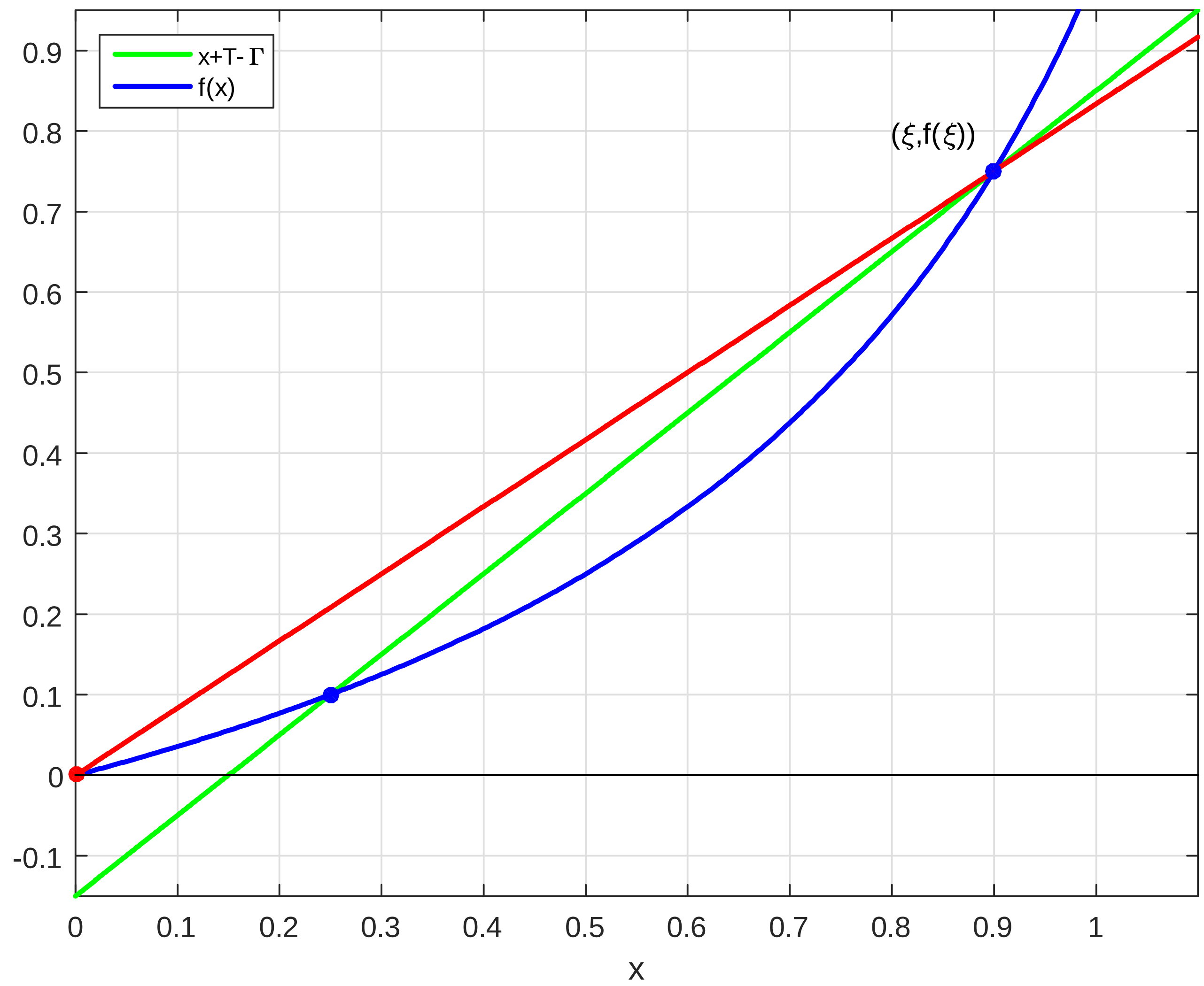
- in terms of savings of the young,

$$x_t(s) = \frac{1+m}{1+i} \times s D_{t-1}, \quad \Gamma_t(x) = T + \left(\frac{m}{1+m} - \frac{i}{1+i}\right)(1+i) x,$$

  and then the recursion becomes

$$\frac{1+i}{1+m} \times x_{t+1} = f\left(x_t\right)$$

a purely nominal policy



- the low-savings steady state is replaced by a *balanced budget trap*
- any $x_0 \in (0, \xi)$ means that x_t converges to that trap

removing the balanced budget trap

- take some small $\varepsilon > 0$ and $\alpha > 0$

$$d_t(s) \;=\; \frac{i}{1+i} \times s + \alpha \max\left\{0, \frac{\varepsilon - sD_{t-1}}{(1+m)D_{t-1}}\right\}, \quad \frac{D_t(s) - D_{t-1}}{D_{t-1}} = m$$

- the admissibility requirement

$$s(D_t(s) - D_{t-1}) + T = d_t(s)D_t(s) + G_t(s)$$

forces

$$G_t(s) = T + \left(\frac{m}{1+m} - \frac{i}{1+i}\right)(1+m)sD_{t-1} - \alpha \max\left\{0, \varepsilon - sD_{t-1}\right\}$$

