Economies with observable types

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1. Introduction

Almost twenty five years after the publication of the seminal analysis of competitive economies with asymmetric information of Prescott and Townsend (1984a, 1984b), some confusion persists on the fundamental issues. In fact, as we are going to argue below, even the fundamental definitions are not clear. Here we hope to clarify two points.

First, in spite of the large variety of models for such economies, one can give a common treatment of a large class of them. We study economies with observable types, that is, economies of asymmetric information where types are observable, but actions or personal states or both are private information of the individuals. We reduce the analysis of these economies to that of economies with individual risk as in Malinvaud (1972, 1973). Thus for existence of equilibrium and Welfare Theorems one does not need to consider separately special subclasses (see for example Bennardo and Chiappori, 2003; Gottardi and Jerez, 2007). Economies of moral hazard and private information are particular cases of this general framework.

Second, the way in which the incentive compatibility (IC) constraints are defined has fundamental implications not only on the nature of the prices (linear or non-linear), but also on the existence and optimality properties. Imposing the IC constraints on the firm instead of on the consumers may make empty the set of equilibria. To see why this is the case, recall that the crucial step of Prescott and Townsend is the introduction of the IC constraints on individual choices. Jerez (2005) observes that this restriction is conceptually problematic by arguing that “the natural interpretation is to view the incentive-compatibility constraints as restrictions on the set of contracts that firms can offer to consumers.” Jerez (2003, 2005) poses an important question. Can we model competitive economies of asymmetric information removing the IC constraints from...
individual choices and placing them on the firm only? What are the properties of the equilibrium set of this economy? Jerez claims that this way of modeling the economy delivers the same allocation set of Prescott and Townsend, although naturally, supporting prices may now be non-linear in commodities, but linear in lotteries. This is wrong, because the equilibrium set may now be empty.

1.1. The root of the problem

The equilibrium set can be empty for two different reasons. First, prices making constrained efficient allocations individuals optimal choices (that is, supporting prices) may fail to exist. Second, even when supporting prices do exist, the firm may find it more profitable to supply something else at those prices. Let us start with the first. Pick a constrained efficient joint lottery that satisfies two properties. For some type, the IC are binding and the lottery gives positive probability to some non-incentive compatible contract. Then, there exists a manipulation that when applied delivers with the same lottery the same utility level of truthful behavior. Thus, the supporting price has now to solve two, possibly incompatible tasks: It has to make the prescribed lottery optimal among the budget feasible lotteries when the individual behaves truthfully and when he optimally manipulates. However, the manipulation transforms the utility function over the contracts into the utility functions over the manipulated contracts, and the latter can have a very different shape. There is no reason why a unique price should support the same lottery with two very different utility functions. It is easy to find necessary and sufficient conditions for price supportability and, then, to find, constrained efficient allocations that violate them. The next step is to build an economy with one type and one constrained efficient allocation that violates the necessary and sufficient conditions. This economy has obviously an empty set of equilibria.

As we said, even economies with price supportable constrained efficient allocations may fail to have equilibria. Consider a constrained efficient lottery. Look for a candidate price that makes it an optimal choice of the individuals and a profit maximizing choice of the firm. Why do individuals at that price do not put any weight on contracts outside the support of the prescribed lottery? Because those contracts are too expensive at the candidate price. Hence, the shape of the utility function puts lower bounds on the values of the candidate supporting price calculated at the contracts outside of the lottery support. However, the profit maximizing nature of the prescribed lottery puts upper bounds on those values, the contracts are too cheap to be supplied. These two restrictions, one coming from the individuals and the other from the firm, can make the set of candidate equilibrium prices empty. We show an economy of private information with one type and one constrained efficient allocation where these two requirements collide. The support of the constrained efficient allocation contains only IC contracts. Thus, the constrained efficient allocation is price supportable, but no price that makes the constrained efficient allocation an optimal choice of the individuals makes it a profit maximization choice of the firm. This economy too has an empty set of equilibria.

There is nothing one can do about the first problem (lack of price supportability), but we can fix the second: if we restrict the firm to supply measures over incentive compatible contracts, then the equilibrium exists and is constrained efficient in the set of feasible lotteries over incentive compatible contracts. We now define formally our problem.

2. Economies with asymmetric information

We study finite economies: states, actions, types and trades are a finite set.

2.1. The economy

Individuals in the economy belong to one of a finite set of types, \( I \equiv \{1, \ldots, n\} \). For each type \( i \) there is a large population of size \( \lambda_i \), with \( \lambda_i > 0 \) and \( \sum_{i=1}^n \lambda_i = 1 \). Each individual chooses an action, \( a \), in a finite set \( \mathcal{A} \). Individuals face aggregate and personal uncertainty. \( \Omega \) is a finite set of states of nature that affect every agent in the economy. Each state \( \omega \) occurs with a fixed probability \( \rho(\omega) \). A personal state \( s \) out of a finite set \( S \) is realized, one for each individual. The probability of such realization depends on type, action and state of nature. For every finite set \( M \), \( \Delta(M) \) denotes the set of probability vectors on \( M \). For each type, action and aggregate state, there exists a probability vector over the set of personal states: that is, a \( q_i(\cdot; a, \omega) \in \Delta(S) \) is given for every \( i \in I \) and every \( (a, \omega) \in \mathcal{A} \times \Omega \). Individuals exchange goods according to a finite set of individual net trades, \( X^i ; 0 \in X^i \), so no trade is always an option. \( X^i \) is a subset of the Euclidean space \( \mathbb{R}^L \), where \( L \geq 1 \) is the number of physical commodities. The preferences of type \( i \) are represented by a utility function \( u^i : X^i \times \Omega \to \mathbb{R} \), for every \( i \in I \).

2.2. Contracts

The set of net trade policies is the set \( Z^i \) of state contingent net trades. It is the finite set of maps \( z : S \times \Omega \to X^i \). The set of contracts is the set \( C^i \) of pairs of action and net trade policy. A contract \( c = (a, z) \) assigns an action \( a \) and stipulates the provision of a state contingent net trade \( z \), which describes for every realization of the pair of states \( (s, \omega) \) a net trade vector \( z(s, \omega) \in X^i \). The set of contracts is finite. The utility function \( v^i \) induces a utility function \( u^i \) over \( C^i \) that takes the expected utility form.
2.3. Feasible lottery profiles

Lotteries on deterministic contracts are also traded: a lottery \( \tau \) is an element of \( \Delta(C^i) \). A different description of a lottery is given by a pair \((\tau_1, \tau_2)\) where \( \tau_1 \in \Delta(A^i) \) and \( \tau_2 \) is a vector \((\sigma_{b\omega})_{b\omega \in \Lambda}\) of conditional probabilities on \( Z^i \), one for each action. The two descriptions are equivalent: for every \( \tau \in \Delta(C^i) \) there is a pair \((\tau_1, \tau_2)\), and vice versa. A lottery profile is a vector \( \sigma = (\sigma^i)_{i \in I} \), assigning the same lottery \( \sigma^i \) to each individual of type \( i \). Individual utility functions are extended over the set of lotteries \( \Delta(C^i) \) assuming that they are linear in lotteries. Let \( U^i \) denote the row vector of dimension \( 1 \times C^i \) with entries \( u^i(c) \), \( c \in C^i \). Using the convention that individual lotteries are column vectors, the utility of an individual of type \( i \) generated by a lottery \( \sigma^i \) is

\[
U^i \sigma^i \equiv \sum_{c \in C^i} u^i(c) \sigma^i(c).
\]

All the economies we consider are economies with individual risk (as in the classical analysis of Malinvaud (see Malinvaud, 1972, 1973), we model the individual risk with the variable \( s \)). The models differ for the information publicly available. This information may be different for two variables: the action \( a \) and the personal state \( s \). By making each of these variables private information of the individuals we obtain different types of economies.

2.4. Information and time

By the law of large numbers, a fraction \( q^i(s; a, \omega) \) of type \( i \) individuals that have adopted the action \( a \) is at each aggregate state \( \omega \) in personal state \( s \). Thus a lottery profile \( \sigma = (\sigma^i)_{i \in I} \) is feasible if for every commodity \( \ell \) and aggregate state \( \omega \) the sum of net trades is not positive:

\[
\sum_{i \in I} \lambda^i (a, z) \sum_{s \in S} q^i(s; a, \omega) z^i(s, \omega) \leq 0.
\]

To have a more compact notation, for \((a, z) \in C^i, \omega \in \Omega \) and \( i \in I \), let

\[
T^i((a, z); \omega) \equiv \sum_{s \in S} q^i(s; a, \omega) z^i(s, \omega)
\]

be the column vector of dimension \( L \times 1 \) of type \( i \) aggregate net trade in state \( \omega \) generated by \((a, z)\). Let \( T^i(\omega) \) to be the matrix of dimension \( L \times C^i \) whose columns are the vectors \( T^i((a, z); \omega) \), \((a, z) \in C^i \), and finally let \( T^i \) be the matrix of dimension \( L \Omega \times C^i \) obtained by stacking together the matrices \( T^i(\omega), \omega \in \Omega \). Thus, the feasibility condition (2) can be rewritten, with \( \lambda^i(0) = 0 \in \mathbb{R}^{L \Omega} \):

\[
\sum_{i \in I} \lambda^i T^i \sigma^i = \left(\sum_{i \in I} \lambda^i T^i(\omega)\sigma^i\right)_{\omega \in \Omega} \leq 0.
\]

2.4. Information and time

The type of an individual is publicly observed. The public information about either personal states or actions of individuals of type \( i \) is either completely revealing (the variable is observed) or completely non-revealing. When a variable is publicly observed, individuals have to be truthful: the action chosen is the prescribed action if the action is observed and the reported personal state is the true state if the state is observed. The complete time sequence of events is the following. Individuals trade, and get a lottery \( \tau \). The action \( a \) is chosen according to the lottery \( \tau_1 \), and this outcome is communicated to the individual. The individual chooses the action \( b \), possibly different from \( a \). The public information available on the chosen action \( b \) is revealed. Then first the state of nature \( \omega \) is determined, and the personal state is realized, according to \( q^i(\cdot; b, \omega) \). The personal state \( s \) is communicated, and the public information on \( s \) is revealed. Then individuals report the personal state \( t \), possibly different from \( s \). Finally the aggregate state is communicated, and the net trade \( z^i(t, \omega) \) realizes according to \( \tau_2(\cdot; a) \). Obviously, we could envision different time sequence of events. For instance, the personal state could be revealed before the action is chosen. However, the results of this paper do not depend on the adopted time sequence.

