Economies with Observable Types *

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Abstract

We study competitive equilibria of economies of asymmetric information with observable types. Individuals trade in lotteries and the incentive compatibility constraints are imposed either on the individual consumption set, as in the seminal work of Prescott and Townsend ([9], [10]), or on the production set of the intermediaries, as in Jerez ([3], [4]).

The first approach makes economies of asymmetric information with observable types isomorphic to competitive economies of individual risk (as in Malinvaud). Thus, competitive equilibria supported by type dependent and fair prices exist, even in the presence of aggregate risk, and the constrained version of the two Fundamental Theorems of Welfare Economics hold true.

The second approach substantially reduces the set of equilibrium allocations. The equilibrium set may be empty even if constrained efficient allocations exist and the Second Welfare Theorem may fail.
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1 Introduction

Almost twenty five years after the publication of the seminal contributions of Prescott and Townsend ([9], [10]) to the analysis of competitive economies with asymmetric information some confusion seems to persist on the fundamental features of these economies, such as existence and optimality of the competitive equilibria. In fact, as we are going to argue below, even the fundamental definitions are not clear. Here we hope to clarify two points.

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First, in spite of the large variety of models for such economies, one can give a common treatment of a large class of them. We establish existence and constrained optimality for economies with observable types, that is economies of asymmetric information where types are observable, but actions or personal states or both are private information of the individuals. The method we use reduces the analysis to that of economies with individual risk as in Malinvaud ([7], [8]). Therefore, one does not need to look, if he is interested in existence and optimality, to special subclasses (see for example [1], [5]). Economies of moral hazard and private information are particular cases of this general framework.

Second, the way in which the incentive compatibility (IC) constraints are defined has fundamental implications not only on the nature of the prices (linear or non linear) but on the existence and optimality propertied. Imposing the IC constraints on the firm instead of on the consumer may make empty the set of equilibria.

To see the details with clarity, we need some background.

Reduction to economies of individual risk

In the economies we consider, individual preferences are defined over contracts, which are pairs of actions and state-contingent trades of physical commodities. Individuals can manipulate contracts using their private information, and can transform a contract into a new one by taking either actions different than the ones prescribed or by mis-reporting realizations of personal states or both. A contract is *incentive compatible* if individuals find optimal to not manipulate it.

The set of incentive compatible contracts needs not to be convex, so the standard analysis of competitive equilibria does not apply directly. Prescott and Townsend devise a framework to extend the general equilibrium analysis of competition to such economies. Their modeling transforms economies of asymmetric information in convex economies of individual risk. One can derive immediately and with no further assumption that equilibria exist under very general specification of the economy. Also the equilibrium prices are fair and personalized (type dependent). Finally, the two Welfare theorems hold. Let us look how this is accomplished.

Many elements characterize the Prescott and Townsend formulation. First, they introduce lotteries over contracts and define preferences over lotteries linearly. Second, trade is restricted to incentive compatible lotteries. There are three important implications of these two steps. First, the problem of asymmetric information is removed: Only incentive compatible lotteries are traded and, hence, individuals do not have incentive to manipulate. Second, since preferences over lotteries are linear, the set of incentive compatible lotteries is convex and, hence, non-convexities are eliminated. Alternatively, trade could take place on incentive compatible contracts and the large number of individuals used as a convexifying device. However, and this is the third implication, the introduction of lotteries enhances the welfare of the economy. Indeed, the set of incentive compatible lotteries contains the set of lotteries over incentive compatible contracts thereby potentially increasing the individual utilities of the constrained efficient allocations.

Hence, the economies are now economies of individual risk *à la* Malinvaud. However, the price domain is the set of linear functionals over lotteries and, hence, prices can be non linear in commodities. Many allocations (including inefficient ones) can be decentralized as competitive equilibrium. On the other hand if prices are restricted to be fair (linear in commodities) the model delivers all the results of the General Equilibrium analysis. This is an obvious consequence of the isomorphisms of these economies with the classical individual risk economies. The restriction to fair prices is made endogenous by the introduction of the Prescott and Townsend intermediary. The latter is a profit maximizing firm whose production set coincides with the set of feasible (signed) measures. The constant return to scale nature of the firm technology has two satisfying implications. First, the fact that the firm is unique is without loss of generality. Second, profits at equilibrium are zero. The latter property forces prices to be fair and makes irrelevant the distribution of property rights across individuals.

The crucial step of Prescott and Townsend is the introduction of the IC constraints on individual choices. Jerez ([4]) observes that this restriction is conceptually problematic. ([4]) argues "the natural interpretation is to view the incentive-compatibility constraints as restrictions on the set of contracts that firms can offer to consumers, rather than as consumers self-imposing these constraints." This the route taken by both the mechanism design literature (where the firm is identified with the principal) as well as the partial equilibrium literature of asymmetric information economies. Jerez ([3], [4]) poses an important question. Can we model competitive economies of asymmetric information removing the IC constraints from individual choices and placing it on the firm only? What are the properties of the equilibrium set of this economy? Jerez claims that this way of modeling the economy delivers the same allocation equilibrium set of Prescott and Townsend , although naturally, supporting prices may now be non linear in commodities, although linear in lotteries. This is not quite correct: the equilibrium set may now be empty.

The definition of competitive equilibria

The key question to address first is: what is the correct definition of equilibria in competitive economies with asymmetric information? Consider an economy with simple moral hazard, where the firm can only sell contract which are incentive compatible. A consumer is free to choose from the set of all the budget feasible lotteries, included the non incentive compatible ones. After he has made his choice, the outcome of the lottery over actions is determined and he is told the recommended action. The consumer can manipulate the contracts in the support of the lottery by choosing actions that do not conform to the recommendations. The question is at which stage will the consumer contemplate this possibility. He can do that ex-post, that is after having bought the lottery. Then, since the contracts supplied by the firm are incentive compatible, the consumer will find it convenient to take the recommended actions. This is of course the reason for imposing the restriction to incentive compatible lotteries in the first place: to insure that the consumer does not manipulate the lottery he purchased.

With ex-post manipulations, the consumer ranks (and hence chooses) lotteries without contemplating manipulations, but after, once the lottery has been bought, he deviates from the recommended actions only if the lottery is not incentive compatible. This is the formulation adopted by Jerez. It is inconsistent: if for some moral reason, or for strict adherence to the rules of Walrasian behavior he does not contemplate deviations at the moment of choosing the lottery, he should not do so at the moment of choosing the action, and then the asymmetric information problem would vanish entirely. We therefore assume that at the moment of choosing the lottery the consumer anticipates the possibility of deviating later. By behaving this way, not only the consumers takes all profitable ex-post manipulations, but since he considers the manipulations ex-ante, that is before buying the lotteries, he changes the ranking of the (budget feasible) lotteries. The utility attached to a lottery is the expected utility generated by the optimal manipulations of the contracts in its support, that is by the contracts obtained by changing the recommended actions with their best alternatives. This captures the asymmetric information problem that these economies face. Consumers use strategically their private information changing ex-ante and ex-post the nature of the budget feasible contracts: they select the budget feasible lotteries that maximize the expected utility generated by their optimal manipulation. If they behave this way, and the IC constraint is imposed only on the firm, the equilibrium set may be empty.

The root of the problem

The equilibrium set can be empty for two different reasons. The first has to do with the individual optimal choices, and the second with the profit maximization of the firm. Let us start with the first. What allocations can be supported by prices as optimal choices of the individuals? If Jerez were right, the answer should be "all the constrained efficient allocations." Pick a constrained efficient joint lottery that satisfies two properties. For some type, first the IC are binding and second the lottery gives positive probability to some contract that is not incentive compatible. Then, there exists a manipulation that when applied delivers with the same lottery the same utility level of truthful behavior.

The latter immediately implies that the supporting price has now to solve two, possibly incompatible tasks: It has to make the prescribed lottery optimal among the budget feasible lotteries when the individual behaves truthfully and when he optimally manipulates. However, the manipulation transforms the utility function over the contracts into the utility functions over the manipulated contracts, and the latter can have a very different shape. There is no reason why a unique price should support the same lottery with two very different utility functions.

It can do so if the lottery puts weights only on IC contracts. The reason is simple. Suppose indeed that the support of the prescribed lottery contains only IC contracts. Then, since the IC constraints are binding, the utility function and the manipulated utility function must be identical over the support of the prescribed lottery. Outside they are not, but the price values there can be chosen high enough to make those contracts undesirable for any possible utility function obtained through manipulations. However, if the support of the prescribed lottery contains non-IC contracts, the utility function and the manipulated utility have different shapes in the support of the prescribed lottery and a supporting price may fail to exist.

It is to easy to find necessary and sufficient conditions for price supportability and, then, to find, constrained efficient allocations that violate them. The next step is to build an economy with one type and one constrained efficient allocation that violates the necessary and sufficient conditions. This economy has obviously an empty set of equilibria.

As said, even economies with price supportable constrained efficient allocations may fail to have equilibria. Consider a constrained efficient lottery. Look for a candidate price that makes it an optimal choice of the individuals and a profit maximizing choice of the firm. Why do individuals at that price do not put any weight on contracts outside the support of the prescribed lottery? Because those contracts are too expensive at the candidate price. Hence, the shape of the utility function puts lower bounds on the values of the candidate supporting price calculated at the contracts outside of the lottery support. However, the profit maximizing nature of the prescribed lottery puts upper bounds on those values, the contracts are too cheap to be supplied. These two restrictions, one coming from the individuals and the other from the firm, can make the set of candidate equilibrium prices empty. We show an economy of private information with one type and one constrained efficient allocation where these two requirements collide. The support of the constrained efficient allocation contains only IC contracts. Thus, the constrained efficient allocation is price supportable, but no price that makes the constrained efficient allocation an optimal choice of the individuals makes it a profit maximization choice of the firm. This economy too has an empty set of equilibria.

Outline

In section 2 we define the basic structure of the economies we study. In section 3, we isolate a subset of economies (those with observable types) for which a unified analysis can be provided. In section 4 we study Prescott and Townsend economies. Competitive equilibria exists and the two Fundamental Theorems of Welfare Economics hold true even in the presence of aggregate uncertainty. In section 5 we define economies in which the IC constraints are imposed on the firm. We first identify necessary and sufficient conditions for the existence of supporting prices, and then, using this characterization, we show that these economies may have an empty equilibrium set of equilibria. Finally, section 6 concludes the paper.

2 Economies with Asymmetric Information

To keep the technical side of the analysis simple we study finite economies: the states, the actions, the types and the trades are a finite set.

2.1 The economy

Individuals in the economy belong to one of a finite *set of types*, $I \equiv \{1, \ldots, n\}$. For each type i there is a large population of size λ^i, with $\lambda^i > 0$ and $\sum_{i \in I} \lambda^i = 1$. Each individual chooses an *action*, a, out of a finite set A. For example, in models of principal agent the action is the effort of the agent.

Individuals face aggregate and personal uncertainty. Ω is a finite set of *states of nature* that affect every agent in the economy. Each state ω occurs with a fixed probability $\rho(\omega)$. For example in insurance models a state may be a flood, or an earthquake. A *personal state* s out of a finite set S is realized, one for each individual. In insurance models an s may be an accident.