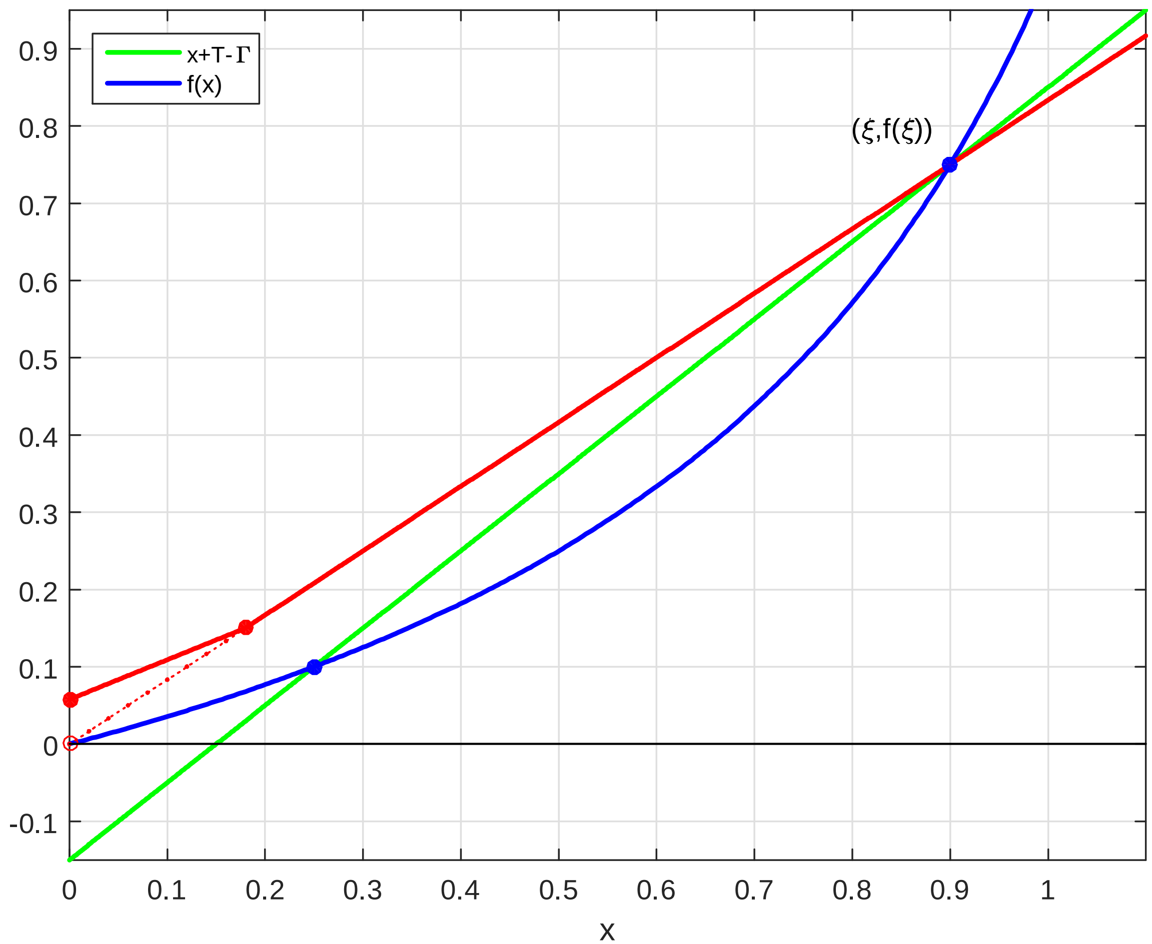
- again, the savings of the young are monotone in s,

$$x_t(s) = \frac{1+m}{1+i} \times sD_{t-1} - \alpha \max\left\{0, \varepsilon - sD_{t-1}\right\},$$

and thence

$$\Gamma_t(x) = T + \left(\frac{m}{1+m} - \frac{i}{1+i}\right)(1+i)x - \alpha \max\left\{0, \frac{\varepsilon - \frac{1+i}{1+m} \times x}{\frac{1+m}{1+i} + \alpha}\right\}$$

uniqueness for almost purely nominal policies



- the recursion is

$$x_{t+1} = \frac{1+m}{1+i} \times y_{t+1}, \quad \max\left\{ y_{t+1}, \frac{\frac{1+m}{1+i} \times y_{t+1} + \alpha\varepsilon}{\frac{1+m}{1+i} + \alpha} \right\} = f(x_t)$$

conclusion

- the government can uniquely implement permanent primary deficits
  - if the offer curve is not too steep at the origin
  - using Markov policies that are continuous in current prices
  - key: surplus when the price of government stock is too low
- so the government can avoid the indeterminacy pointed out by Bassetto and Cui [2018]
  - the fiscal theory of the price level survives $r - g < 0$
- but, things are more complicated when there are also private-sector bubble assets
  - see Amol and Luttmer [2024]
  - Minneapolis Federal Reserve Bank working paper 807
  - unique implementation requires a sharply discontinuous policy
  - or a large enough tax on private-sector bubble assets
  - else there is a continuum of equilibria and a balanced budget trap

References

[1] Amol, A. and E.G.J. Luttmer, "Unique Implementation of Permanent Primary Deficits?", *Federal Reserve Bank of Minneapolis*, working paper 807 (2024).

[2] Bassetto, M. and W. Cui, "The Fiscal Theory of the Price Level in a World of Low Interest Rates, *Journal of Economic Dynamics and Control*, vol. 89 (2018), 5-22.

[3] Brunnermeier, M.K., S. Merkel, and Y. Sannikov, "The Fiscal Theory of the Price Level with a Bubble," working paper (2023).

[4] Cochrane, J.H., "Money as Stock," *Journal of Monetary Economics*, vol. 52, no. 3 (2005), 501-528.

[5] Cochrane, J.H., "The Fiscal Theory of the Price Level," *Princeton University Press* (2023).