We now illustrate how the different cases are all a special case of our general framework.

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1 For two vectors \( x \) and \( y \) of identical dimension, we write \( x \succ y \) to indicate that \( x \) is strictly larger than \( y \) in every component; \( x > y \) to indicate that \( x \) is strictly larger than \( y \) for at least one component; and \( x \geq y \) to indicate that \( x \) is greater or equal than \( y \) in every component.
2.5. Economies with individual risk

This is the basic model. In this economy all variables (action chosen by the individual and personal states) are observed, so there is no incentive compatibility constraint. The set of incentive compatible lotteries is \( IC^i \equiv \Delta(C^i) \), for all \( i \in I \), and the set of incentive compatible joint lotteries is the product over types of the set of lotteries \( IC = \times_{i \in I} IC^i \). As known (see Malinvaud, 1972, 1973; Cass et al., 1994), an equilibrium with type dependent fair prices, where each type \( i \) faces a different price \( p^i \), exists.

2.6. Economies with moral hazard

In these economies the personal states are observed, while the action is private information. Recall that by the adopted time sequence of events, the prescribed action is communicated, and chosen before the net trade policy, the personal and the aggregate states are revealed; so the individual can make his action choice depend only on the prescribed action. Thus, the set of incentive compatible lotteries for each type \( i \) is the set of lotteries such that the individual of each type indeed prefers the action assigned by the lottery to any other action. Lottery \( \sigma \) is incentive compatible if

\[
\sum_z \sigma(a, z)[u^i(a', z) - u^i(a, z)] \leq 0, \quad \text{for all } a' \in A^i. \tag{4}
\]

This is the standard representation of (ad-interim) IC constraints for moral hazard economies as in Bennardo and Chiappori (2003), Jerez (2003, 2005), and Prescott and Townsend (1984a). Again, for convenience, we rewrite the IC constraints by making explicit how an individual type can “manipulate” any given contract. Let \( \Phi^i \) be the finite set of all functions from \( A^i \) to \( A^i \). Each \( \phi \) corresponds to a manipulation, that is, to a list of deviations from the prescribed actions. For \( \phi \in \Phi^i \), define the manipulated contracts \( \phi(c) = (\phi(a), z) \) and then the utility function \( U^i(\phi) \) as

\[
U^i(\phi) = (u^i(\phi(c)))_{c \in \Phi^i} = (u^i(\phi(a), z))_{(a, z) \in \Phi^i}.
\]

Then, the set of incentive compatible lotteries for type \( i \) can be written as:

\[
IC^i \equiv \{ \tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi^i \}, \tag{5}
\]

and it is a non-empty, closed and convex subset of \( \Delta(C^i) \). Our formulation of the set of incentive compatible lotteries is identical to the ad-interim representation adopted in the literature, that is, to the set of lotteries satisfying inequality (4). This is shown in the next lemma.

Lemma 1. Let \( IC^i_+ \equiv \{ \sigma \in \Delta(C^i) : \sigma \text{ satisfies (4)} \} \), then \( IC^i = IC^i_+ \).

The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is \( IC = \times_{i \in I} IC^i \).

2.7. Economies with private information

In these economies actions are observed, while the realization of the personal state is not. Recall that individuals report the personal state after the observable action has been reported and the aggregate state of nature has been realized. Hence, a lottery \( \sigma \) is incentive compatible if for all \( (a, \omega) \in A^i \times \Omega \)

\[
\forall s' \in S \sum_z \sigma(a, z)q^i(s; a, \omega)(v^i(a, s, \omega, z(s', \omega)) - v^i(a, s, \omega, z(s, \omega))) \leq 0. \tag{6}
\]

This is the standard formalization of the IC constraints of economies of private information, as for instance in Kehoe et al. (2002), Prescott and Townsend (1984a) and Prescott and Townsend (1984b). As before, we find convenient to rewrite the IC constraints by making explicit how individuals can manipulate contracts. Since in our private economy, individuals can misreport the personal state realization, given the information they have on the action and the aggregate state, for any function \( \phi : A^i \times \Omega \times S \rightarrow S \), they can manipulate a contract \( c = (a, z) \) into a contract \( \phi(c) = (a, \phi(z)) \), where \( \phi(z)(a, s, \omega) \equiv z(a, \phi(a), s, \omega) \). Let \( \Phi \) be the finite set of all functions from \( A^i \times \Omega \times S \) to \( S \). For any \( \phi \in \Phi \), let

\[
U^i(\phi) = (u^i(\phi(c)))_{c \in \Phi} = (u^i(a, \phi(z)))_{(a, z) \in \Phi}.
\]

The set of incentive compatible lotteries for each type \( i \) can be written as:

\[
IC^i \equiv \{ \tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi \}. \tag{7}
\]

This set too is non-empty, closed and convex. Our formulation of the set of incentive compatible lotteries is identical to the interim representation adopted in the literature, that is, to the set of lotteries satisfying inequalities (6).
Lemma 2. Let $IC^i = \{\sigma \in \Delta(C^i) : \sigma \text{ satisfies (6)}\}$, then $IC^i = IC_{ig}^i$.

The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is $IC = \times_{i \in I} IC_i$.

2.8. General economies

The most general formulation allows different types to have different variables of private information. For instance, for some types, actions or a subset of actions may be not observable, while for others, the same may hold true for personal states. Also, it may be possible that for some types, the variables of private information may be subsets of the both actions and personal states. Thus, in its most general formulation, the set of manipulations available to type $i$ is $\Phi^i$, a subset of the maps $\phi : A^i \times S \rightarrow A^i \times S$, and the set of incentive compatible lotteries for type $i$ can be written as:

$$IC^i \equiv \{ \tau : (U_i(\phi) - U_i)^\tau \leq 0, \text{ for every } \phi \in \Phi^i \}.$$  \hspace{1cm} (8)

For any specification of $\Phi^i$, $IC^i$ is non-empty, closed and convex. It should be evident by now that this way of formulating the IC constraints is nothing else than the standard ad-interim formulation used in the literature.

3. Equilibrium

The Prescott and Townsend construction identifies the consumption set of the individuals with the set of incentive compatible lotteries. Thus the consumption set is convex, while preferences are linear. This formulation makes economies of observed types with asymmetric information isomorphic to standard general equilibrium economies with individual risk. Under standard and well understood assumptions, the equilibrium set is nonempty and the Welfare Theorems hold true. This construction, however, requires an exogenous restriction on consumption sets. We follow the alternative construction of Jerez (2003, 2005): the IC constraints are removed from the consumption set of the individuals, while the profit maximizing firm is restricted to supply incentive compatible allocations. Let $IC^*$ denote the set of IC vectors:

$$IC^* = \times_{i \in I} \{ \beta^i \in R_{+}^{C^i} : (U_i(\phi) - U_i)^{\beta^i} \leq 0, \text{ for every } \phi \in \Phi^i \}.$$  \hspace{1cm} (9)

Notice that $IC \subset IC^*$, since the former is a subset of the joint probability distributions over contracts, while the latter is a subset of positive measures (vectors) over the same domain. The production set of the firm $Y_U$ is identified with $Y_U = Y \cap IC^*$, for $Y \equiv \{(\beta^i)_{i \in I} \in \mathbb{R}^{\sum C_i} : \sum_i \beta^i \leq 0\}$ denoting the set of feasible vectors. Since the production set of the firm is a cone, at equilibrium profits must be zero. Then, the latter restricts the price domain to $P_U$, the set of prices generating zero profits:

$$P_U = \left\{ \hat{p} \in R_{+}^{\sum C_i} : \sum_i \hat{p}^i \beta^i \leq 0, \text{ for all } (\beta^i)_{i \in I} \in Y_U \right\}.$$  \hspace{1cm} (10)

The formalization of the individual behavior is our point of departure from Jerez’s analysis. In our economies, individuals may prefer to take an action different than the action prescribed by the lottery and/or report a personal state different than the one effectively realized. Thus, when evaluating a lottery at the ex-ante trading stage, an individual anticipates that lottery can be optimally manipulated at the ad-interim stage. Formally, the ex-ante utility attached to a lottery $\tau$ is $V^i(\tau) = \max_{\phi \in \Phi^i} U_i(\phi)\tau$. The manipulation map $\phi$ prescribes a deviation for each action recommended or personal state realized, that is, it prescribes a contingent plan of deviations. Naturally, manipulations that are optimal ex-ante prescribe a collection of deviations that are optimal at the ad-interim stage. Thus, preferences as described by $V^i$ are time consistent. Equivalently, the ex-ante utility function is just the expected interim utility function computed at the optimal ad-interim deviations, where the expectation is taken over the elements of the domain of the manipulation maps. To avoid any possible confusion, let us illustrate this point within a pure moral hazard economy. The utility function $V^i(\tau)$ is $V^i(\tau) = \max_{a \in A^i} \sum_{a'} U_i(a', z)T_2(z; a)$. Equivalently, $V^i(\tau) = \max_{\phi \in \Phi^i} U_i(\phi)\tau$, where $\phi^* \in \arg \max_{\phi \in \Phi^i} U_i(\phi)\tau$ is a contingent plan of the optimal ad-interim deviations $\phi^*(a)$ solution of the problem $\max_{\phi \in \Phi^i} \sum U_i(a', z)T_2(z; a)$. Thus the ex-ante utility function $V^i$ is just the expected value of the ad-interim utility functions $\max_{\phi} \sum U_i(a', z)T_2(z; a)$ evaluated at the optimal ad-interim deviation. These conclusions generalize to any general economy with observable types. Hence, for given $p \in P_U$, the maximization problem of the type $i$ individual is:

$$\max_{\phi \in \Phi^i, \sigma \in \Delta(C^i)} U_i(\phi)\sigma^i, \text{ subject to } \hat{p}^i \sigma^i \leq 0.$$  \hspace{1cm} (11)

Denoting by $id \in \Phi^i$ the identity or truth telling map, a competitive equilibrium is defined as follows.