The probability of such realization depends on type, action and state of nature, as we now describe. For every finite set M, $\Delta(M)$ denotes the set of probability vectors on M. For each type, action and aggregate state, there exists a probability vector over the set of personal states: that is, a $q^i(\cdot; a, \omega) \in \Delta(S)$ is given for every $i \in I$ and every $(a, \omega) \in A \times \Omega$. Individuals exchange goods according to a finite set of *individual net trades*, X, which is independent of types, states and actions; $0 \in X$, so no trade is always an option. X is a subset of the Euclidean space $\mathbb{R}^L$, where $L \geq 1$ is the number of physical commodities. The preferences of type i are represented by a utility function

$$v^i : A \times S \times \Omega \times X \to \mathbb{R}, \text{ for every } i \in I.$$

2.2 Contracts

The set of *net trade policies* is the set Z of state contingent net trades. It is the finite set of maps $z : S \times \Omega \to X$. The set of *contracts* is the set C of pairs of action and net trade policy. A contract $c = (a, z)$ assigns an action a and stipulates the provision of a state contingent net trade z, which describes for every realization of the pair of states (s, ω) a net trade vector $z(s, \omega) \in X$. The set of contracts is finite, is type invariant and so it is its cardinality. The utility function v^i induces a utility function u^i over C that takes the expected utility form

$$u^i(c) = u^i(a, z) \equiv \sum_{\omega \in \Omega} \rho(\omega) \sum_{s \in S} q^i(s; a, \omega) v^i(a, s, \omega, z(s, \omega)).$$

Lotteries on deterministic contracts are also traded: a lottery τ is an element of $\Delta(C)$. A different description of a lottery is given by a pair (τ_1, τ_2) where $\tau_1 \in \Delta(A)$ and τ_2 is a vector $(\tau_2(\cdot; a)_{a \in A})$ of conditional probabilities on Z, one for each action. The two descriptions are equivalent: for every $\tau \in \Delta(C)$ there is a pair (τ_1, τ_2) , and vice versa. A *lottery profile* is a vector

$$\sigma = (\sigma^i)_{i \in I},$$

assigning the same lottery σ^i to each individual of type i. Individual utility functions are extended over the set of lotteries $\Delta(C)$ assuming that they are linear in lotteries. Let U^i denote the row vector of dimension $1 \times C$ with entries $u^i(a,z)$, for every $(a,z) \in C$. Using the convention that individual lotteries are column vectors, the utility of an individual of type i generated by a lottery σ^i is

$$U^i \sigma^i \equiv \sum_{(a,z)\in C} u^i(a,z)\sigma^i(a,z). \tag{1}$$

All the economies we consider are economies with individual risk (as in the classical analysis of Malinvaud (see ([7], [8]), we model the individual risk with the variable s). The models differ for the information publicly available. This information may be different for two variables: the action a and the personal state s. By making each of these variables private information of the individuals we obtain different types of economies.

2.3 Feasible lottery profiles

By the law of large numbers, a fraction $q^i(s; a, \omega)$ of type i individuals that have adopted the action a is at each aggregate state ω in personal state s. Thus a lottery profile $\sigma = (\sigma^i)_{i\in I}$ is *feasible* if for every commodity ℓ and aggregate state ω the sum of net trades is not positive:

$$\sum_{i\in I} \lambda^i \sum_{(a,z)\in C} \sigma^i(a,z) \sum_{s\in S} q^i(s; a,\omega) z_\ell(s,\omega) \leq 0. \tag{2}$$

To have a more compact notation, for $(a,z) \in C$, $\omega \in \Omega$ and $i \in I$, let

$$T^i((a,z);\omega) \equiv \sum_{s\in S} q^i(s;a,\omega) z(s,\omega)$$

be the column vector of dimension $L \times 1$ of type i aggregate net trade in state ω generated by (a,z). Let $T^i(\omega)$ to be the matrix of dimension $L \times \sharp C$ whose columns are the vectors $T^i((a,z);\omega)$, $(a,z) \in C$, and finally let T^i be the matrix of dimension $L\sharp\Omega \times \sharp C$ obtained by stacking together the matrices $T^i(\omega)$, $\omega \in \Omega$. Thus, the feasibility condition (2) can be rewritten as[1]:

$$\sum_{i\in I} \lambda^i T^i \sigma^i = \left(\sum_{i\in I} \lambda^i T^i(\omega)\sigma^i \right)_{\omega\in\Omega} \leq 0 \tag{3}$$

where $0 \in \mathbb{R}^{L\Omega}$.

[1]For two vectors x and y of identical dimension, we write $x \gg y$ to indicate that x is strictly larger than y in every component; $x > y$ to indicate that x is strictly larger than y for at least one component; and $x \geq y$ to indicate that x is greater or equal than y in every component.

2.4 Information and time

The type of an individual is publicly observed. The public information about either personal states or actions of individuals of type i is either completely revealing (the variable is observed) or completely non revealing. When a variable is publicly observed, individuals have to be truthful: the action chosen is the prescribed action if the action is observed and the reported personal state is the true state if the state is observed.

The complete time sequence of events is the following. Individuals trade, and get a lottery τ. The action a is chosen according to the lottery τ_1, and this outcome is communicated to the individual. The individual chooses the action b, possibly different from a. The public information available on the chosen action b is revealed. Then first the state of nature ω is determined, and the personal state is realized, according to $q(\cdot; a, \omega)$. The personal state s is communicated, and the public information on s is revealed. Then individuals report the personal state t, possibly different from s. Finally the aggregate state is communicated, and the transfer made. Obviously, we could envision different time sequence of events. For instance, the personal state could be revealed before the action is chosen. However, neither the positive nor the negative results of this paper depend on the adopted time sequence.

2.5 Prices in economies with individual risk

We begin with a simple exposition adapted to our problem, of the main idea of economies with individual risk. We use the simplest setup to keep the main idea in focus. Consider an economy where all information is public, so there are no incentive compatibility constraints. An efficient lottery profile is determined as solution of the problem (with $\alpha \in \Delta(I)$):

$$\max_{\{\sigma=(\sigma^i)_{i\in I}\}} \sum_{i\in I} \alpha^i U^i \sigma^i \tag{4}$$

subject to

$$\sum_{i\in I} \lambda^i T^i \sigma^i \leq 0.$$

Let π^i be the multiplier for the constraint that the vector σ is a probability vector, and $\nu \equiv (((\nu_\ell(\omega))_{\ell=1}^L)_{\omega\in\Omega}$ the multipliers for the feasibility constraint. For $\nu(\omega) \equiv (\nu_\ell(\omega))_{\ell=1}^L$, the first order conditions give

$$u^i(c) \leq \pi^i + \lambda^i \sum_{\omega} v(\omega) T^i(c, \omega), \tag{5}$$

with equality for every c for which the optimal $\hat{\sigma}$ has $\hat{\sigma}_c > 0$. If we compare the conditions (5) with the first order condition of the individual of type i in the competitive economy, we find that prices that support the efficient allocation have to be different for every type, that is have to be i dependent; and of the form

$$p^i = \lambda^i \nu T^i.$$

In conclusion, it is clear that in economies with individual risk prices have to be type dependent. A vector of *type dependent prices* is a vector $p \equiv (p^i)_{I \in I}$: each p^i is a vector in $\mathbb{R}^C$, where the entry $p^i(a, z)$ defines the value of the contract (a, z).

3 Three Types of Asymmetric Information

The economy so far described contains as a particular cases three different and well known types of economies: for each one there is a corresponding incentive compatibility constraint on the set of lottery profiles.

Individual risk

This is the basic model. In this economy all variables (action chosen by the individual and personal states) are observed, so there is no incentive compatibility constraint. For each type i the set of incentive compatible lotteries is

$$IC^i \equiv \Delta(C), \text{ for all } i \in I$$

and the set of incentive compatible strategy profiles is the product over types of the set of lotteries $IC = \times_{i \in I} IC^i$. It is well known (see ([7], [8], [2])) that an equilibrium with *type dependent prices*, where each type i faces a different price p^i, exists.

Moral hazard

In these economies the personal states are observed, while the action is private information. Also, recall that by the adopted time sequence of events, the prescribed action is communicated, and the action chosen before the net trade policy, the personal and the aggregate states are revealed is revealed: so the individual can make his action choice depend only on the prescribed action. Thus, the set of incentive compatible lotteries for each type i is the set of lotteries such that the individual of each type indeed prefers the action assigned by the lottery to any other action. A lottery σ is incentive compatible if

$$\sum_z \sigma(a, z)[u^i(a', z) - u^i(a, z)] \leq 0, \text{ for all } a' \in A \tag{6}$$

equivalently, by using the convention that $\sigma_2(z; a) = 0$ if $\sum_z \sigma(a, z) = 0$, the latter can be rewritten as

$$\sum_z \sigma_2(z; a)[u^i(a', z) - u^i(a, z)] \leq 0, \text{ for all } a' \in A \tag{7}$$

The two inequalities above are the standard representation of (ad interim) incentive constraints for Moral Hazard economies as in [1], [3], [4], and [9]. However, for the competitive analysis that follows, it will be convenient to rewrite the incentive constraints by making explicit how an individual type can *"manipulate"* any given contract. Let $\Phi(MH)$ be the finite set of all functions from A to A. Each ϕ corresponds to a *manipulation,* that is, to a deviation from the prescribed action. For $\phi \in \Phi(MH)$, define the new vector $U^i(\phi)$ by

$$U^i(\phi)(a,z) = u^i(\phi(a), z).$$

Then, the set of incentive compatible lotteries for type i can be written as:

$$IC^i \equiv \{\tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi(MH)\}, \tag{8}$$

and it is a non-empty, closed and convex subset of $\Delta(C)$. Our formulation of the set of incentive compatible lotteries is identical to the interim representation adopted in the literature, that is, to the set of lotteries satisfying inequalities (6) or (7). This is shown in the next lemma. Let

$$IC^i_* \equiv \{\sigma \in \Delta(C) : \sigma \text{ satisfies (6) or (7)}\}.$$

Lemma 1 $IC^i = IC^i_*$

The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is $IC \equiv \times_{i\in I} IC^i$.

Private information

In these economies actions are observed, while the realization of the personal state is not. Recall that individuals report the personal state after the observable action has been reported and the aggregate state of nature has been realized. Hence, a lottery σ is incentive compatible if for all $(a,\omega) \in A \times \Omega$

$$\sum_z \sigma(a,z) q^i(s;a,\omega)[v^i(a,s,\omega,z(s',\omega)) - v^i(a,s,\omega,z(s,\omega))] \leq 0, \text{ for all } s' \in S \tag{9}$$

or, equivalently (again by using the convention spelled out above),

$$\sum_z \sigma_2(z;a) q^i(s;a,\omega)[v^i(a,s,\omega,z(s',\omega)) - v^i(a,s,\omega,z(s,\omega))] \leq 0, \text{ for all } s' \in S \tag{10}$$

This is the formalization of the incentive constraints adopted in [6], [9] and [10] for economies of private information once it is taken into account that in all these works both actions and states of aggregate uncertainty are missing.