Definition 3. A competitive equilibrium is an array $(\hat{p}, \hat{\sigma}, \hat{\tau})$ such that:

\begin{enumerate}
  \item $\hat{p} = (\hat{p}^i)_{i \in I} \in P_U$;
\end{enumerate}
(2) For every type $i$, $(\text{id}, \hat{\sigma}^i)$ is the solution of the consumer problem (9) at $\hat{p}^i$;
(3) $\hat{\tau} = (\hat{\tau}^i)_{i \in I}$ is profit maximizing at $\hat{p}$, i.e., $(\hat{\tau}^i)_{i \in I} \in \arg \max_{\hat{p} \in \mathbb{R}^{|\mathcal{C}|}} \sum_i \hat{p}^i \hat{\tau}^i$;
(4) Markets clear, that is: $\hat{\tau}^i = \lambda_i \hat{\sigma}^i$, for all $i \in I$.

By the definition of $Y_U$, the market clearing condition implies that the equilibrium allocation $\hat{\tau} = (\lambda_i \hat{\sigma}^i)_{i \in I}$ is feasible according to Definition 3.

The competitive equilibrium allocations satisfy the First Fundamental Theorem of Welfare Economics: if an equilibrium exists, its allocation is constrained efficient. Indeed, suppose otherwise. Then, there exists a collection of lotteries $\tau = (\lambda_i \hat{\sigma}^i)_{i \in I} \in Y_U$ that Pareto dominates the competitive allocation $\hat{\tau} = (\lambda_i \hat{\sigma}^i)_{i \in I}$. Individuals do not acquire $\hat{\sigma}^i$ because at $\hat{p}^i$, $\hat{p}^i \hat{\sigma}^i \geq 0$, for $i \in I$, with at least one strict inequality. Thus, $\sum_i \lambda_i \hat{p}^i \hat{\sigma}^i > \sum_i \lambda_i \hat{p}^i \hat{\sigma}^i$. Hence, since $\tau \in Y_U$, the last inequality implies that $(\lambda_i \hat{\sigma}^i)_{i \in I}$ is not a profit maximizing choice at $\hat{p}$. A contradiction.

3.1. Price supportability

We come now to the next fundamental question: whether or not this way of modeling asymmetric information economies with observable types leads to the same conclusions of the Prescott and Townsend construction. Do equilibria exist under standard assumptions? Does the Second Fundamental Theorem of Welfare Economics hold true? Jerez (2005) shows that this is the case, but, in Jerez (2005), individuals do not optimize over $\phi \in \Phi^i$, that is, their objective function in (9) is just $U^i \sigma^i$. Thus, in Jerez’s formulation individuals evaluate lotteries at the ex-ante trading stage without contemplating any deviations even though the lottery under evaluation may not be incentive compatible. It is only at the ad-interim stage that they exploit their private information deviating from the prescriptions of the lottery and/or misreporting personal states if they find convenient to do so. For instance, in the context of a pure moral hazard economy, their ex-ante utility function is

$$
\hat{u}_i(\phi) = \sum_{\tau \in \mathcal{C}} \mathbb{E}_{\phi}[\tau(z)]
$$

where $\mathbb{E}_{\phi}[\tau(z)]$ is the expected value of the ad-interim utility function. Equivalently, individual preferences display time inconsistency.

Unfortunately, once the consistency between ex-ante and ad-interim utility functions is reestablished or, equivalently, once individuals behave according to the programming problem (9), the equilibrium set may be empty and the Second Fundamental Theorem of Welfare Economics may fail. The potential emptiness of the equilibrium set has two different origins. First, constrained efficient allocations may fail to be price supportable. Second, they might be price supportable, but the supporting prices may not make them profit maximizing choices of the firm. We investigate the first issue in the current section and the second in Section 3.3.

By definition, if $(\hat{p}, \hat{\sigma}, \hat{\tau})$ is a competitive equilibrium, then $(\hat{p}, \hat{\sigma})$ must be as well an equilibrium of the pure exchange economy, that is an equilibrium of the economy without the firm. We call the latter a pure exchange equilibrium. Formally:

**Definition 4.** A pure exchange equilibrium is a pair $(\hat{p}, \hat{\sigma})$, with $\hat{p} = (\hat{p}^i)_{i \in I} \in \mathbb{R}^{|\mathcal{C}|}$, such that:

(1) For every type $i$, $(\text{id}, \hat{\sigma}^i)$ is the solution of the consumer problem (9) at $\hat{p}^i$; and
(2) Markets clear, that is, $\sum_i \lambda_i \hat{T}^i \hat{\sigma}^i \leq 0$.

An incentive compatible individual lottery $\sigma^i$ is *price supportable* if $(\text{id}, \sigma^i)$ is a solution to the individual programming problem (9) at some price $p^i$, and a feasible and incentive compatible joint allocation $\sigma$ is *price supportable* if the individual lotteries $\sigma^i$ are *price supportable*, for all $i$. Evidently, if a feasible and incentive compatible joint allocation $\sigma$ is price supportable, then it is an equilibrium allocation of the pure exchange economy.

We provide the intuition for the failure of price supportability of constrained efficient allocations. Suppose that at some constrained efficient allocation $\sigma^i$, the IC constraints for type $i$ are binding. Thus, by definition, there exists a manipulation $\phi \in \Phi^i \setminus \{\text{id}\}$ such that $U^i \sigma^i = U^i(\phi) \sigma^i$. The latter implies that the supporting prices must make $\sigma^i$ the most preferred budget feasible lottery with both the utility function $U^i$ and $U^i(\phi)$. However, $U^i$ and $U^i(\phi)$ may have quite different shapes thereby making impossible to find a common supporting price. Indeed the ex-ante utility function $V^i(\sigma)$ is the objective function appearing in the individual programming problem (9). Since $V^i(\sigma^i) = \max_{\phi \in \Phi^i} U^i(\phi) \sigma^i$, $V^i$ is, by construction, the upper envelope of the (finite) family of linear maps $U^i(\phi) \sigma^i$, $\phi \in \Phi^i$, and, thus, may fail to be concave. If $\sigma^i$ is a non-concave area of the map $V^i$, $\sigma^i$ may not be price supportable.

We now determine necessary and sufficient conditions making a joint, feasible and incentive compatible lottery $\sigma$ price supportable. We assume that preferences $U^i(\phi)$, $\phi \in \Phi^i$, are locally non-satiated. This assumption excludes the possibility that some Kuhn–Tucker multipliers may be zero.

**Assumption 5 (Local non-satiation).** For all types $i \in I$, there is an incentive compatible contract $c^i = (a^i, z^i)$, with $z^i \gg 0$, such that

$$
u^i(\phi(a^i)) > \nu^i(\phi)(a^i)
$$

for all $\phi \in \Phi^i$, for every incentive compatible, and feasible joint lottery $\sigma = (\sigma^i)_{i \in I}$.
Assumption 5 is satisfied if \( z^{i*} \) is very large and there is enough monotonicity of preferences in net trades, as in all examples of private information and moral hazard economies in the literature.

An equilibrium \((p, \sigma^{*})\) of the pure exchange economy must satisfy two requirements. First, given the fact that individuals behave truthfully, \( \sigma^{*} \) is an optimal solution at \( p^{*} \), for all types \( i \in I \), and, second, behaving truthfully is optimal. As already mentioned, it is requirement (2) that is both missing from the analysis in Jerez (2005) and that creates problems.

The utility functions \( U^{i}(\phi) \tau^{i} \) are linear in lotteries. Thus, requirement (1) has a well-known characterization based on the necessary and sufficient conditions for optimality of linear programming problems. Given a price \( p^{i} \), the individual lottery \( \sigma^{*i} \) is an optimal solution to

\[
\max_{\beta^{i} \in \Delta(C)} U^{i}\beta^{i}, \quad \text{subject to} \quad p^{i}\beta^{i} \leq 0
\]

(10)

if and only if there exists a scalar \( \alpha^{i} \geq 0 \) such that, for \( C(\sigma^{*i}) \subset C^{i} \) denoting the support of the lottery \( \sigma^{*i} \):

\[
U^{i}(c) - U^{i}\sigma^{*i} = \alpha^{i} p^{i}(c), \quad \text{if} \ c \in C(\sigma^{*i});
\]

\[
U^{i}(c) - U^{i}\sigma^{*i} \leq \alpha^{i} p^{i}(c), \quad \text{if} \ c \notin C(\sigma^{*i});
\]

\[
p^{i} \sigma^{*i} = 0.
\]

(11)

By Assumption 5, \( U^{i}(c^{i}) - U^{i}\sigma^{*i} > 0 \). Thus, \( \alpha^{i} > 0 \). Therefore, requirement (1) for the optimality of \((id, \sigma^{*})\) restricts the set of prices that can potentially support the lottery \( \sigma^{*i} \), to satisfy conditions (11). We call \( P^{i}(\sigma^{*}) \) the set of prices satisfying (11) and exploiting the homogeneity of the budget constraints we write it, without loss of generality, as

\[
P^{i}(\sigma^{*}) = \{ p^{i}: p^{i} \text{ satisfies (11) for } \alpha^{i} = 1 \}.
\]

The second requirement for the price supportability of \( \sigma^{*i} \) is more complicated. Roughly speaking, in principle, \( \sigma^{*i} \) may fail to be price supportable for two rather different and independent reasons:

(A) there is a manipulation \( \phi \) such that \( U^{i}(\phi)\tau^{i} > U^{i}\sigma^{*i} \), for some \( \tau^{i} \) budget feasible at \( p^{i} \in P^{i}(\sigma^{*i}) \), whose support \( C(\sigma^{*i}) \subset C^{i} \), contained in the support of \( \sigma^{*i} \), i.e., \( C(\tau^{i}) \subset C(\sigma^{*i}) \);

(B) there is a manipulation \( \phi \) such that \( U^{i}(\phi)\tau^{i} > U^{i}\sigma^{*i} \), for some lottery \( \tau^{i} \) budget feasible at \( p^{i} \in P^{i}(\sigma^{*}) \) such that \( p^{i} \sigma^{*i} = 0 \).

By definition, prices in \( P^{i}(\sigma^{*i}) \) coincide over the set of contracts \( C(\sigma^{*i}) \), while they can take different values for contracts outside that set. Therefore if \( \phi \) satisfies (A) for some \( \tilde{p}^{i} \in P^{i}(\phi) \), it does so for all \( p^{i} \in P^{i}(\phi) \). To the contrary, (B) is a price dependent phenomenon: some manipulations \( \phi \) may satisfy (B) for some \( \tilde{p}^{i} \in P^{i}(\phi) \), while may fail to do so for other \( p^{i} \in P^{i}(\sigma^{*}) \). Thus, the search for necessary and sufficient conditions for the price supportability of \( \sigma^{*} \) naturally partitions the set of manipulations \( \Phi^{i} \) into two disjoint and exhaustive subsets, \( \Phi(\sigma^{*}) \) and its complement. Manipulations in \( \Phi(\sigma^{*}) \) are potential candidates for generating (A), while manipulations outside \( \Phi(\sigma^{*}) \) can upset \( \sigma^{*i} \) only through (B), but not (A). More precisely, for given \( i \) and \( \phi \in \Phi^{i} \) consider the following programming problem:

\[
\max_{\mu^{i} \in \Delta(C(\sigma^{*i})))} U^{i}(\phi)\mu^{i}, \quad \text{subject to} \quad \sum_{c \in C(\sigma^{*i})} (U^{i}(c) - U^{i}\sigma^{*i})\mu^{i}(c) \leq 0.
\]

(12)

Let \( [\mu^{i}_{\phi}] \) denote the set of optimal solutions, \( \mu^{i}_{\phi} \) one of its elements, and let \( V^{i}_{\phi} \) be the value of the programming problem, that is, \( V^{i}_{\phi} = U^{i}[\mu^{i}_{\phi}] \). The programming problem (12) restricts choices to lotteries over contracts in the set \( C(\sigma^{*i}) \), and defines the budget set by pricing these contracts with any \( p \in P^{i}(\sigma^{*i}) \). By the definition of \( P^{i}(\sigma^{*i}) \), it is \( V^{i}_{id} = U^{i}(\sigma^{*i}) \). Price supportability requires that \( V^{i}_{\phi} \leq U^{i}(\sigma^{*i}) \), all \( \phi \in \Phi^{i} \). Otherwise, some manipulation \( \phi \) delivers a value, \( V^{i}_{\phi} \), higher than \( U^{i}(\sigma^{*i}) \) with the restricted choice set \( \Delta(C(\sigma^{*i})) \), for each \( p \in P^{i}(\sigma^{*i}) \). Then, a fortiori, the same holds true for the standard unrestricted programming problem (9) whose optimal manipulation must be therefore some \( \phi \neq id \). Thus, the set \( \Phi(\sigma^{*}) \) is naturally defined as the set of non-trivial manipulations yielding values \( V^{i}_{\phi} \) greater or equal than \( U^{i}\sigma^{*i} \), that is:

\[
\Phi(\sigma^{*}) = \{ \phi: V^{i}(\phi) \geq U^{i}\sigma^{*i} \text{ and } U^{i}(\phi(c)) - U^{i}(c) > 0 \text{, for some } c \in C(\sigma^{*i}) \}.
\]

Let \( C^{i}_{\phi} \) denote the set of incentive compatible and deterministic contracts for individuals of type \( i \). By definition \( U^{i}(c) \geq U^{i}(\phi(c)) \), for all \( c \in C^{i}_{\phi} \) and \( \phi \in \Phi^{i} \). Thus, if \( C(\sigma^{*i}) \), the support of \( \sigma^{*i} \), is contained \( C^{i}_{\phi} \), the set \( \Phi(\sigma^{*}) \) is empty.

We now proceed as follows. First we look for price restrictions on the contracts \( c \in C^{i}_{\phi} \) that make \((id, \sigma^{*i})\) immune from manipulations \( \phi \in \Phi^{i}\Phi(\sigma^{*}) \) and then we search for necessary and sufficient conditions for price supportability.
3.1.1. Manipulations in \( \Phi^i \backslash \Phi(\sigma^*) \)

Consider \( \sigma^* \), a feasible and incentive compatible allocation. We look at manipulations in \( \Phi^i \backslash \Phi(\sigma^*) \) and we ask the following question: can we find prices \( p = (p^i)_{i \in I} \times_{i \in I} P^i(\sigma^*) \) such that \( U^i(\phi) \tau^i \leq U^i(\sigma^*) \), for all budget feasible lotteries at \( p^i \), and manipulations \( \phi^i \in \Phi^i \backslash \Phi(\sigma^*) \), and \( i \in I \)? The answer is affirmative and the intuition is simple. The definition of \( P^i(\sigma^*) \) only pins down the prices of contracts in the set \( C(\sigma^*) \), while it just puts lower bound for the prices of the contracts in the set \( C(\sigma^*) \). It is quite intuitive that we can select values of \( p(c), c \in C^i \backslash C(\sigma^*) \), so high to make suboptimal any manipulation \( \phi^i \in \Phi^i \backslash \Phi(\sigma^*) \). This we show in Proposition 6, that therefore establishes that price supportability is just a search for necessary and sufficient conditions ruling out condition (A).

For the details: It suffices to limit attention to a simple pricing rule define by the following subset of \( P^i(\sigma^*) \):

\[
\bar{p}^i(\sigma^*) = \{ p^i \in P^i(\sigma^*): p^i(c) = \bar{p}^i, c \in C^i \backslash C(\sigma^*) \}
\]

Each \( p^i \in \bar{p}^i(\sigma^*) \) is uniquely identified by a scalar \( \bar{p}^i \). Hereafter, we constantly use this identification and, with some abuse of notation, sometimes we write \( \bar{p}^i \in P(\sigma^*) \) and \( \phi \in \Phi^i \). Consider programming problem:

\[
\max_{\mu \in \Delta(C)} U^i(\phi) \mu^i, \quad \text{subject to } \bar{p}^i \mu \leq 0.
\]

Let \( [\mu^i_\phi(\bar{p}^i)] \) be the set of optimal solutions, \( \mu^i_\phi(\bar{p}^i) \) be one of its elements, and \( V^i_\phi(\bar{p}^i) \) the value of the program.

**Proposition 6.** Let \( \sigma^* \) be a feasible and incentive compatible allocation. Then, there exists \( (\bar{p}^i)_{i \in I} \in \times_{i \in I} \bar{p}^i(\sigma^*) \) such that \( V^i_\phi(\bar{p}^i) \leq U^i(\sigma^*) \), for all \( \bar{p}^i \geq \bar{p}^i, \phi \in \Phi^i \backslash \Phi(\sigma^*) \), and \( i \in I \).

Proposition 6 immediately implies that if \( \Phi^i(\sigma^*) = \emptyset \), for all \( i \), \( \sigma^* \) is price supportable. As already observed, \( \Phi^i(\sigma^*) \) is indeed empty if \( C(\sigma^*) \subset C^i_{IC} \). Thus:

**Corollary 7.** Let \( \sigma^* \) be an incentive compatible and feasible allocation. Suppose that \( C(\sigma^*) \subset C^i_{IC} \), for all \( i \). Then, \( \sigma^* \) is price supportable.

Thus, by Corollary 7, the set of equilibria of the pure exchange economy is non-empty. All feasible lotteries whose support is contained, for each individual type, in the set of deterministic and incentive compatible lotteries are price supportable.

3.1.2. Manipulations in \( \Phi(\sigma^*) \)

We are looking for necessary and sufficient conditions for the price supportability of an incentive compatible, feasible joint lottery \( \sigma^* \) when \( \Phi(\sigma^*) \neq \emptyset \), for some \( i \in I \). The latter implies that the support of \( \sigma^* \) contains contract that are not incentive compatible. There are two steps to the argument. The first obvious step is to observe that the incentive compatible and feasible lottery \( \sigma^* \) is price supportable if and only if \( V^i(\phi) = U^i(\sigma^*) \), for all \( \phi \in \Phi(\sigma^*) \) and \( i \). The second more interesting step is to translate the conditions \( V^i(\phi) = U^i(\sigma^*) \), for all \( \phi \in \Phi(\sigma^*) \), and all \( i \), into conditions that the utility functions \( U^i(\phi)(c), c \in C(\sigma^*) \) must satisfy. This we do by exploiting the necessary and sufficient conditions for optimality of the programming problems (12). This is the substance of the next proposition. Let

\[
C(\phi) = \{ c: \mu^i_\phi(c) > 0, \text{ for some } \mu^i_\phi \in [\mu^i_\phi] \}.
\]

Remember that \( [\mu^i_\phi] \) is the set of optimal solutions to the programming problems (12). Therefore, \( C(\phi) \subset C(\sigma^*) \).