As before, we find convenient to rewrite the incentive constraints in an equivalent by making explicit how individuals can manipulate contracts. Since in our private economy, individuals can miss-report the personal state realization, given the information they have on the action and the aggregate state, for any function $\phi : A \times \Omega \times S \to S$ they can manipulate a contract (a,z) into a contract $(a, z(\phi(a,\cdot)))$, where $z(\phi(a,\cdot))(s,\omega) \equiv z(\phi(a,s,\omega),\omega)$. Let $\Phi(PI)$ be the finite set of all functions from $A \times \Omega \times S$ to S. For any $\phi \in \Phi(PI)$, let

$$U^i(\phi)(a,z) = u^i(a, z(\phi(a,\cdot)))$$

The set of incentive compatible lotteries for each type i can be written as:

$$IC^i \equiv \{\tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi(PI)\} \tag{11}$$

This set too is non-empty, closed and convex. Once again, our formulation of the set of incentive compatible lotteries is identical to the interim representation adopted in the literature, that is, to the set of lotteries satisfying inequalities (9) or (10). This is shown in the next lemma. Let

$$IC^i_* = \{\sigma \in \Delta(C) : \sigma \text{ satisfies (9) or (10)}\}.$$

Lemma 2 $IC^i = IC^i_*$.

The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is $IC(PI) \equiv \times_{i \in I} IC(PI)^i$.

General economies

The most general formulation allows different types to have different variables of private information. For instance, for some types, actions or a subset of actions may be not observable, while for others, the same may hold true for personal states. Also, it may be possible that for some types, the variables of private information may be subsets of the both actions and personal states. Thus, in its most general formulation, the set of manipulations available to type i is Φ^i, a subset of the maps $\phi : A \times \Omega \times S \to A \times S$, and the set of incentive compatible lotteries for type i can be written as:

$$IC^i \equiv \{\tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi^i\} \tag{12}$$

For any specification of Φ^i, IC^i is non-empty, closed and convex. It should be evident by now that this way of formulating the incentive constraints is nothing else that the standard ad interim formulation used in the literature.

4 The Prescott and Townsend Economy

The first step in the Prescott and Townsend ([9], [10]) construction restricts individual trade over lotteries to take place in the incentive compatible set. Each individual of type i is constrained to trade in the i-th market, that is to choose a lottery in the set IC^i and pay it at the price p^i. Thus the consumption set of type i individuals is:

$$\Delta(C) \cap IC^i$$

The domain of individual prices is the set of linear functionals over lotteries, described by vectors in $\mathbb{R}^C$. In principle, even though prices are linear in lotteries, they do not need to be linear or even affine in commodities. Thus, for instance, if both (a, z) and $(a, 2z)$ are in C, $p^i(a, 2z)$ may be different from

$2p^i(a,z)$, and $p^i(a,0)$ maybe different from 0. However, without restricting the price domain, any feasible and incentive compatible allocation, $(\hat{\sigma}^i)_{i\in I}$, is price supportable as a competitive equilibrium allocation: it suffices to define $\hat{p}^i(a,z) = u^i(a,z) - U^i\hat{\sigma}^i$. At $\hat{p}$, the cost of a lottery $\bar{\sigma}^i$ is $\hat{p}^i\bar{\sigma}^i = U^i\bar{\sigma}^i - U^i\hat{\sigma}^i$. Thus, every lottery $\bar{\sigma}^i$ preferred to $\hat{\sigma}^i$ is not budget feasible, that is, $\hat{p}$ supports $\hat{\sigma}$ as a competitive equilibrium. Prescott and Townsend narrow down the price domain by introducing a firm, which is a price taking profit maximizing intermediary. Its production set, Y, coincides with the set of collections of individual vectors (signed measures) $\beta = (\beta^i)_{i\in I} \in \mathbb{R}^{CI}$ that are feasible, that is,

$$Y \equiv \{\beta = (\beta^i)_{i\in I} \in \mathbb{R}^{CI} : \sum_i T^i\beta^i \leq 0\}.$$

Note that the β's are not required to be non negative, or to add to one. Since the technology exhibits constant returns to scale, considering just one firm is without loss of generality. Furthermore, by the linearity of the firm's production set, at equilibrium profits must be zero. The latter has two important implications. First, since the profits are zero the individual property rights of the firm do not need to be specified. Second, the presence of the firm restricts the price domain to P, the set of prices generating zero profits. Thus, the Farkas' alternative theorem provides an immediate characterization of the price restriction induced by the presence of the firm: That is, $\hat{p} = (\hat{p}^i)_{i\in I} \in P$ if and only if:

$$\sum_i \hat{p}^i\beta^i > 0 \text{ and } \sum_i T^i\beta^i \leq 0$$

does not have a solution $(\beta^i)_{i\in I} \in \mathbb{R}^{CI}$, and, hence, by the Farkas' alternative theorem,

$$P = \{(p^i)_{i\in I} \in \mathbb{R}^{CI} : p^i = \theta T^i \text{ for } \theta \in \mathbb{R}^{L\Omega}_+\} \tag{13}$$

Thus, P, the price domain of the Prescott and Townsend economy, coincides with the set of personalized prices of the classical analysis of the Individual Risk Economies ([7], [8], [2]). Prices in P are defined by two elements. First, by $\theta \in \mathbb{R}^{L\Omega}_+$, a type invariant, commodity price contingent on the aggregate state of nature: $\theta_{\ell\omega}$ is the type-invariant price of one unit of commodity ℓ to be delivered in aggregate state ω, independently of the realization of individual states. Then each pricing of the state contingent commodities extends to a price of pairs of action-contingent commodities (that is contingent on aggregate and personal states) according to the rule

$$p^i_{\ell s\omega}(a) = q^i(s;a,\omega)\theta_{\ell\omega}$$

Prices in P are "fair" because the type i's personalized value of a contract $(a,z) \in C$,

$$p^i(a,z) = \sum_{\ell=1}^{L}\sum_{\omega\in\Omega}\theta_{\ell\omega}\rho(\omega)\sum_s q^i(s;a,\omega)z(s,\omega)$$

is proportional to the value of the effective net trades it generates. For given type invariant price θ, the latter depends on the action a as well as on the type dependent probability vectors q^i.

Definition 3 *A competitive equilibrium is an array $(\hat{p}, \hat{\sigma}, \hat{\tau})$ such that:*

1. $\hat{p} = (\hat{p}^i)_{i\in I} \in P$.
2. *For every type i, the lottery $\hat{\sigma}^i$ is the solution of the consumer problem:*

$$\max_{\tau\in\Delta(C)\cap IC^i} U^i\tau, \text{ subject to } \hat{p}^i\tau \leq 0. \tag{14}$$

3. $\hat{\tau} = (\hat{\tau}^i)_{i\in I}$ *is profit maximizing, i.e.,*

$$(\hat{\tau}_i)_{i\in I} \in \arg\max_{\beta}\{\sum_i \hat{p}^i\beta^i \text{ subject to } \beta \in Y\}. \tag{15}$$

4. *Markets clear:*

$$\hat{\tau}_i = \lambda_i\hat{\sigma}_i, \text{ for all } i \in I. \tag{16}$$

By the definition of the firm's production set Y, the market clearing condition implies that the the equilibrium allocation $\hat{\tau} = (\lambda^i\hat{\sigma}^i)_{i\in I}$ is feasible according to the definition (3).

As usual, the existence of a competitive equilibrium and the two Fundamental Theorem of Welfare Economics require some minimal assumptions. Given the linearity of the preferences, we just need to make sure that local non satiation and the minimum wealth conditions, for $p \in P$, are satisfied. These conditions are stated formally in the following two assumptions.

Assumption 4 ***(Local non satiation)*** *There is an incentive compatible contract $\hat{c}^1 = (a^1, z^1)$, with $z^1 \gg 0$, such that*

$$u^i(\hat{c}) > U^i\sigma^i, \text{ for all } i$$

and for every incentive compatible and feasible joint lottery $\sigma = (\sigma^i)_{i\in I}$.

Assumption 5 ***(Minimum wealth condition)*** *There is an incentive compatible contract $c^2 = (a^2, z^2)$, with $\hat{z}^2 \ll 0$.*

By assumption (4), there cannot be an incentive compatible and feasible allocation where the contract c^1 is assigned to some types i with probability 1. Thus, the economy is not satiated within the set of incentive compatible and feasible allocations. Therefore, if assumption (4) holds true, $0 \notin P$ or, more precisely, the type invariant, ω-contingent prices θ defining the price domain P are non zero, that is, $\theta \in \mathbb{R}^{L\Omega}_+ \setminus \{0\}$ in (13). Then, if assumption (5) holds true, the minimum wealth condition is satisfied for all types, since $0 > p^ic^2$, for all $p \in P$. As known, if condition (5) holds, the optimal consumption problem of an individual of type i satisfies all the assumptions of the maximum theorem and, hence, the demand for lottery of any type i individual is a non-empty, compact-valued, and upper-hemi continuous correspondence of $p \in P$. Since the utility is linear in lotteries, it is convex-valued as well. Furthermore, if condition (4) holds, the optimal solution to the individual consumption problem satisfies the budget constraint with an equality.

Since individual trade is restricted in the incentive compatible set and prices are fair, the Prescott and Townsend economy is a convex economy of individual risk. Thus, under assumptions (5) and (4) the existence of a competitive equilibrium even in presence of aggregate risk follows from a standard argument. This is proved next.

Theorem 6 *Under assumptions (5) and (4), a competitive equilibrium exists.*

If the local non-satiation condition in assumption (4) holds then at any feasible allocations each individual is locally non-satiated and, at equilibrium, the budget constraints of the individuals are satisfied with an equality. Thus, each competitive allocation $\hat{\sigma}$ is constrained Pareto optimal, that is, there is no other feasible and incentive compatible allocation τ such that,

$$U^i(\tau^i) \geq U^i(\hat{\sigma}^i) \text{ for all } i, \text{ with at least one strict inequality.}$$

This is proved in the next theorem.

Theorem 7 *Under assumption (4), competitive allocations are constrained efficient.*

Also, the Second Fundamental Theorem of Welfare Economics holds. Every constrained efficient allocation can be decentralized as a quasi-equilibrium with supporting prices in P^* (and, as usual, the quasi equilibrium is an equilibrium if at the supporting price the minimum wealth condition holds true). The argument is standard and it is therefore omitted.

5 The Unconstrained Economy

In the economic environment studied so far the consumption set of the individuals is identified with the set of incentive compatible lotteries over contracts in C, i.e., with $\Delta(C) \cap IC^i$. This construction makes economies of observed types with asymmetric information isomorphic to standard general equilibrium economies with individual risk: this is the substance of Theorems (6) and (7). This, however, requires an exogenous restriction on consumption sets. Alternatively, as in Jerez ([3], [4]), the economy is formalized by removing the incentive compatibility constraints from the consumption set of the individuals and by restricting the firm to supply incentive compatible allocations.

Let IC^* denote the set of incentive compatible vectors, that is:

$$IC^* \equiv \times_{i \in I} \{\beta^i \in \mathbb{R}^C_+ : (U^i(\phi) - U^i)\beta^i \leq 0, \text{ for every } \phi \in \Phi^i\}$$

The production set of the firm Y_U is identified with

$$Y_U = Y \cap IC^*.$$

Since the production set of the firm is a cone, at equilibrium profits must be zero. Then, the latter restricts the price domain to P_U, the set of prices generating zero profits:

$$P_U = \{(p^i)_{i\in I} \in \mathbb{R}^{CI} : \sum_i \hat{p}^i \beta^i \leq 0, \text{ for all } (\beta^i)_{i\in I} \in Y_U\}.$$

Again, we can use the Farkas' alternative theorem to provide a sharper characterization of the set P_U. We postpone this to section 5.3.