**Proposition 8.** Let \( \sigma^* \) be a feasible and incentive compatible joint lottery. \( \sigma^* \) is price supportable if and only if, for all \( i \in I \) and \( \phi \in \Phi(\sigma^*) \), there exist scalars \( b^i_\phi > 0 \) such that \( U^i(\phi(c)) - U^i(\sigma^*) = b^i_\phi(U^i(c) - U^i(\sigma^*)) \), for \( c \in C(\phi) \).

The intuition behind the argument for Proposition 8 is pretty simple. If \( \Phi(\sigma^*) \neq \emptyset \), \( \sigma^* \) is price supportable if only and only if \( V^i(\phi) = U^i(\sigma^*) \), for all \( \phi \in \Phi(\sigma^*) \). Then, since by definition, \( C(\phi) \subset C(\sigma^*) \), the first order conditions for optimality of the programming problems (12) imply the equalities in the statement of the proposition.

Proposition 8 provides simple instructions to build economies with some or all non-price-supportable constrained efficient allocations. Pick an economy with a constrained efficient allocation \( \sigma^* \) such that 1) the IC constraints are binding, that is, \( U^i(\phi) = U^i(\sigma^*) \), for some \( i \) and \( \phi \in \Phi(\sigma^*) \), but 2) the conditions of Proposition 8 are violated for \( i \) at \( (\phi, \sigma^*) \). Then, \( \sigma^* \) is not price supportable. Evidently, if none of the constrained efficient allocations of some economy are price supportable then the equilibrium set of the economy is empty.
3.2. Economies with non-price-supportable constrained efficient allocations

In this section, we look first at economies of private information and then at economies of moral hazard. We build two simple examples showing, first, the failure of the Second Fundamental Theorem of Welfare Economics and, the second, the lack of price supportable constrained efficient allocations. Obviously, the second example shows as well an economy with an empty equilibrium set.

3.2.1. Economies with private information

The example below is taken from Kehoe et al. (2002). It shows an economy of private information where one of the constrained efficient allocations is not price supportable providing therefore an example of a pure exchange economy and therefore of an economy where the Second Fundamental Theorem of Welfare Economics fails.


There is only one type, there are no actions, no aggregate states, one physical commodity and two equiprobable personal states, \( s = g, b \). Endowments are \( e = (e_g, e_b) = (30, 10) \) and the utility function of the individuals is \( U(z_g + e_g, z_b + e_b) = \frac{1}{2}V(z_g + 30) + \frac{1}{2}V(z_b + 10) \), for \( V(x) = 78x - x^2 \).

In Kehoe et al. (2002), it is shown that these two are optimal contracts:

(a) \( \sigma_1(-1, -5) = \frac{7}{10} \) and \( \sigma_1(-1, 7) = \frac{3}{10} \); and
(b) \( \sigma_2(-1, -7) = \frac{9}{32} \) and \( \sigma_2(-1, 1) = \frac{14}{32}, \sigma_2(-1, 9) = \frac{9}{32} \).

All incentive compatible, feasible, and deterministic contracts must satisfy the equality \( z_g = z_b \). Thus, none of the contracts in \( C(\sigma) \), \( k = 1, 2 \), is incentive compatible. Let \( \phi_g \) be the manipulation defined as \( \phi_g(g) = b \) and \( \phi_g(b) = b \). The utility function \( U(\phi_g) \) is defined by \( U(\phi_g)(z_g, z_b) = \frac{1}{2}V(z_g + 30) + \frac{1}{2}V(z_b + 10) \). It is easily verified that \( U(\phi_g)\sigma_2 = U\sigma_2 \), \( k = 1, 2 \). Thus, \( \Phi(\phi_g) \neq \emptyset \), \( k = 1, 2 \).

We claim that \( \sigma_1 \) is price supportable, while \( \sigma_2 \) is not. Start with \( \sigma_1 \). It is easy to show that \( V(\phi_g) \geq V(\phi) \), for all \( \phi \), where recall that \( V(\phi) \) is the value of the problem (12) with manipulation \( \phi \in \Phi \). Furthermore, \( U(\phi_g)\sigma_1 = U\sigma_1 \). Then, since the support of \( \sigma_1 \) contains only two points, \( \sigma_1 \in [\delta_{\phi_g}, \bar{\delta}_{\phi_g}] \) or, equivalently, there is \( b_\delta > 0 \) such that \( (U(\phi_g)(z_g, z_b) - U\sigma_1) = b_\delta(U(z_g, z_b) - U\sigma_1), \) for all \( (z_g, z_b) \in C(\sigma_1) \). Thus, by Proposition 8, \( \sigma_1 \) is price supportable. Consider next \( \sigma_2 \). Once again \( U(\phi_g)\sigma_2 = U\sigma_2 \). By trivial computations:

\[
(U(\phi_g)(z_g, z_b) - U\sigma_2)_{(z_g, z_b) \in C(\sigma_2)} = (-316, 36, 260)
\]

while

\[
(U(z_g, z_b) - U\sigma_2)_{(z_g, z_b) \in C(\sigma_2)} = (-238, 18, 210).
\]

The vectors \( (U(\phi_g)(z_g, z_b) - U\sigma_2)_{(z_g, z_b) \in C(\sigma_2)} \) and \( (U(z_g, z_b) - U\sigma_2)_{(z_g, z_b) \in C(\sigma_2)} \) are therefore linearly independent. Therefore, Proposition 8 implies that \( \sigma_2 \) is not price supportable.

3.2.2. Economies with moral hazard

Recall that for moral hazard economies, manipulations are maps \( \phi : A^l \to A^l \), that is, a manipulation \( \phi \) changes a contract \( c = (a, z) \) into the contract, \( \phi(c) = (\phi(a), z) \). Also, recall that an individual lottery \( \sigma^a \) entails ex-ante randomization if \( \sum_a \sigma^a(a, z) < 1 \) for some \( a \in A \), while it entails ex-post randomization if the conditional lottery \( \sigma^a(\{z \}; a^\ast) = \sum_{z^\ast} \sigma^a(a, z^\ast) \in (0, 1) \), for some \( (a^\ast, z^\ast) \in C \). The bottom line is that, under minor qualifications, incentive binding lotteries entailing both ex-ante and ex-post randomization are not price supportable. We explain the result by discussing the problem within a simple economy, that is, an economy of moral hazard with one type and two actions, say, \( H \) and \( L \). Choose the fundamentals of the economy so that three conditions are satisfied. First, the unique constrained efficient lottery \( \sigma \) entails both ex-ante and ex-post randomization, that is \( \sum_a \sigma(a, z) > 0 \), for \( a = L, H \). Second, the IC constraints are binding: there exists a manipulation \( \phi^* \) such that \( U\sigma = U(\phi^*)\sigma \) and \( \phi^*(L) = L \) while \( \phi^*(H) = L \). Third, the support of \( \sigma \) contains deterministic contracts that are not incentive compatible and \( U(L, z) > U(H, z) \), for some \( (H, z, L, z) \in C(\sigma) \).

Then, by the second and third conditions, \( \phi^* \in \Phi(\sigma^*) \). Thus, by Proposition 8, \( \sigma \) is price supportable if and only if \( U(\phi^*)(c) - U\sigma = b_{\phi^*}[U(c) - U\sigma], \) for \( b_{\phi^*} > 0 \) and \( c \in C(\sigma) \). However, since \( \phi^*(L) = L \), if \( \phi^* \) is generically the case then \( U(L, z) - U\sigma \neq 0 \), for some \( (L, z, L, z) \in C(\sigma) \), then \( b_{\phi^*} = 1 \), while since \( U(L, z) > U(H, z) \), \( b_{\phi^*} \neq 1 \). Thus, \( \sigma \) is not price supportable.

An example of a simple moral hazard economy that exactly satisfy the construction above is found in the companion working paper (Rustichini and Siconolfi, 2010). Evidently, such an economy has an empty equilibrium set.

It should be obvious that the lack of price supportability has nothing to do with the simplifying features of the problem discussed above, but rather with the existence of ex-ante and ex-post randomization. This is precisely stated in Lemma 10 below. The argument is a simple implication of Proposition 8.
Lemma 10. Let σ be a feasible and incentive compatible joint lottery of a moral hazard economy. Suppose σi entails ex-ante and ex-post randomization, for some type i. Suppose that for such a type, there exists φ such that

(i) \( U_i(φ)σ_i = U_iσ_i \),
(ii) \( U_i(φ(a), z) > U_i(a, z) \), for some \( (a, z) ∈ C(σ_i) \),
(iii) \( U_i(φ(a'), z) = U_i(a', z) \neq U_iσ_i \), for some \( (a', z) ∈ C(σ_i) \).

Then, σ is not price supportable.

3.3. Equilibrium allocations

Obviously, the set of equilibrium allocations of the economy with the firm is contained in the set of equilibrium allocations of the pure exchange economy. Furthermore, the profit maximizing behavior of the firm restricts the set of prices that can be used to support the allocations as an equilibrium. Indeed, if \( σ^* \) is an equilibrium allocation not only the necessary and sufficient conditions for price supportability must be satisfied, but also there must exists a price \( p ∈ \times_i P_i(σ^*) \) such that \( (λ^iσ^*)_{i∈I} \) is profit maximizing at \( p \). As well known, the latter is true if and only if \( p ∈ P_U \), the price domain generating zero economic profit for the firm. The profit maximization problem is a linear programming problem. Therefore the first order conditions are necessary and sufficient for optimality. Hence, we can state that \( (λ^iσ^*)_{i∈I} \) is the profit maximizing choice at \( p \) if and only if there exist a vector \( θ > 0 \) (associated to the resource constraints) and scalars \( b^i_φ ≥ 0, φ ∈ Φ^i \) (associated to the IC constraints) such that the following holds:

\[
P^i(c) = θ T^i(c) + \sum_{φ ∈ Φ^i} b^i_φ (U_φ(c) - U(c)), \quad \text{if } c ∈ C(σ^*), \quad i ∈ I,
\]

\[
P^i(c) ≥ θ T^i(c) + \sum_{φ ∈ Φ^i} b^i_φ (U_φ(c) - U(c)), \quad \text{otherwise},
\]

\[
\sum_{i} λ^i T^iσ^* ≤ 0 \quad \text{and} \quad U_iσ^* ≥ U_i(φ)σ^i, \quad (i, φ) ∈ I × Φ^i.
\] (14)

As known, if the incentive constraint associated to \( φ ∈ Φ^i \) is not binding, that is if \( (U_i - U_i(φ))σ^* > 0 \), then \( b^i_φ = 0 \). Indeed, the profits of the firm are

\[
\sum_{i} λ^i p^iσ^* = θ \sum_{i} λ^i T^iσ^* + \sum_{i} λ^i \sum_{c ∈ C(σ^*)} \sum_{φ ∈ Φ^i} b^i_φ (U_φ(c) - U(c))σ^i(c).
\]

Since \( σ^* \) is feasible, \( θ \sum_{i} λ^i T^iσ^* = 0 \), and since profits are zero for \( p ∈ P_U \), our claim follows because profits are equal to:

\[
\sum_{i} λ^i \sum_{c ∈ C(σ^*)} \sum_{φ ∈ Φ^i} b^i_φ (U_φ(c) - U(c)).
\]

As we have already discussed, price supportability restricts the price function in \( C(σ^*) \), while it only puts lower bounds in \( C(σ^*) \). However, profit maximization specifies as well upper bounds for supporting prices. Unfortunately, the upper bounds of the profit maximization may collide with the lower bounds of the utility maximization thereby making the set of competitive equilibria empty for extremely well behaved economies that have price supportable constrained efficient allocations. Indeed, we construct below an example of an economy of private information with just one type and without actions and aggregate states. The economy has a unique Pareto optimum, \( σ^* = δ^*_z \), with non-binding IC constraints. Hence the Pareto optimum is the unique candidate for being an equilibrium allocation. However, it fails to be a competitive equilibrium, or, equivalently, the equilibrium set of this economy is empty. Before going into details, we provide the intuition for such a result. Since the IC constraints are not binding at \( z^* \), by the profit maximizing conditions (14), equilibrium prices are \( p(z) ≤ θ T(z) = \sum_{i=1}^T θ_i \sum_{j∈S} q_i j z_j(s) \). That is, prices are less or equal than fair prices for all contracts (and they are fair and equal to zero for \( z^* \)). Thus, for such an economy, the existence of a profit maximizing firm transforms the existence problem into investigating whether or not fair prices can be supporting prices of \( z^* \) for the individuals. Since \( z^* ∈ C_{IC} \), \( z^* \) is by Corollary 7 price supportable. However, as stated in Proposition 6, supporting \( z^* \) requires to set prices for \( z \neq z^* \) above some lower bound. There is no reason why this lower bound should be below the fair pricing values. In the example it is not and hence the equilibrium set is empty.

Example 11. A private information economy with an empty set of equilibria, but with a unique and price supportable efficient allocation.
There is only one type, there are no actions, no aggregate states, two physical commodities and two equiprobable personal states, \( s = 1, 2 \). Endowments are \( e = (e_1, e_2) = [(30, 10), (10, 30)] \) and the utility function of the individuals is 

\[
U(z_1 + e_1, z_2 + e_2) = \sum_j V(z_j + e_j), \quad \text{with} \quad V(x) = \ln x_1 + \ln x_2.
\]

We restrict, without loss of generality, the set of manipulation to contain two elements \( \phi_s \), \( s = 1, 2 \), with \( \phi_s(s') = s \), for all \( s' \). Thus, there are only two IC constraints. A contract is a pair of net trades \((z_1, z_2)\). A lottery is incentive compatible if, for \( \kappa = 1, 2 \):

\[
0 \leq \sum_{(z_1, z_2) \in C} \sigma(z_1, z_2) \left( \sum_{s=1}^2 V(z_s + e_s) - \sum_{(z_1, z_2) \in C} \sigma(z_1, z_2) \sum_{s=1}^2 V(z_s + e_s) \right).
\]

We write the right-hand side of the above inequality \((U - U(\phi_s)z)\sigma\) for brevity. The degenerate lottery \( \sigma^+ = \delta_{z^*} \), with 

\[ z^* = \{-10, 10\}, \quad \{10, -10\}, \]

is the unique Pareto optimal allocation. Since \( \delta_{z^*} \) is clearly incentive compatible, it is the unique constrained Pareto optimal allocation. Since the support of the efficient lottery is contained in the set of incentive compatible and deterministic allocations, this lottery is price supportable. It is also immediate to check that 

\[ (U - U(\phi_s))\delta_{z^*} > 0, \quad \text{for all} \quad s. \]

Since, the IC constraints are not binding, the multipliers \( b_{\phi_k} \) in (14) must be equal to zero for the two manipulations \( \phi_s \), \( s = 1, 2 \). Therefore, since the personal states are equiprobable, by conditions (14), the candidate unconstrained equilibrium prices are parameterized by \( \theta = (\theta_1, \theta_2) \gg 0 \) and they are:

\[
p(z^*) = \sum_{\ell=1}^2 \theta_\ell (z^*_1 + z^*_2) = 0, \quad \text{and} \quad p(z) = \sum_{\ell=1}^2 \theta_\ell (z_{1\ell} + z_{2\ell}), \quad \text{for} \quad z \neq z^*,
\]

where \( \ell \) denotes the commodity index.

Consider the degenerate lottery \( \delta_{z^*} \), with 

\[ z^* = \{10, 10\}, \quad \{-20, 20\}. \]

Evidently, since \( z^*_1 + z^*_2 = -10 \), for all \( \ell \), \( p(z^*) < 0 \), for all candidate unconstrained equilibrium prices. Thus, \( \delta_{z^*} \) is always budget feasible. Furthermore, the manipulation of the Pareto optimal allocation \( z^* \) cannot be decentralized as a competitive equilibrium and, therefore, the set of equilibria is empty.

### 3.4. Existence and constrained optimality

The analysis developed so far has shown that imposing the IC constraints in the individual choice sets is not equivalent to some form of price non-linearity. Indeed, with unrestricted choice sets, non-linear prices may fail to decentralize constrained efficient allocations, and the equilibrium set is empty. There are two different problems that have to be disentangled. The first and most important is that constrained efficient allocations may not be price supportable. The various examples and propositions suggest that we cannot decentralize much more than individual lotteries with support in \( C^\prime \), the set of incentive compatible and deterministic contracts. The second problem is that, as pointed out by Example 11, even when the support of constrained efficient allocations is contained in that set, non-linear prices may fail to decentralize it as a competitive equilibrium. Thus, limiting the analysis to economies with such a property is not only obviously restrictive, it may also be pointless.

This brings us to investigate the role played by lotteries in asymmetric information economies with observable types. Lotteries have been introduced for two different reasons. First, they eliminate discontinuities in the consumer problem by convexifying the set of incentive compatible contracts. Second, lotteries introduce randomness that may be able to alleviate the IC constraints thereby enhancing the welfare of the economy. In order to have a welfare enhancing role lotteries put weight on contracts outside the incentive compatible set thereby creating decentralization problems. It is this second role of lotteries that has to be removed in order to obtain a positive result. We want obviously to maintain the feature of eliminating the IC constraints from the individuals’ choices. There are two ways of proceeding. They both require to restrict the production set of the intermediary. The first is to restrict the choice sets of both the firm and the individuals to deterministic contracts. Non-convexities are eliminated by exploiting the large numbers of the individuals. The second is to restrict the firm to offer vectors (signed measures) with support in \( C^\prime \), while leaving individuals’ choices unconstrained in the budget set. Both ways conduct to similar results. An equilibrium always exists and it is efficient in the set of incentive compatible contracts with the first approach and in the set of lotteries over incentive compatible contracts with the second. The second approach that we outline below has the advantage to deliver naturally non-linear prices as linear functionals over the individuals consumption set.

Identify the production set of the firm with the intersection of the feasible set \( Y \) with the positive cone generated by the sets of incentive compatible and deterministic vectors:

\[
Y^U = Y \cap \times_{i \in I} \{ \beta^i \in \mathbb{R}^{C^i}: \beta^i(c) = 0, \quad c \in C^i \setminus C^i_{IC} \}.
\]

The constant returns to scale nature of the firm technology and the zero profit condition immediately prices, by Farkas’ Lemma, the contracts within \( C^i_{IC} \) fairly. Thus we get the following price domain:

\[
P^* = \{ (p^i)_{i \in I} \in \mathbb{R}^{\sum_{i \in I} C^i}: p^i(c) = \theta^T^i(c), \quad \text{for} \quad c \in C^i_{IC} \text{ and} \theta \in \Omega^i_{IC} \setminus \{0\} \}.
\]
By construction, the firm cannot supply (signed) measures whose support is not contained in $C^1_C$. Thus, the pricing of the contracts $c \in C^1\setminus C^1_C$ does not affect the firm behavior. As usual, existence of a competitive equilibrium requires two additional restrictions: local non-satiation, as for instance formalized by Assumption 5 and the minimum wealth condition, that we can formalize as follows:

**Assumption 12 (Minimum wealth condition).** For all $i \in I$, there is an incentive compatible contract $\tau^i = (a^i, z^i)$, with $z^i \equiv 0$.