For given $p \in P_U$, the maximization problem of the type i individual is:

$$\max_{\phi\in\Phi^i, \sigma\in\Delta(C)} U^i(\phi)\sigma^i \text{ subject to } \hat{p}^i\sigma^i \leq 0 \tag{17}$$

We call *unconstrained* the equilibria of this economy and we provide below their precise definition. The map $id \in \Phi^i$ denotes the identity or truth telling map.

Definition 8 *An unconstrained equilibrium of the economy with observable types is an array $(\hat{p}, \hat{\sigma}, \hat{\tau})$ such that:*

1. *$\hat{p} = (\hat{p}^i)_{i\in I} \in P^U$;*
2. *For every type i, $(id, \hat{\sigma}^i)$ is the solution of the consumer problem (17);*
3. *$\hat{\tau} = (\hat{\tau}^i)_{i\in I}$ is profit maximizing, i.e., $(\hat{\tau}^i)_{i\in I} \in \arg\max_{\beta\in Y_U} \sum_i \hat{p}^i\beta^i$;*
4. *Markets clear:*
$$\hat{\tau}^i = \lambda_i\hat{\sigma}^i, \textit{ for all } i \in I.$$

Once again, by the definition of Y_U, the market clearing condition implies that the equilibrium allocation $\hat{\tau} = (\lambda^i\hat{\sigma}^i)_{i\in I}$ is feasible according to the definition (3).

Unconstrained equilibrium allocations satisfy the First Fundamental Theorem of Welfare Economics: if an equilibrium exists, its allocation is constrained efficient. Indeed, suppose otherwise. Then, there exists a collection of lotteries $\tau = (\lambda^i\sigma^i)_{i\in I} \in Y_U$ that Pareto dominates the competitive allocation $\hat{\tau} = (\lambda^i\hat{\sigma}^i)_{i\in I}$. Individuals do not acquire σ^i because at $\hat{p}^i$, $\hat{p}^i\sigma^i \geq 0$, for $i \in I$, with at least one strict inequality. Thus, $\sum_i \lambda^i\hat{p}^i\sigma^i > \sum_i \lambda^i\hat{p}^i\hat{\sigma}^i$. Hence, since $\tau \in Y_U$, the last inequality implies that by supplying $(\lambda^i\hat{\sigma})_{i\in I}$, the firm is not maximizing profits at p. A contradiction.

The question of interest is whether or not this way of modelling asymmetric information economies with observable types leads to the same conclusions of the Prescott and Townsend construction. Indeed, Jerez [4] shows that this is the case, but, in [4], individuals do not optimize over $\phi \in \Phi^i$, that is, their objective function in (17) is just $U^i\sigma^i$. Unfortunately, once individuals are allowed to exploit their private information and optimally manipulated contracts, as in (17), the unrestricted equilibrium set may be empty and the Second Fundamental Theorem of Welfare Economics may fail.

The potential emptiness of the unconstrained equilibrium set has two different origins. First, constrained efficient allocations may fail to be price supportable. Second, they might be price supportable, but the supporting prices may not make them profit maximizing choices of the firm. We investigate the first issue in section 5.1 and the second in section 5.3.

5.1 Price supportability

By definition, if (p, σ) is an unconstrained equilibrium, it must be as well an equilibrium of the pure exchange economy, that is an equilibrium of the economy without the firm. We call the latter a pure exchange equilibrium. Formally:

Definition 9 *A pure exchange equilibrium is a pair $(\hat{p}, \hat{\sigma})$, with $\hat{p} = (\hat{p}^i)_{i \in I} \in \mathbb{R}^{CI}$, such that*

1. $(id, \hat{\sigma}^i) \in \arg\{\max_{\phi \in \Phi^i, \sigma \in \Delta(C)} U^i(\phi)\sigma^i$ *subject to* $\hat{p}^i \sigma^i \leq 0\}$, *at* $\hat{p}^i$ *for every* i, *and*

2. *Markets clear, that is,* $\sum^i \lambda^i T^i \sigma^i \leq 0$.

An incentive compatible individual lottery σ^i is *price supportable* if there exists a price p^i such that (id, σ^i) is a solution to the individual programming problem (17) at p^i, and a feasible and incentive compatible joint allocation σ is *price supportable* if there exists $p = (p^i)_{i \in I}$ such that the individual lotteries σ^i are *price supportable*, for all i. Evidently, if a feasible and incentive compatible joint allocation σ is price supportable, then it is an equilibrium allocation of the pure exchange economy.

We provide the intuition for the failure of price supportability of constrained efficient allocations. Suppose that at some constrained efficient allocation σ^*, the incentive constraints for type i are binding. Thus, by definition, there exists a manipulation $\phi \in \Phi^i$ such that $U^i \sigma^{*i} = U^i(\phi)\sigma^{*i}$. The latter implies that the supporting prices must make σ^{*i} be the most preferred budget feasible lottery with both the utility function U^i and $U^i(\phi)$. However, U^i and $U^i(\phi)$ may have quite different shapes thereby making impossible to find a common supporting price. Indeed the objective function appearing in the individual programming problem (17) is $V^i(\sigma^i) = \max_{\phi \in \Phi^i} U^i(\phi)\sigma^i$. V^i is, by construction, the upper envelope of the (finite) family of linear maps $U^i(\phi)\sigma^i$, $\phi \in \Phi^i$, and, thus, may fail to be concave. If σ^{*i} is a non concave area of the map V^i, σ^* may not be price supportable.

5.1.1 Necessary and sufficient conditions for price supportability

We want to determine necessary and sufficient conditions making a joint, feasible and incentive compatible lottery σ price supportable. To simplify the arguments that follow it will be convenient to strengthen the local non satiation assumption also extending it to the utility functions $U^i(\phi)$. The assumption below excludes the possibility that some Kunh-Tucker multipliers may be zero.

Assumption 10 ***(Local non satiation)*** *There is an incentive compatible contract $c^1 = (a^1, z^1)$, with $z^1 \gg 0$, such that*

$$u^i(\phi(c^1)) > U^i(\phi)\sigma^i$$

for all i, for all $\phi \in \Phi^i$, for every incentive compatible and feasible joint lottery $\sigma = (\sigma^i)_{i \in I}$.

Notice that assumption (10) is stronger than assumption (4). Indeed, by the former, the inequalities $u^i(\phi(c^1)) > U^i(\phi)\sigma^i$ must be satisfied for all $\phi \in \Phi$, while, by the latter, only for $\phi = id$. Moreover, assumption (10) is satisfied if z^1 is very large and there is enough monotonicity of preferences in net trades. These conditions are surely satisfied in all examples of private information and moral hazard economies present in the literature.

An equilibrium (p, σ^*) of the pure exchange economy must satisfy two requirements:

1) given the fact that individuals behave truthfully, σ^{*i} is an optimal solution at p^{*i}, for all types $i \in I$, and

2) behaving truthfully is optimal.

As already mentioned, it is requirement 2) that is both missing from the analysis in [4] and that creates problems.

Given any manipulation function $\phi \in \Phi^i$, truth telling included, the individual problem (17) is a linear programming problem. Thus, requirement 1) has a well known characterization based on the necessary and sufficient conditions for optimality. Then, given p^i, σ^{*i} is is an optimal solution to

$$\max_{\beta^i \in \Delta(C)} U^i \beta^i, \text{ subject to } p^i \beta^i \leq 0 \tag{18}$$

if and only if there exists a scalar $\alpha^i \geq 0$ such that:

$$\begin{aligned} U^i(c) - U^i\sigma^{*i} &= \alpha^i p^i(c) \text{ if } c \in C(\sigma^{*i}) \\ U^i(c) - U^i\sigma^{*i} &\leq \alpha^i p^i(c), \text{ if } c \notin C(\sigma^{*i}) \\ p^i\sigma^{*i} &= 0. \end{aligned} \tag{19}$$

By assumption (4) (or (10)), $U^i(c^1) - U^i(\sigma^{*i}) > 0$. Thus, $\alpha^i > 0$. Therefore, requirement 1) for the optimality of (id, σ^{*i}) restricts the set of prices that can potentially support the lottery σ^{*i}, to satisfy conditions (19). A price p^i that does not satisfy (19) cannot possibly support σ^{*i}, since at that price $U^i\tau^i > U^i\sigma^{*i}$, for some budget feasible lottery τ^i. We call "*potentially supporting*" the set of prices satisfying (19), we denote it by $P^i(\sigma^{*i})$, and exploiting the homogeneity of the budget constraints, we write it, without loss of generality, as

$$P^i(\sigma^{*i}) = \{p^i : p^i \text{ satisfies (19) for } \alpha^i = 1\}.$$

The second requirement for the price supportability of σ^{*i} is more complicated. Roughly speaking, in principle, σ^{*i} may fail to be price supportable for two rather different and independent reasons:

A) for each $p^i \in P^i(\sigma^{*i})$, there is a manipulation ϕ such that $U^i(\phi)\tau^i > U^i\sigma^{*i}$, for some τ^i budget feasible at p^i, whose support is contained in the support of C^i, i.e., $C(\tau^i) \subset C(\sigma^{*i})$;

B) for each $p^i \in P^i(\sigma^{*i})$, there is a manipulation ϕ such that $U^i(\phi)\tau^i > U^i\sigma^{*i}$, for some lottery τ^i budget feasible at p^i whose support contains contracts outside the support of σ^{*i}, i.e., $C(\tau^i) \cap \{C \backslash C(\sigma^{*i})\} \neq \varnothing$.

By definition, prices in $P^i(\sigma^{*i})$ coincide over the set of contracts $C^i(\sigma^{*i})$, while they can take different values for contracts outside that set. Therefore if a pair ϕ satisfies A) for some $\bar{p}^i \in P^i(\sigma^*)$, it does so for all $p^i \in P^i(\sigma^*)$. To the contrary, B) is a price dependent phenomenon: some manipulations ϕ may satisfy B) for some $\bar{p}^i \in P^i(\sigma^*)$, while may fail to do so for other $p^i \in P^i(\sigma^*)$. Thus, the search for necessary and sufficient conditions for the price supportability of σ^* naturally partitions the set of manipulations Φ^i into two disjoint and exhaustive subsets, $\Phi^i(\sigma^{*i})$ and its complement. Manipulations in $\Phi^i(\sigma^{*i})$ are potential candidates for generating A), while manipulations outside $\Phi^i(\sigma^{*i})$ can upset σ^{*i} only through B), but not A). More precisely, for given i and $\phi \in \Phi^i$ consider the following programming problem:

$$\max_{\mu^i \in \Delta(C(\sigma^{*i}))} U^i(\phi)\mu, \text{ subject to } \sum_{c \in C(\sigma^{*i})} (U^i(c) - U^i\sigma^{*i})\mu^i(c) \leq 0. \tag{20}$$

Let $[\mu^i_\phi]$ denote the set of optimal solutions, μ^i_ϕ one of its elements, and let V^i_ϕ be the value of the programming problem, that is, $V^i_\phi = U^i[\mu^i(\phi)]$.

Now, the set $\Phi^i(\sigma^{*i})$ is naturally defined as the set of non trivial manipulations yielding values V^i_ϕ greater or equal than $U^i\sigma^{*i}$, that is:

$$\Phi^i(\sigma^{*i}) = \{\phi : V^i(\phi) \geq U^i\sigma^{*i} \text{ and } U^i(\phi(c)) - U^i(c) > 0, \text{ for some } c \in C(\sigma^{*i})\}.$$

We now proceed as follows. First we look for price restrictions on the contracts $c \in C \backslash C(\sigma^{*i})$ that make (id, σ^{*i}) *immune* from manipulations $\phi \in \Phi \backslash \Phi^i$ and then we search for necessary and sufficient conditions for price supportability.