If Assumption 12 holds, the minimum wealth condition is satisfied for all types, since $0 > p'c^2$, for all $p \in P$. Call the feasible, joint lottery $\pi^*$ weakly efficient if it does not exist a feasible joint allocation $\pi$ with $C(\pi) \subset C^1_C$ Pareto dominating $\pi^*$. Then we can state the following proposition.

**Proposition 13.** If Assumptions 12 and 5 hold, there always exists an equilibrium $(\hat{p}, \hat{\pi}, \hat{z})$ of the economy with production set $Y_U$. Furthermore, under Assumption 5, the equilibrium allocation $\hat{\pi}$ is weakly constrained efficient.

For the proof, consider the economy $(Y_U, (C^1_C, U^j)_{j \in I})$. By the definition of the sets $C^1_C$, this is an economy of individual risk without any informational issue. Therefore, under the stated assumptions an equilibrium exists and its allocation is constrained efficient (and hence weakly constrained efficient). Call $\pi^*$ the equilibrium allocation and $p^*$ the equilibrium price. Bear in mind that $p^{*i}$ assigns price only to contracts in $C^1_C$, while it is silent for contracts outside that set. Hence, we just need to show that we can assign prices $p'(c)$ to contracts outside $C^1_C$ so that at the extended price $(p^{*i}, (p'(c)_{c \in C^1_C})$ individuals of type $i$ optimally choose (id. $\pi^{*i}$) when solving the programming problem (9). Obviously, by the definition of $Y_U$, the price extension does not change the behavior of the firm. However, since by construction $C'(\pi^*) \subset C^1_C$, the set of manipulations $\Phi(\pi^*)$ is empty. Thus, Proposition 6 concludes the argument.

Proposition 13 implies that, at such an equilibrium, the only exchanged contracts are incentive compatible and all incentive compatible contracts are priced fairly. Finally it is quite obvious that the Second Fundamental Theorem of Welfare Economics is true. Every weakly constrained efficient allocation can be decentralized as a quasi-equilibrium with supporting prices in $P^*$ (and that, as usual, the quasi-equilibrium is an equilibrium if at the supporting price the minimum wealth condition holds).

**Example 14.** The price domain of the economy with the production set $Y_U$.

By Proposition 13, the private information economy with the production set $Y_U$ of Example 11 has weakly constrained efficient competitive equilibria. Since such an economy has unique Pareto optimal allocation, $z^* = (-10, 10), (10, -10)$, which is also incentive compatible, $z^*$ is the unique equilibrium allocation.

We construct the equilibrium prices of all contracts for such private information economy when the production set is $Y_U$. By taking into account the set of available manipulations, the incentive compatible set is

$$C^1_C = \{z_1, z_2: U(z_1, z_2) \geqslant U(z_1, z_1) \text{ and } U(z_1, z_2) \geqslant U(z_2, z_2)\}.$$

A contract $z$ is optimally manipulated into at most two contracts $\phi^k_z(z)$, $k = 1, 2$, defined as follows: if $z \in C^1_C$, then $\phi^k_z(z) = z$, for both $k$, similarly if $U(z_1, z_2) > \max(\{U(z_1, z_1), U(z_2, z_2)\})$, $s \neq s'$, $s = 1, 2$, then $\phi^k_z(z) = (z_s, z_s)$, for both $k$, while if $U(z_1, z_2) > U(z_1, z_2)$, then $\phi^k_z(z) = (z_s, z_s)$, $k = 1, 2$.

For each contract $z$, the manipulated contracts $\phi^k_z(z)$ are incentive compatible and $U(\phi^k_z(z)) = U(\phi^k_z(z)) \geqslant U(\phi(z))$, for each $z$ and $\phi \in \Phi$. Notice that choosing $\phi^1$ or $\phi^2$ makes no difference in terms of utility, but, depending on what the price is, could make a difference in terms of costs. We claim that the equilibrium allocation is supported by any price in the set

$$P(z^*) = \left\{p: p(z) = \sum_{\ell} \left(\frac{(z_{1\ell} + z_{2\ell})}{2}\right), \text{ if } z \in C^1_C\right\},$$

while $p(z) > \max_{\ell}(p(\phi^k_z(z))) = \max_{\ell} \sum_{\ell} \left(\frac{U^k_{1\ell}z_{1\ell} + U^k_{2\ell}z_{2\ell}}{2}\right)$.

First, for $p \in P(z^*)$, it is $p(z^*) = 0$ implying that $z^*$ is budget feasible and the firm profits are zero at $z^*$. Second, a measure $\beta$ is feasible if and only if $\sum_{\ell} \beta(z^{1{\ell}+2\ell}) \leqslant 0$, $\ell = 1, 2$. Thus, since $C(\beta) \subset C^1_C$, for $p \in P(z^*)$ and $\beta \in Y_U$, profits are $\sum z(\beta(z)) = \sum \beta(z) \sum_{\ell}(\frac{U^k_{1\ell}z_{1\ell} + U^k_{2\ell}z_{2\ell}}{2})$. Therefore, for $p \in P(z^*)$, profits are non-positive and $z^*$ is a profit maximizing choice. Third, for $p \in P(z^*)$, individuals behave truthfully. Indeed, by the definition of $\phi^k_z$ and $P(z^*)$, it is

$$\sum z U(\phi(z)) \sigma(z) \leqslant \sum z U(\phi^k_z(z)) \sigma(z) = \sum z U(\phi^k(z)) \sigma(z)$$
and
\[ \sum z p(z)\sigma(z) \geq \max_k \sum z p(\phi_k(z))\beta(z) \]

for all lotteries \( \sigma \), \( p \in P(z) \) and \( \phi \). Then, since \( \phi_k(z) \in CIC \), for all \( z \) and \( k \), for \( p \in P(z^*) \), individuals optimally choose incentive compatible lotteries. However, if \( C(\sigma) \subset CIC \), the budget constraint is just \( \sum z \sigma(z) \sum_i \frac{(\beta_i - \beta_{i+1})}{2} \). Since individuals are strictly risk averse, the optimal solution is a degenerate lottery, and since prices are state and commodity invariant the optimal solution is \( z^* \).

4. Conclusions

A useful technique to prove existence of equilibria in competitive economies is to proceed on the basis of the Welfare Theorems, reducing the equilibrium problem to an optimization problem, and deriving equilibrium prices as shadow prices in the maximization problem. An important and in our opinion so far misunderstood feature of the economies with asymmetric information (even with observable types) is that this technique fails if the incentive compatibility constraint is imposed as a constraint on the technology of the firm. The incentive constraint on the firm does affect prices, but cannot decentralize the constraint on the consumer to limit the choice to incentive compatible contracts. When the consumer is offered the price derived from the firm maximization problem he will typically deviate from the recommended course of action in the contract. The reason is clear: those prices were designed to prevent deviation ad-ante (occurring potentially at the moment in which the recommended action is communicated, after the contract is chosen), not the plan of an ex-ante deviation at the moment of the choice of the contract.

To restore existence and Welfare Theorems the production set of the intermediary has to be restricted to supply measures over incentive compatible contracts.

Appendix A

Proof of Lemma 1. We first prove that \( IC_i^* \subset IC_i \). Let \( \bar{\sigma} \in IC_i^* \), but suppose by contradiction that \( \bar{\sigma} \notin IC_i \). Then, there exists \( \phi : A^i \to A^i \) such that
\[ (U^i(\phi) - U^i)\bar{\sigma} = \sum a z (a, z)(u^i(\phi(a), z) - u^i(a, z)) > 0. \]

Then, for some \( \bar{a} \) with \( \sum z \bar{\sigma}(a, z) > 0 \), it is \( \sum z \bar{\sigma}(a, z)(u^i(\phi(a'), z) - u^i(a, z)) > 0 \) contradicting \( \bar{\sigma} \notin IC_i^* \). For the inclusion \( IC_i \subset IC_i^* \). To each pair of action \( (a, a') \) associates a map \( \phi_{a,a'} \in \phi \) defined as \( \phi_{a,a'}(a) = \bar{a} \), if \( a \neq a' \), \( \phi_{a,a'}(a) = a' \). If \( \sigma \in IC_i \), then \( (U^i(\phi_{a,a'}(a) - U^i)\sigma \leq 0 \), for all \( a, a' \). The latter reads:
\[ \sum a z (a, z)(u^i(\phi(a), z) - u^i(a, z)) = \sum z (a, z)(u^i(a', z) - u^i(a, z)) \leq 0. \]

Thus, the claim is proved. \( \square \)

The proof of Lemma 2 is a trivial modification of the proof of Lemma 1 and is therefore omitted.

Proof of Proposition 6. Since prices are personalized, it suffices to show that the claim holds true for just one type \( i \). We begin with a lemma.

Lemma 15. The function \( V_i^\phi(\bar{p}) \) with domain \( \bar{p}^i(\sigma^{*i}) \) satisfies: \( V_i^\phi(\bar{p}) \geq V_i^\phi(\bar{p}) \), \( V_i^\phi(\bar{p}) \) is non-increasing in \( \bar{p}^i \), \( \lim_{\bar{p}^i \to +\infty} V_i^\phi(\bar{p}) = V_i^\phi \).