5.1.2 Manipulations in $\Phi^i \backslash \Phi^i(\sigma^{*i})$

We look at manipulations in $\Phi^i \backslash \Phi^i(\sigma^*)$ and we ask the following question: can we find a prices $p = (p^i)_{i \in I} \in \times_{i \in I} P^i(\sigma^{*i})$ such that $U^i(\phi)\tau^i \leq U^i(\sigma^{*i})$, for all budget feasible lotteries at p^i, and manipulations $\phi^i \in \Phi^i \backslash \Phi^i(\sigma^*)$, and $i \in I$? The answer is affirmative and the intuition is simple. The definition of $P^i(\sigma^{*i})$ only pins down the prices of contracts in the set $C^i(\sigma^{*i})$, while it just puts lower bound for the prices of the contracts in the set $C \backslash C(\sigma^{*i})$. It is quite intuitive that we can select values of $p^i(c)$, $c \in C \backslash C(\sigma^{*i})$, so high to make suboptimal any manipulation $\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})$. This we show in Proposition 12, that therefore establishes that price supportability is just a search for necessary and sufficient conditions ruling out condition A). Now, the details.

It suffices to limit attention to a simple pricing rule define by the following set:

$$P'^i(\sigma^{*i}) = \{p^i : p^i(c) = U^i(c) - U^i\sigma^i,\ c \in C(\sigma^i),\ p^i(c) = \bar{p}^i,\ c \in C\backslash C(\sigma^i)\}$$

Each $p^i \in P'^i(\sigma^{*i})$ is uniquely identified by a scalar $\bar{p}^i$. Hereafter, we constantly use this identification and, with some abuse of notation, sometimes we write $\bar{p}^i \in P'^i(\sigma^{*i})$. The next, easy lemma establishes that limiting attention to prices in $P'^i(\sigma^{*i})$ is without loss of generality.

Lemma 11 *If σ^* is price supportable, then there exists a supporting price $\bar{p} = (\bar{p}^i)_{i\in I} \in \times_{i\in I} P'^i(\sigma^{*i})$.*

Consider σ^*, a feasible and incentive compatible allocation. Let $\bar{p} = (\bar{p}^i)_{i\in I} \in \times_{i\in I} P'^i(\sigma^{*i})$ be given. For given $\phi \in \Phi^i$, consider the programming problem:

$$\max_{\mu\in\Delta(C)} U(\phi)\mu, \text{ subject to } \bar{p}^i\mu \leq 0. \tag{21}$$

Let $[\mu^i_\phi(\bar{p}^i)]$ be the set of optimal solutions, $\mu^i_\phi(\bar{p}^i)$ be one of its elements, and $V^i_\phi(\bar{p}^i)$ the value of the program.

Proposition 12 *Let σ^* be a feasible and incentive compatible allocation. Then, there exists $\bar{p} \in \times_{i\in I} P'^i(\sigma^{*i})$ such that $V^i_\phi(\bar{p}'^i) \leq U^i\sigma^{*i}$, for all $\bar{p}'^i \geq \bar{p}^i$, all $\phi \in \Phi^i\backslash\Phi^i(\sigma^{*i})$, and $i \in I$.*

Proposition 12 immediately implies that if $\Phi^i(\sigma^*) = \varnothing$, for all i, σ^* is price supportable. Evidently, if the contracts in the support of $C(\sigma^{*i})$ are contained in C^i_{IC}, the set of deterministic and incentive compatible contracts for i, $\Phi^i(\sigma^{*i})$ is empty. Thus:

Corollary 13 *Let σ^* be an incentive compatible and feasible allocation. Suppose that $C(\sigma^{*i}) \subset C_{IC}$, for all i. Then, σ^* is price supportable.*

Thus, by Corollary 13, the set of equilibria of the pure exchange economy is non empty. All feasible lotteries whose support is contained, for each individual type, in the set of deterministic and incentive compatible lotteries are price supportable.

5.1.3 Manipulations in $\Phi^i(\sigma^{*i})$

We are looking for necessary and sufficient conditions for the price supportability of an incentive compatible, feasible joint lottery σ^* when $\Phi^i(\sigma^{*i}) \neq \varnothing$, for some $i \in I$. To state the obvious: the incentive compatible and feasible lottery σ^* is price supportable if and only if $V^i(\phi) = U^i\sigma^{*i}$, for all $\phi \in \Phi^i(\sigma^{*i})$, and all i. The interesting problem is to translate the latter into conditions that the contracts $c \in C(\sigma^{*i})$ must satisfy. This we do by exploiting the necessary and sufficient conditions for optimality of the programming problems (20). This is the substance of the next proposition. Let

$$C^i(\phi) = \{c : \mu^i_\phi(c) > 0, \text{ for some } \mu^i_\phi \in [\mu^i_\phi]\}.$$

Remember that $[\mu^i_\phi]$ is the set of optimal solutions to the programming problems (20). Therefore, $C^i(\phi) \subset C^i(\sigma^{*i})$.

Proposition 14 *Let σ^* be a feasible and incentive compatible joint lottery. σ^* is price supportable if and only if, for all $i \in I$ and $\phi \in \Phi^i(\sigma^{*i})$, there exist scalars $b^i_\phi > 0$ such that $U^i(\phi(c)) - U^i\sigma^* = b^i_\phi[U^i(c) - U^i\sigma^{*i}]$, for $c \in C(\phi)$.*

The next corollary states a necessary condition for the price supportability of joint lotteries with binding incentive constraints.

Corollary 15 *Let σ^* be a feasible and incentive compatible joint lottery. If σ^{*i} is price supportable, then for all ϕ such that $U^i(\phi)\sigma^{*i} = U^i\sigma^{*i}$, it must be $\sigma^{*i} \in [\mu^i_\phi]$, or equivalently, $U^i(\phi(c)) - U^i\sigma^* = b^i_\phi[U^i(c) - U^i\sigma^{*i}]$, for $c \in C(\sigma^{*i})$.*

Corollary 15 provides simple instructions to build economies with some or all non price supportable constrained efficient allocations. It suffices to pick an economy with a constrained efficient allocation violating the conditions in the Corollary 15. Evidently, if none of the constrained efficient allocations of some economy are price supportable then the unconstrained equilibrium set is empty. This we do in the next section.

5.2 Economies with non price supportable constrained efficient allocations

In this section, we look first at economies of Private Information and then at economies of Moral Hazard. We build simple examples showing the failure of the Second Fundamental Theorem of Welfare Economics and the lack of price supportable constrained efficient allocations. Obviously, the latter shows an economy with an empty set of unconstrained equilibria

Private information

The example below is taken from [6]. It shows an economy of private information where one of the constrained efficient allocations is not price supportable providing therefore an example of a pure exchange economy and therefore of an unconstrained economy where the Second Fundamental Theorem of Welfare Economics fails.

Example 16 *Failure of the Second Fundamental Theorem of Welfare Economics.*

There is only one type, there are no actions, no aggregate states, one physical commodity and two equiprobable personal states, $s = g, b$. Endowments are

$e = (e_g, e_b) = (30, 10)$ and the utility function of the individuals is $U(z_g + e_g, z_b + e_b) = \frac{1}{2}V(z_g + 30) + \frac{1}{2}V(z_b + 10)$, for $V(x) = 78x - (x)^2$.

In [6], it is shown that these two are optimal contracts:
a) $\sigma_1(-1, -5) = \frac{1}{2}$ and $\sigma_1(-1, 7) = \frac{1}{2}$; and
b) $\sigma_2(-1, -7) = \frac{9}{32}$ and $\sigma_2(-1, 1) = \frac{14}{32}$, $\sigma_2(-1, 9) = \frac{9}{32}$.

All incentive compatible, feasible, and deterministic contracts must satisfy the equality $z_g = z_b$. Thus, none of the contracts in $C(\sigma_k)$, $k = 1, 2$, is incentive compatible. Let ϕ_g be the manipulation defined as $\phi_g(g) = b$ and $\phi_g(b) = b$. The utility function $U(\phi_g)$ is defined by $U(\phi_g)(z_g, z_b) = \frac{1}{2}V(z_b + 30) + \frac{1}{2}V(z_b + 10)$. It is easily verified that $U(\phi_g)\sigma_k = U\sigma_k$, $k = 1, 2$. Thus, $\Phi^i(\sigma_k) \neq \varnothing$, $k = 1, 2$.

We claim that σ_1 is price supportable, while σ_2 is not. Start with σ_1. It is easy to show that $V_{\phi_g} \geq V_\phi$, for all ϕ, where recall that V_ϕ is the value of the problem (20) with manipulation $\phi \in \Phi$. Furthermore, $U(\phi_g)\sigma_1 = U\sigma_1$. Then, since the support of σ_1 contains only two points, $\sigma_1 \in [\mu_{\phi_g}]$ or, equivalently, there is $b_\phi > 0$ such that $(U(\phi_g)(z_g, z_b) - U\sigma_1) = b_\phi(U(z_g, z_b) - U\sigma_1)$, for all $(z_g, z_b) \in C(\sigma_1)$. Thus, by Corollary 15, σ_1 is price supportable.

Move to σ_2. Once again $U(\phi_g)\sigma_2 = U\sigma_2$. By trivial computations:

$$(U(\phi_g)(z_g, z_b) - U\sigma_2)_{(z_g,z_b)\in C(\sigma_2)} = (-316, 36, 260)$$

while

$$(U(z_g, z_b) - U\sigma_2)_{(z_g,z_b)\in C(\sigma_2)} = (-238, 18, 210)$$

Thus, the two vectors $(U(\phi_g)(z_g, z_b)-U\sigma_2)_{(z_g,z_b)\in C(\sigma_2)}$ and $(U(z_g, z_b)-U\sigma_2)_{(z_g,z_b)\in C(\sigma_2)}$ are linearly independent. Therefore, Corollary 15 implies that σ_2 is not price supportable.■

Moral hazard

Recall that for moral hazard economies, manipulations are maps $\phi : A \to A$, that is, a manipulation ϕ changes a contract $c = (a, z)$ into the contract, $\phi(c) = (\phi(a), z)$. Also, recall that an individual lottery σ^i entails ex-ante randomization if $\sum_z \sigma^i(a, z) < 1$ for all $a \in A$, while it entails ex-post randomization if the conditional lottery $\sigma^i(z^*; a^*) \equiv \frac{\sigma^i(a,z^*)}{\sum_z \sigma^i(a^*,z^*)} \in (0, 1)$, for some $(a^*, z^*) \in C$. The bottom line is that, under minor qualifications, contracts entailing both ex-ante and ex-post randomization are not price supportable. We explain the result by discussing the problem within a simple economy with one type and two actions.

Consider an economy of moral hazard with one type and two actions, say, H and L. Pick the fundamentals of the economy so that:

1) the unique constrained efficient allocation is a lottery, σ, that entails both ex-ante and ex-post randomization, that is $\sum_z \sigma(a, z) > 0$, for $a = L, H$.

2) the incentive constraints are binding: there exists a manipulation ϕ^* such that $U\sigma = U(\phi^*)\sigma$ and $\phi^*(L) = L$ while $\phi^*(H) = L$

3) the support of σ contains deterministic contracts that are not incentive compatible and $U(L, z) > U(H, z)$, for some $(H, z) \in C(\sigma)$.

Then, by conditions 2) and 3), $\phi^* \in \Phi(\sigma^*)$. Thus, by Corollary 15, σ is price supportable if and only if $U(\phi)(c) - U\sigma = b_\phi[U(c) - U\sigma]$, for $b_\phi > 0$ and

$c \in C(\sigma^i)$. However, since $\phi^*(L) = L$, if (as it will be generically the case) $U^i(L,z) - U^i\sigma \neq 0$, then $b_\phi = 1$, while since $U(L,z) > U(H,z)$, $b_\phi \neq 1$. Thus, σ is not price supportable.

It should be obvious that the lack of price supportability has nothing to do with the simplifying features of the problem discussed above, but rather with the existence of ex-ante and ex-post randomization. This is precisely stated in Lemma 17 below. The argument, as for the problem discussed above, is a trivial implication of Corollary 15 and it is, therefore, omitted.

Lemma 17 *Let σ be a feasible and incentive compatible joint lottery of a moral hazard economy. Suppose σ^i entails ex-ante and ex-post randomization, for some type i. Suppose that for such a type, there exists ϕ such that i) $U^i(\phi)\sigma^i = U^i\sigma^i$, ii) $U^i(\phi(a),z) > U^i(a,z)$, for some $(a,z) \in C(\sigma^i)$, iii) $U^i(\phi(a'),z) = U^i(a',z) \neq U^i\sigma^i$, for some $(a',z) \in C(\sigma^i)$. Then, σ^i is not price supportable.*

For moral hazard economies with one type, both [1] and [3] provide conditions under which optimal contracts entail randomization across different actions (levels of effort). However, both papers provide sufficient conditions that eliminate ex-post randomization, that is, such that the conditional lotteries $\sigma(z|a)$ are degenerate. The latter translates into the property that the optimal contracts of these economies, σ, satisfy $C(\sigma) \subset C_{IC}$. Therefore, $\Phi(\sigma) = \varnothing$, and hence, by Proposition 12, σ is price supportable. The example below is a simple moral hazard economy where the optimal contract entails both ex-ante and ex-post randomization and satisfies conditions *i*)-*iii*) of Lemma 17. The optimal contract is therefore not price supportable and, as a consequence, this economy has an empty set of unconstrained equilibria.

Example 18 *A moral hazard economy without price supportable efficient allocations.*

There is one type and two actions, $a = L, H$. There is one physical commodity, no aggregate uncertainty and two personal states, $s = 1, 2$. The probability of the states are $q(\cdot; L) = (\frac{9}{10}, \frac{1}{10})$ and $q(\cdot; H) = (\frac{1}{10}, \frac{9}{10})$. Endowments are state dependent, but action independent and they are $e = (1, 21)$. Given a pair of state contingent net trades $z = (z(1), z(2))$ and an action a, the utility function is $U_a(z) = \sum_s q(s;a)u_a(e(s) + z(s))$. We assume that

$$u_L(e(s)+z(s)) = 8\sqrt{(1 + e(s) + z(s))}, \text{ while } u_H(e(s)+z(s)) = (e(s)+z(s)-1).$$

The set of net trades is state contingent and it is $z(s) \in [-e, 99]$. We show that every optimal contract of this economy, σ, satisfies conditions 1)-3).

We have picked the values of the fundamentals in order to satisfy the sufficient conditions for ex-ante randomization (as stated for instance in [4]). We identify a lottery without ex-post randomization with σ^*, denoting the probability of adopting the action L, $z^*_a = (z^*_a(1), z^*_a(2))$, $a = L, H$, denoting the pair of state contingent trades when the action a is chosen. Routine computations show that the optimal contract without ex-post randomization is

$$\sigma^* \approx 0.466, \; z^*_L \approx (11.02, -8.92), \; z^*_H \approx (-1, -8.64)$$

At (σ^*, z_L^*, z_H^*) the resource constraint is binding. The H-incentive constraint

$$U_H(z_H^*) \geq U_L(z_H^*)$$

is binding, while the L-action constraint is not. However, U_L displays risk aversion while U_H does not. The introduction of ex-post randomization releases the H-incentive constraint making more resources available for consumption associated to the event of the L-action is taken. This is clearly a welfare improving maneuver. Thus, the optimal contract, σ^*, entails randomization across three contracts (L, z_L^*), (H, z_H^{*1}) and (H, z_H^{*2}). By routine computations:

$$z_L^* \approx (11.2, -8.8),\ z_H^{*1} = -e \text{ and } z_H^{*2} = (-1, 99)$$

and

$$\sigma^*(L, z_L^*) \approx 0.5,\ \sigma^*(H; z_H^{*1}) = [1-\sigma^*(L, z_L^*)]\frac{91}{100},\ \sigma^*(H; z_H^{*2}) = [1-\sigma^*(L, z_L^*)]\frac{9}{100}$$

Let ϕ^* be the manipulation defined as $\phi^*(L) = L$ and $\phi^*(H) = L$. At the optimal lottery, σ^*, the H-incentive constraint is binding, while the L is not, and thus $U\sigma^* = U(\phi^*)\sigma^*$. Furthermore, since $U_a(z_H^{*1}) = U_a(0)$, for $a = H, L$, and, since $U_L(0) > U_H(0)$, $U_H(z_H^{*1}) < U_{\phi^*(H)}(z_H^{*1}) = U_L(z_H^1)$ and, hence, the contract (H, z_H^{*1}) is not incentive compatible. Furthermore, $U_L(z_L) - U^i\sigma^* \approx 9.9 \neq 0$. Thus, conditions *i)*-*iii)* of Lemma 17 are met and σ^* is not price supportable. It should also be obvious that the linearity of U_H simplifies the computations, but does not play a role for the result. Indeed it suffices to pick appropriately a function U_H that displays less risk aversion than U_L. Finally, the upper bound $\bar{z}$ allows for the existence of an optimal contract. Indeed, since U_H is linear, without an upper bound on net trades an optimal contract does not exist. ∎

5.3 Unconstrained equilibrium allocations

Obviously, the set of unconstrained equilibrium allocations is contained in the set of pure exchange equilibrium allocations. Furthermore, the profit maximizing behavior of the firm restricts the set of prices that can be used to support the allocations as an unconstrained equilibrium. Indeed, if σ^* is an unconstrained equilibrium allocation not only the necessary and sufficient conditions for price supportability must be satisfied, but also there must exists a price $p \in \times_i P^i(\sigma^{*i})$ such that $(\lambda^i\sigma^{*i})_{i\in I}$ is profit maximizing at p. As well known, the latter is true if and only if $p \in P_U$, the price domain generating zero economic profit for the firm. Thus, by exploiting the the necessary and sufficient conditions for optimality, we can state that $(\lambda^i\sigma^{*i})_{i\in I}$ is the profit maximizing choice at p if and only if there exist a vector $\theta > 0$ (associated to the resource constraints) and scalars $b_\phi^i \geq 0$, $\phi \in \Phi^i$, (associated to the incentive constraints) such that following holds true:

$$
\begin{aligned}
p^i(c) &= \theta T^i(c) + \sum_{\phi\in\Phi^i} b^i_\phi (U(\phi(c)) - U(c)),\ \text{if } c \in C(\sigma^{*i}),\ i \in I \quad (22)\\
p^i(c) &\leq \theta T^i(c) + \sum_{\phi\in\Phi^i} b^i_\phi (U(\phi(c)) - U(c)), \text{ otherwise},\\
\sum_i \lambda^i T^i \sigma^{*i} &\leq 0 \text{ and } U^i\sigma^{*i} \geq U^i(\phi)\sigma^{*i},\ (i,\phi) \in I \times \Phi^i
\end{aligned}
$$

As known, if the incentive constraint associated to $\phi \in \Phi^i(\sigma^{*i})$ is not binding, that is if $(U^i - U^i(\phi))\sigma^{*i} > 0$, then $b_\phi = 0$. Indeed, the profits of the firm are

$$
\sum_i \lambda^i p^i \sigma^{*i} = \theta \sum_i \lambda^i T^i \sigma^{*i} + \sum_i \lambda^i \sum_{c\in C(\sigma^{*i})} \sum_{\phi\in\Phi^i} b^i_\phi (U(\phi(c)) - U(c))\sigma^{*i}(c)
$$

Since σ^* is feasible and $\theta \sum_i \lambda^i T^i \sigma^{*i} = 0$, profits are equal to

$$
\sum_i \lambda^i \sum_{c\in C(\sigma^{*i})} \sum_{\phi\in\Phi^i} b^i_\phi (U(\phi(c)) - U(c)),
$$

which implies our claim.

As we have already discussed, price supportability restricts the price function in $C(\sigma^{*i})$, while it only puts lower bounds in $C\backslash C(\sigma^{*i})$. However, profit maximization specifies as well upper bounds for supporting prices. Unfortunately, the upper bounds of the profit maximization may collide with the lower bounds of the utility maximization thereby making the set of unconstrained equilibria empty for extremely well behaved economies that have price supportable constrained efficient allocations. Indeed, we construct below an example of an economy of private information with a unique Pareto optimum. At the Pareto optimal allocation, the incentive constraints are not binding, and hence the Pareto optimum is both the unique candidate for being an unconstrained equilibrium and it is, obviously, price supportable. However, it fails to be an unconstrained equilibrium, or, equivalently, the unconstrained equilibrium set of this economy is empty. This example should clarify the very large extent to which this equilibrium notion is inadequate

Example 19 *A private information economy with an empty set of unconstrained equilibria, but with a unique and price supportable efficient allocation.*

There is only one type, there are no actions, no aggregate states, two physical commodities and two equiprobable personal states, $s = 1, 2$. Endowments are $e = (e_1, e_2) = [(30, 10), (10, 30)]$ and the utility function of the individuals is $U(z_1 + e_1, z_2 + e_2) = \sum_s V(z_s + e_s)$, with $V(x) = \ln x_1 + \ln x_2$. We restrict, without loss of generality, the set of manipulation to contain two elements ϕ_s, $s = g, b$, with $\phi_s(s') = s$, for all s'. Thus, there are only two incentive constraints. A contract is pair of net trades (z_1, z_2). A lottery σ is incentive compatible if:

$$0 \leq \sum_{(z_1,z_2)\in C(\sigma)} \sigma(z_1, z_2)U(z_1, z_2) - \sum_{c\in C(\sigma)} \sigma(z_1, z_2)U(\phi_s(z_1, z_2)), s = 1, 2.$$

This economy has a unique Pareto optimal allocation that is incentive compatible and it is therefore a constrained Pareto optimal allocation. Since the support of the efficient lottery is contained in the set of incentive compatible and deterministic allocations, this lottery is price supportable.