Proof. \( V_i^\phi(\bar{p}) \) is non-increasing in \( \bar{p}^i \), for all \( \phi \), since for \( \bar{p}^i > \bar{p}^{i2} \), \([\sigma^{*i} : \bar{p}^i \sigma^{*i} \leq 0] \subset [\sigma^{*i} : \bar{p}^{i2} \sigma^{*i} \leq 0] \). Furthermore, for \( \bar{p}^i \in \bar{p}^i \), since both \( \bar{p}^i \mu_i^\phi(\bar{p}) \leq 0 \) and \( 0 \leq \mu_i^\phi(\bar{p}) \), the budget constraint implies that
\[ 0 \leq \sum_{c \in C(\sigma^{*i})} \mu_i^\phi(\bar{p})(c) \leq \sum_{c \in C(\sigma^{*i})} \frac{(U^i(c) - U^i(\sigma^{*i})\mu_i^\phi(\bar{p})(c))}{\bar{p}^i}. \]

By the Maximum Theorem, \( V_i^\phi(\bar{p}) \) is a continuous function in \( \bar{p}^i \), while \( [\mu_i^\phi(\bar{p})] \) is upper hemi-continuous. Since as \( \bar{p}^i \to +\infty, \sum_{c \in C(\sigma^{*i})} \mu_i^\phi(\bar{p})(c) \to 0 \), it is \( \lim_{\bar{p}^i \to +\infty} V_i^\phi(\bar{p}) = V_i^\phi \). \( \square \)

By Lemma 15 and the continuity in \( \bar{p}^i \) of the function \( V_i^\phi(\bar{p}) \), if \( V_i^\phi(\phi) < U^i(\sigma^{*i}) \), there exists a scalar \( \bar{p}^i_0 > 0 \) such that \( V_i^\phi(\bar{p})_0 \leq U^i(\sigma^{*i}) \), for all \( \bar{p}^i \geq \bar{p}^i_0 \). Thus, in order to conclude the argument we need to show that such a scalar \( \bar{p}^i_0 \) exists even
when \( V'(\phi) = U'_i \sigma^{i} \) and \( \phi \notin \Phi(\sigma^{i}) \). Let \( \phi \in \Phi(\sigma^{i}) \) be given. Consider the programming problem (12). Recall that \( C(\phi) = [c: \mu(c) > 0, \text{for some } \mu^{i}_\phi \in [\mu^{i}_\phi]] \subset C(\sigma^{i}) \). Then the following necessary and sufficient conditions hold true for all \( \mu^{i}_\phi \in [\mu^{i}_\phi] \) and some multiplier \( \alpha^i_\phi \geq 0 \):

\[
\begin{align*}
U^i(\phi(c)) - V^i(\phi) &= \alpha^i_\phi (U^i(c) - U^i' \sigma^{i}) , \quad \text{if } c \in C(\phi), \\
U^i(\phi(c)) - V^i(\phi) &\leq \alpha^i_\phi (U^i(c) - U^i' \sigma^{i}) , \quad \text{if } c \in (C(\sigma^{i})) \setminus C(\phi), \\
\sum_{c \in C(\sigma^{i})} (U^i(c) - U^i' \sigma^{i}) \mu^i_\phi(c) &\leq 0. \\
\end{align*}
\]

(A.1)

By the definition of the set \( \Phi(\sigma^{i}) \), it is \( U^i(\phi(c)) \leq U^i(c) \), for all \( c \in C(\sigma^{i}) \). If \( V^i_\phi = U^i' \sigma^{i} \), conditions (A.1) are satisfied by setting \( \alpha^i_\phi = 1 \).

**Lemma 16.** Let \( \phi \in \Phi(\sigma^{i}) \) be such that \( V^i_\phi = U^i' \sigma^{i} \). Then, \( V^i_\phi(\bar{p}^i) = U^i' \sigma^{i} \), for all \( \bar{p}^i \geq \bar{p}^i_\phi \).

**Proof.** It suffices to show that at \( \bar{p}^i \geq \bar{p}^i_\phi \) the first order conditions of the programming problem (13) are solved by the multiplier \( \alpha^i_\phi = 1 \), the lotteries \( \mu^{i}_\phi \in [\mu^{i}_\phi] \), and the value \( V^i(\phi) \), that is, we need to show that the following inequalities are satisfied:

\[
\begin{align*}
U^i(\phi(c)) - V^i(\phi) &= (U^i(c) - U^i' \sigma^{i}) , \quad \text{if } c \in C(\phi), \\
U^i(\phi(c)) - V^i(\phi) &\leq (U^i(c) - U^i' \sigma^{i}) , \quad \text{if } c \in \Phi(\sigma^{i}) \setminus C(\phi), \\
U^i(\phi(c)) - V^i(\phi) &\leq \bar{p}^i , \quad \text{if } c \in \Phi(\sigma^{i}) \setminus C(\phi), \\
\bar{p}^i_\phi \mu^i_\phi = \sum_{c \in C(\sigma^{i})} (U^i(c) - U^i' \sigma^{i}) \mu^i_\phi(c) &\leq 0.
\end{align*}
\]

The first, second conditions, and the budget constraints are nothing else than (A.1) and therefore they hold true. The third set of inequalities are satisfied by the definition of the scalars \( \bar{p}^i \) and \( \bar{p}^i_\phi \). Thus, \([\mu^{i}_\phi] \subset [\mu^{i}_\phi(\bar{p}^i_\phi)] \) and therefore \( V^i_\phi(\bar{p}^i_\phi) = V^i(\phi) \). □

We have shown that for each \( \phi \in \Phi(\sigma^{i}) \), there exists \( \bar{p}^i_\phi \) such that \( V^i_\phi(\bar{p}^i) \leq U^i' \sigma^{i} \), for all \( \bar{p}^i \geq \bar{p}^i_\phi \). Therefore, \( V^i_\phi(\bar{p}^i) \leq U^i' \sigma^{i} \), for all \( \phi \in \Phi(\sigma^{i}) \) and for \( \bar{p}^i \geq \max_{\phi \in \Phi(\sigma^{i})} p\phi \). □

**Proof of Proposition 8.** (\( \Rightarrow \)) If \( \sigma^{i} \) is price supportable, (id, \( \sigma^{i} \)) is an optimal solution to the programming problem (9) at some \( p^i \in P(\sigma^{i}) \). Then, \( V^i_\phi = U^i' \sigma^{i} \), for all \( \phi \in \Phi(\sigma^{i}) \), and, therefore, \([\mu^{i}_\phi] \subset [\mu^{i}_\phi(\bar{p}^i_\phi)] \), for all \( \phi \in \Phi(\sigma^{i}) \), that is, \([\mu^{i}_\phi] \) is an optimal solution to the programming problem (13) at \( \bar{p}^i \in P(\sigma^{i}) \). Then, the following necessary and sufficient conditions are satisfied for some \( \alpha^i_\phi > 0 \):

\[
\begin{align*}
U^i(\phi(c)) - U^i' \sigma^{i} &\leq \alpha^i_\phi [U^i(c) - U^i' \sigma^{i}] , \quad \text{if } c \in \Phi(\phi), \\
U^i(\phi(c)) - U^i' \sigma^{i} &\leq \alpha^i_\phi [U^i(c) - U^i' \sigma^{i}] , \quad \text{if } c \in \Phi(\sigma^{i}) \setminus C(\phi), \\
U^i(\phi(c)) - U^i' \sigma^{i} &\leq \alpha^i_\phi p^i(c) , \quad \text{if } c \in \Phi(\sigma^{i}) \setminus C(\phi), \\
p^i_\phi \mu^i_\phi &\leq 0.
\end{align*}
\]

Consider the contract \( c^{i} \) defined in Assumption 5. Since \( V^i_\phi = U^i' \sigma^{i} \), by Assumption 5, \( U^i(\phi(c)) - U^i' \sigma^{i} > 0 \). Thus, it is \( \alpha^i_\phi > 0 \) concluding the argument.

(\( \Leftarrow \)) First we show that the equations \( U^i(\phi(c)) - U^i' \sigma = b^i_\phi[U^i(c) - U^i' \sigma] \), for \( c \in C(\phi) \), and \( b^i_\phi > 0 \), implies that \( V^i(\phi) = U^i' \sigma^{i} \). Indeed, pick any \( \mu^{i}_\phi \in [\mu^{i}_\phi] \), then \( C(\phi) \subset C(\phi) \) and therefore

\[
\sum_{c \in C(\phi)} \mu^i_\phi(c)[U^i(\phi(c)) - U^i' \sigma^{i}] = V^i(\phi) - U^i' \sigma^{i} = b^i_\phi \sum_{c \in C(\phi)} \mu^i_\phi(c)[U^i(c) - U^i' \sigma^{i}].
\]

Since \( \phi \in \Phi(\sigma^{i}) \), \( V^i(\phi) - U^i' \sigma^{i} \geq 0 \), and since \( \mu^{i}_\phi \in [\mu^{i}_\phi] \), then, by the budget constraint, \( \sum_{c \in C(\phi)} \mu^i_\phi(c)[U^i(c) - U^i' \sigma^{i}] \leq 0 \). Therefore, \( V^i(\phi) = U^i' \sigma \). Now let \( \bar{p}^i_\phi = \max_{c \in C(\sigma^{i})} \frac{U^i(\phi(c)) - V^i(\phi)}{b^i_\phi} \). As already argued for all \( \bar{p} \geq \bar{p}^i_\phi \), \( V^i_\phi(\bar{p}) = V^i_\phi \). By the same argument used for Proposition 6 (Lemma 16) for all \( \bar{p}^i \geq \bar{p}^i_\phi \), \( V^i_\phi(\bar{p}^i) = V^i_\phi \). Thus, for \( \bar{p}^i \geq \max_{\phi \in \Phi(\sigma^{i})} \bar{p}^i_\phi \), \( V^i_\phi(\bar{p}^i) = V^i_\phi \), for all \( \phi \in \Phi(\sigma^{i}) \). The latter concludes the argument. □
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