It is trivial to check that the Pareto optimal allocation is a degenerate lottery $\sigma^* = \delta_{z^*}$, for $z^* = [(-10, 10); (10, -10)]$. It is also immediate to check that $(U - U(\phi_s))\delta_{z^*} > 0$, for all s. Since, the incentive constraints are not binding, the multipliers b_{ϕ_s} in (22) must be equal to zero for the two manipulations ϕ_s, $s = g, b$. Therefore, since the personal states are equiprobable and by conditions (22), the candidate unconstrained equilibrium prices are parameterized by $\theta = (\theta_1, \theta_2) \gg 0$ and they are:

$$\begin{aligned} p(z^*) &= \sum_c \theta_c(z^*_{1c} + z^*_{2c}) = 0, \\ p(z) &\leq \sum_c \theta_c(z_{1c} + z_{2c}), \text{ otherwise.} \end{aligned}$$

Consider the degenerate lottery $\delta_{z'}$, with $z' = [(10, 10); (-20, -20)]$. Evidently, since $z'_{1c} + z'_{2c} = -10$, for all c, $p(z') < 0$, for all candidate unconstrained equilibrium prices. Thus, $\delta_{z'}$ is always budget feasible. Furthermore, the manipulation ϕ_1 transforms the allocation z' into $\phi_1(z') = [(10, 10); (10, 10)]$. it is obvious that $U(\phi_1)\delta_{z'} > U\delta_{z^*}$. Hence, the Pareto optimal allocation z^* cannot be decentralized as an unconstrained equilibrium and, equivalently, the set of unconstrained equilibria is empty. ∎

6 Conclusion

Within the class of economies analyzed in this paper, a fundamental difference arises between economies where the IC constraints are imposed on the consumer and those in which they are imposed on the firm. We consider the core feature of economies with private information the fact that individuals can make optimal use of the private information and may deviate from the recommended course of action. A minimal consistency requirement is that if they are aware of this possibility, they will do so at any opportunity in which they can. This is of course the very reason for introducing incentive compatibility constraints in the economy. For example, if in economies with moral hazard consumers choose the lottery anticipating that they will later follow the recommendation, then the economies are Arrow-Debreu economies: Existence, First and Second Welfare theorems are standard.

A useful technique to prove existence of equilibria in competitive economies is to proceed on the basis of the Welfare Theorems, reducing the equilibrium problem to an optimization problem, and deriving equilibrium prices as shadow prices in the maximization problem. An important and in our opinion so far misunderstood feature of the economies with asymmetric information (even with observable types) is that this technique fails if the incentive compatibility constraint is imposed as a constraint on the technology of the firm. The incentive constraint on the firm does affect prices, but cannot decentralize the constraint on the consumer to limit the choice to incentive compatible contracts. When the consumer is offered the price derived from the firm maximization problem he will typically deviate from the recommended course of action in the contract. The reason is clear: those prices were designed to prevent deviation ex-post (occurring potentially at the moment in which the recommended action is communicated, after the contract is chosen), not the plan of an ex-ante deviation at the moment of the choice of the contract.

7 Appendix

Proof of Lemma (1)

($IC^i_* \subset IC^i$)

Let $\bar\sigma \in IC^i_*$, but suppose by contradiction that $\bar\sigma \notin IC^i$. Then, there exists $\bar\phi : A \to A$ such that

$$(U^i(\bar\phi) - U^i)\bar\sigma = \sum_a \sum_z \bar\sigma(a,z)[u^i(\bar\phi(a),z) - u^i(a,z))] > 0$$

The latter implies that for some $\bar a$ with $\sum_z \sigma(\bar a, z) > 0$ and $a' \equiv \bar\phi(\bar a) \neq \bar a$, it is

$$\sum_z \bar\sigma(a,z)[u^i(a',z) - u^i(a,z)] > 0.$$

However, the latter implies that $\bar\sigma \notin IC^i_*$, a contradiction.

($IC^i \subset IC^i_*$)

To each pair of action (a,a') associates a map $\phi_{a.a'} \in \Phi(MH)$ defined as $\phi_{a,a'}(\bar a) = \bar a$, if $\bar a \neq a$, $\phi_{a,a'}(a) = a'$. If $\sigma \in IC^i$, then $(U^i(\phi_{a,a'}) - U^i)\sigma \leq 0$, for all a, a'. The latter reads:

$$\sum_a \sum_z \sigma(a,z)[u^i(\phi_{a,a'}(a),z) - u^i(a,z)] = \sum_z \sigma(a,z)[u^i(a',z) - u^i(a,z)] \leq 0.$$

Thus, the claim. ■

Proof of Lemma (2)

($IC^i_* \subset IC^i$)

Let $\bar\sigma \in IC^i_*$, but suppose by contradiction that $\bar\sigma \notin IC^i$. Then, there exists $\bar\phi : A \times \Omega \times S \to A$ such that

$$(U^i(\bar\phi) - U^i)\bar\sigma = \sum_{a,z} \bar\sigma(a,z)[u^i(a, z(\phi(a,\cdot))) - u^i(a,z))] > 0$$

where remember that $u^i(a, z(\phi(a,\cdot))) = \sum_\omega \rho(\omega) \sum_s q^i(s; a, \omega)[v^i(a, s, \omega, z(\phi(a,\omega,s),\omega))]$. The latter implies that for some $(\bar a, \bar s, \bar\omega)$ with $\sum_z \sigma(\bar a, z) > 0$ and $s' \equiv \bar\phi(\bar a, \bar s, \bar\omega) \neq \bar a$, it is

$$\sum_z \sigma(a,z) q^i(s; a, \omega)[v^i(\bar a, s, \bar\omega, z(s', \bar\omega)) - v^i(\bar a, s, \bar\omega, z(s, \bar\omega))] > 0.$$

However, the latter implies that $\bar\sigma \notin IC^i_*$, a contradiction.

($IC^i \subset IC^i_*$)

Consider a pair of personal states (s,s'), an action a and an aggregate state ω. To each $((s,s'),a,\omega)$ associates a map $\phi_{((s,s'),a,\omega)} \in \Phi(PI)$ defined as $\phi_{((s,s'),a,\omega)}(\bar a, \bar s, \bar\omega) = \bar s$, if $(\bar a, \bar s, \bar\omega) \neq (s, a, \omega)$, while $\phi_{((s,s'),a,\omega)}(a, s, \omega) = s'$.

If $\sigma \in IC^i$, then $(U^i(\phi_{((s,s'),a,\omega)}) - U^i)\sigma \leq 0$, for all $((s,s'),a,\omega)$. The latter reads:

$$\sum_z \sigma(a,z)q^i(s_1;a,\omega)[v^i(a,s',\omega,z(s_2,\bar{\omega})) - v^i(a,s,\omega,z(s_1,\omega))] \leq 0,$$

for all $((s,s'),a,\omega)$. Thus, the claim. ■

Proof of theorem 6

Let $\Theta = \{\theta \in \mathbb{R}_+^{L\Omega} : \sum_{\omega\in\Omega}\sum_{\ell=1}^{L}\theta_{\ell\omega} = 1\}$. For $\theta \in \Theta$, the pricing map p_θ is defined as:

$$p_\theta^i \equiv \theta T^i. \tag{23}$$

Then, $p_\theta \in P$, for $\theta \in \Theta$. Also let Ξ be a correspondence from the product $\Theta \times \Delta(C)^I$ to itself defined by $\Xi = \Xi_1 \times \Xi_2$ where

$$\Xi_1(\sigma) \equiv \text{ argmax }_{\eta\in\Theta} \sum_i \lambda^i(p_\eta^i \sigma^i) \tag{24}$$

and

$$\Xi_2^i(\theta) \equiv \text{ argmax }_{\xi\in\Delta(C)\cap IC^i} U^i\xi \text{ subject to } p_\theta^i \xi \leq 0. \tag{25}$$

By assumption (5), the correspondences $\Xi_2^i(\theta)$, $i \in I$, are upper-hemi continuous, convex (and compact valued). The same conditions are obviously satisfied by the correspondence $\Xi_1(\sigma)$. Thus, the conditions of Kakutani's fixed point theorem are satisfied. Take a fixed point $(\hat{\theta},\hat{\sigma})$: we claim that the pair $(p_{\hat{\theta}},\hat{\sigma})$ is a competitive equilibrium. The optimality of consumer's choice follows from the definition of Ξ_2^i for every i, so $\hat{\sigma}^i$ satisfies condition (14) at $p_{\hat{\theta}}$, for all i. By assumption (4), for every type i, $\hat{\sigma}_i$ satisfies at $p_{\hat{\theta}}^i$ the budget constraint with equality, that is:

$$p_{\hat{\theta}}^i \sigma^i = \hat{\theta}T^i\hat{\sigma}^i = 0. \tag{26}$$

and hence

$$\sum_{i\in I}\lambda^i\hat{\theta}T^i\hat{\sigma}^i = 0.$$

Suppose now that for some pair (ℓ^*,ω^*)

$$\sum_{i\in I}\lambda^i\sum_{s,a,z} z_{\ell^*}(s,\omega)q^i(s;a,\omega)\hat{\sigma}^i(a^*,z) > 0.$$

Pick $\bar{\theta}$ defined as $\bar{\theta}(\ell^*,\omega^*) = 1$, while $\bar{\theta}(\ell,\omega) = 0$, for $(\ell,\omega) \neq (\ell^*,\omega^*)$. Then, $\bar{\theta} \in \Theta$, but $\sum_i \lambda^i\bar{\theta}T^i\hat{\sigma}^i = \sum_{i\in I}\lambda^i\sum_{s,a,z} z_{\ell^*}(s,\omega)q^i(s;a,\omega)\hat{\sigma}^i(a^*,z) > 0$, and therefore, $\sum_i \lambda^i\bar{\theta}T^i\hat{\sigma}^i > \sum_i \lambda^i\hat{\theta}T^i\hat{\sigma}^i = 0$ contradicting $\hat{\theta} \in \Xi_1(\hat{\sigma})$. Thus, $\hat{\sigma}$ is feasible, that is, $\sum_{i\in I}\lambda^i\hat{\theta}T^i\hat{\sigma}^i = 0$ or equivalently $(\lambda^i\hat{\sigma}^i)_{i\in I} \in Y$. Finally, since $\hat{\theta} \in \Theta$, $p_{\hat{\theta}} \in P$ and therefore the profits of the firm are zero at $p_{\hat{\theta}}$. Therefore, $\hat{\beta} \equiv (\lambda_i\hat{\sigma}_i)_{i\in I}$ satisfies condition (15) at $p_{\hat{\theta}}$, and trivially $\hat{\beta}$ and $\hat{\sigma}$ satisfy condition (16). Thus, $(\hat{p},\hat{\sigma},\hat{\tau})$ is a competitive equilibrium. ■

Proof of Theorem 7

Suppose, by contradiction, that a feasible and incentive compatible joint lottery $\tau = (\tau_i)_{i \in I}$ Pareto dominates the equilibrium allocation. By the local non-satiation assumption (4), at the competitive price $\hat{p}$, $\hat{p}^i \tau^i \geq 0$, for all i , with at least one strict inequality. Since $\hat{p} \in P$, $\hat{p}^i = \hat{\theta} T^i$. Multiply the budget constraint of every type by λ^i and add to get:

$$\hat{\theta} \sum_i \lambda^i T^i \tau^i > 0$$

and therefore τ is not feasible. ∎

Proof of Lemma 11

Let $B^i(p^i) = \{\tau^i \in \Delta(C) : \hat{p}^i \tau^i \leq 0\}$ be the budget set. Observe that if $\hat{p}^i > p^i$, then $B^i(\hat{p}^i) \subset B^i(p^i)$. If σ^{*i} is price supportable, (id, σ^{*i}) is an optimal solution to the programming problem 17 at some $p^{*i} \in P^i(\sigma^{*i})$. The latter means that the lotteries $\tau^i \in [\tau_\phi(p^{*i})]$, the set of optimal solutions to the programming problems $\{\max_{\beta^i \in \Delta(C)} U^i(\phi)\beta^i,\, \beta^i \in B^i(p^{*i})\}$, satisfy $U^i(\phi)[\tau_\phi(p^{*i})] \leq U^i \sigma^{*i}$, for all $\phi \in \Phi^i$. Furthermore, since $B^i(\hat{p}^i) \subset B^i(p^i)$, for $\hat{p}^i > p^i$, then $U^i(\phi)[\tau_\phi(p^i)]$ is non increasing in p^i. Set $\bar{p}^i = \max_{c \in C \setminus C(\sigma^i)} p^{*i}(c)$ and observe that $\bar{p} \in P'^i(\sigma^{*i})$ and that at such a price $\bar{p} \in P'^i(\sigma^{*i})$, σ^{*i} and the multiplier $\alpha = 1$ satisfy the necessary and sufficient first order conditions (19). Thus, σ^{*i} is an optimal solution to 18 at $\bar{p}^i$. Moreover, since $U^i(\phi)[\tau_\phi(p^i)]$ is non increasing in p^i and since $\bar{p}^i > p^{*i}$, $U^i(\phi)[\tau_\phi(\bar{p}^i)] \leq U^i(\phi)[\tau_\phi(p^{*i})] \leq U^i \sigma^{*i}$. Thus the claim. ∎

Proof of Proposition 12

Since prices are personalized, it suffices to show that the claim holds true for just one type i.

Lemma 20 *For all $\bar{p} \in P'^i(\sigma^{*i})$, the function $V^i(\bar{p})$ satisfies: $V^i_\phi(\bar{p}) \geq V^i_\phi$, $V^i(\phi, \bar{p})$ is non increasing in $\bar{p}$, $\lim_{\bar{p} \to +\infty} V^i_\phi(\bar{p}) = V^i_\phi$.*

Proof: As already observed in the proof of Lemma 11, $V^i_\phi(\bar{p})$ is non increasing in $\bar{p}$, for all ϕ. Furthermore, for $\bar{p} \in P^i(\sigma^{*i})$, since both $\bar{p}\mu^i_\phi(\bar{p}) \leq 0$ and $0 \leq \mu^i_\phi(\bar{p})(c) \leq 1$, for all $c \in C$, and $\mu^i_\phi(\bar{p}) \in [\mu^i_\phi(\bar{p})]$, the budget constraint implies that

$$0 \leq \sum_{c \in C \setminus C(\sigma^{*i})} \mu^i_\phi(\bar{p})(c) \leq \frac{-\sum_{c \in C(\sigma^{*i})} (U^i(c) - U^i \sigma^{*i}) \mu^i_\phi(\bar{p})(c)}{\bar{p}}.$$

By the Maximum Theorem, $V^i_\phi(\bar{p})$ is a continuous function in $\bar{p}$, while $[\mu^i_\phi(\bar{p})]$ is upper hemi continuous. Thus, since as $\bar{p} \to +\infty$, $\sum_{c \in C \setminus C(\sigma^{*i})} \mu^i_\phi(\bar{p})(c) \to 0$, therefore, $\lim_{\bar{p} \to +\infty} V^i_\phi(\bar{p}) = V^i_\phi$. ∎

By Lemma 20 and the continuity in $\bar{p}$ of the function $V^i_\phi(\bar{p})$, if $V^i(\phi) < U^i\sigma^{*i}$, there exists a scalar $\bar{p}_\phi > 0$ such that $V^i_\phi(\bar{p}) \leq U^i\sigma^{*i}$, for all $\bar{p} \geq \bar{p}_\phi$. Thus, in order to conclude the argument we need to show that such a scalar $\bar{p}$ exists even when $V^i(\phi) = U^i\sigma^{*i}$ and $\phi \notin \Phi^i(\sigma^{*i})$.

Let $\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})$ be given. Consider the programming problem (20). Recall that $C(\phi) = \{c : \mu(c) > 0, \text{ for some } \mu^i_\phi \in [\mu^i_\phi]\} \subset C(\sigma^{*i})$. Then the following necessary and sufficient conditions hold true for all $\mu^i_\phi \in [\mu^i_\phi]$ and some multiplier $\alpha_\phi \geq 0$:

$$\begin{aligned}
U^i(\phi(c)) - V^i(\phi) &= \alpha_\phi(U^i(c) - U^i\sigma^{*i}), \text{ if } c \in C(\phi), \qquad (27)\\
U^i(\phi(c)) - V^i(\phi) &\leq \alpha_\phi(U^i(c) - U^i\sigma^{*i}), \text{ if } c \in C(\sigma^{*i})\backslash C(\phi),\\
\sum_{c \in C(\sigma^{*i})} (U^i(c) - U^i\sigma^{*i})\mu^i_\phi(c) &\leq 0.
\end{aligned}$$

By the definition of the set $\Phi^i \backslash \Phi^i(\sigma^{*i})$, it is $U^i(\phi(c)) \leq U^i(c)$, for all $c \in C(\sigma^{*i})$. Then, since $V^i_\phi = U^i\sigma^{*i}$, conditions 27 are satisfied by setting $\alpha_\phi = 1$. Now define

$$\bar{p}_\phi = \max_{c \in C \backslash C(\sigma^{*i})} U^i(\phi(c)) - U^i\sigma^{*i}.$$

Lemma 21 *Let $\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})$ be such that $V^i_\phi = U^i\sigma^{*i}$. Then, $V^i_\phi(\bar{p}) = U^i\sigma^{*i}$, for all $\bar{p} \geq \bar{p}_\phi$.*

Proof: It suffices to show that at $\bar{p}_\phi$, the first order conditions of the programming problem 21 are solved by the multiplier $\alpha_\phi = 1$ and the value $V^i(\phi)$, that is, we need to show that the following inequalities are satisfied:

$$\begin{aligned}
U^i(\phi(c)) - V^i(\phi) &= (U^i(c) - U^i\sigma^{*i}), \text{ if } c \in C(\phi),\\
U^i(\phi(c)) - V^i(\phi) &\leq (U^i(c) - U^i\sigma^{*i}), \text{ if } c \in C(\sigma^{*i})\backslash C(\phi),\\
U^i(\phi(c)) - V^i(\phi) &\leq \bar{p}_\phi, \text{ if } c \in C \backslash C(\sigma^{*i}),\\
\bar{p}_\phi \mu^i_\phi &= \sum_{c \in C(\sigma^{*i})} (U^i(c) - U^i\sigma^{*i})\mu^i_\phi(c) \leq 0.
\end{aligned}$$

The first and second conditions, and the budget constraints are nothing else than 27 and therefore they hold true. The third set of inequalities are satisfied by the definition of the scalar $\bar{p}_\phi$. Thus, $[\mu^i_\phi] \subset [\mu^i_\phi(\bar{p}_\phi)]$ and therefore $V^i_\phi(\bar{p}_\phi) = V^i(\phi)$. ∎

We have shown that for each $\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})$, there exists $\bar{p}_\phi$ such that $V^i_\phi(\bar{p}) \leq U^i\sigma^{*i}$, for all $\bar{p} \geq \bar{p}_\phi$. Therefore, $V^i_\phi(\bar{p}) \leq U^i\sigma^{*i}$, for all $\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})$ and for $\bar{p} \geq \max_{\phi \in \Phi^i \backslash \Phi^i(\sigma^{*i})} p_\phi$. ∎

Proof of Proposition 14

($\Longrightarrow$) By Lemma 11, if σ^{*i} is price supportable, (id, σ^{*i}) is an optimal solution to the programming problem (17) at some $\bar{p}^i \in P'^i(\sigma^{*i})$. Then, $V^i_\phi = U^i\sigma^{*i}$, for all $\phi \in \Phi^i(\sigma^{*i})$, and, therefore, $[\mu^i_\phi] \subset [\mu^i_\phi(\bar{p}^i)]$, for all $\phi \in \Phi^i(\sigma^{*i})$, that is, $[\mu^i_\phi]$ is an optimal solution to the programming problem (21) at $\bar{p}^i \in P^i(\sigma^{*i})$.

Then, the following necessary and sufficient conditions are satisfied for some $\alpha_\phi \geq 0$:

$$\begin{aligned}
U^i(\phi)\mu^i_\phi - U^i\sigma^{*i} &= b_\phi[U^i(c) - U^i\sigma^{*i}], \text{ if } c \in C(\mu^i_\phi), \\
U^i(\phi(c)) - U^i\sigma^{*i} &\leq b_\phi[U^i(c) - U^i\sigma^{*i}], \text{ if } c \in C(\sigma^{*i})\backslash C(\mu^i_\phi), \\
U^i(\phi(c)) - U^i(\phi)\mu^i_\phi &\leq b_\phi\bar{p}^i, \text{ if } c \in C\backslash C(\sigma^{*i}), \\
p^i\mu^i_\phi &\leq 0.
\end{aligned}$$

Since $U^i(\phi)\mu^i_\phi = U^i\sigma^{*i}$, if $b_\phi > 0$, the claim is proved. Hence, we just have to rule out the case $\alpha_\phi = 0$. Consider the contract c^1 defined in assumption 10. If $U^i(\phi(c^1)) = V^i_\phi = U^i\sigma^{*i}$, then assumption 10 is violated. Thus, $U^i(\phi(c^1)) - U^i\sigma^{*i} > 0$ and, therefore, $b_\phi > 0$.

($\Longleftarrow$) First we show that the equations $U^i(\phi(c)) - U^i\sigma = b_\phi[U^i(c) - U^i\sigma^{*i}]$, for $c \in C^i(\phi)$, and $b_\phi > 0$, implies that $V^i(\phi) = U^i\sigma^{*i}$. Indeed, pick any $\mu^i_\phi \in [\mu^i_\phi]$, then $C^i(\phi) \subset C(\phi)$ and therefore

$$\sum_{c\in C^i(\phi)} \mu^i_\phi(c)[U^i(\phi(c)) - U^i\sigma^{*i}] = V^i(\phi) - U^i\sigma^{*i} = b_\phi \sum_{c\in C^i(\phi)} \mu^i_\phi(c)[U^i(c) - U^i\sigma^{*i}]$$

Since $\phi \in \Phi^i(\sigma^{*i})$, $V^i(\phi) - U^i\sigma^{*i} \geq 0$, and since $\mu^i_\phi \in [\mu^i_\phi]$, then, by the budget constraint, $\sum_{c\in C^i(\phi)} \mu^i_\phi(c)[U^i(c) - U^i\sigma^{*i}] \leq 0$. Therefore, $V^i_\phi = U^i\sigma$. Now let $\bar{p}_\phi = \max_{c\in C\backslash C(\sigma^{*i})} \frac{U^i(\phi(c)) - V^i_\phi}{b_\phi}$. As already argued for all $\bar{p} \geq \bar{p}_\phi$, $V^i_\phi(\bar{p}) = V^i_\phi$. Thus, for $\bar{p} \geq \max_{\phi\in\Phi^i(\sigma^{*i})} \bar{p}_\phi$, $V^i_\phi(\bar{p}) = V^i_\phi$, for all $\phi \in \Phi^i(\sigma^{*i})$. Since we have shown that $V^i_\phi = U^i\sigma^{*i}$, for all $\phi \in \Phi^i(\sigma^{*i})$, the latter concludes the argument.■

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