1. (a) With sequential market structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A sequential markets equilibrium is sequences of rental rates \( \hat{r}_0^k, \hat{r}_1^k, \ldots \), interest rates \( \hat{r}_0^b, \hat{r}_1^b, \ldots \), wages \( \hat{w}_0, \hat{w}_1, \ldots \), consumption levels \( \hat{c}_0, \hat{c}_1, \ldots \), capital stocks \( \hat{k}_0, \hat{k}_1, \ldots \), and bond holdings \( \hat{b}_0, \hat{b}_1, \ldots \), such that

- Given \( \hat{r}_0^k, \hat{r}_1^k, \ldots, \hat{r}_0^b, \hat{r}_1^b, \ldots \), and \( \hat{w}_0, \hat{w}_1, \ldots \), the consumer chooses \( \hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots \), and \( \hat{b}_0, \hat{b}_1, \ldots \) to solve

\[
\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t + \left(1 + \hat{r}_t^k - \delta\right)k_t + \left(1 + \hat{r}_t^b\right)b_t, \quad t = 0, 1, \ldots
\]

\[k_0 = \hat{k}_0\]

\[b_t \geq -B\]

\[c_t, k_t \geq 0.\]

Here \( b_t \geq -B \), where \( B > 0 \), rules out Ponzi schemes but does not otherwise bind in equilibrium. [It is also possible to define a labor choice \( \hat{\ell}_t \) if we impose the constraint \( 0 \leq \ell_t \leq 1 \).]

- \( \hat{r}_t^k = F_k(\hat{k}_t, 1), \quad t = 0, 1, \ldots \)
- \( \hat{w}_t = F_k(\hat{k}_t, 1), \quad t = 0, 1, \ldots \)

[A good answer would explain that these are the profit maximization conditions for constant returns.]

- \( \hat{c}_t + \hat{k}_{t+1} - (1 - \delta)\hat{k}_t = F(\hat{k}_t, 1), \quad t = 0, 1, \ldots \)

- \( \hat{b}_t = 0, \quad t = 0, 1, \ldots \)

(b) A Pareto efficient allocation is sequences \( \hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots \) that are feasible,

\[
\hat{c}_t + \hat{k}_{t+1} - (1 - \delta)\hat{k}_t \leq F(\hat{k}_t, 1), \quad t = 0, 1, \ldots
\]
\[ \hat{k}_0 \leq \bar{k}_0, \]

and such that there exists no alternative allocation \( \tilde{c}_i, \tilde{k}_i \) that is also feasible and such that

\[
\sum_{t=0}^{\infty} \beta' u(\tilde{c}_t) > \sum_{t=0}^{\infty} \beta' u(\hat{c}_t).
\]

In other words, the allocation \( \hat{c}_i, \hat{k}_i \) solves

\[
\max \sum_{t=0}^{\infty} \beta' u(c_t)
\]

s.t. \( c_t + k_{t+1} - (1-\delta)k_i \leq F(k_i,1), \ t = 0, 1, \ldots \)
\[ k_0 \leq \bar{k}_0 \]
\[ c_t, k_i \geq 0. \]

[It is also possible to include \( \ell_t \) in \( F(k_i, \ell_t) \) and impose the constraints \( 0 \leq \ell_t \leq 1 \).]

(c) Suppose that \( \hat{r}_1^k, \hat{r}_1^b, \hat{w}_t, \hat{c}_i, \hat{k}_i, \hat{b}_i \) is an equilibrium. Then

\[
\hat{c}_i + \hat{k}_{t+1} - (1-\delta)\hat{k}_i = F(\hat{k}_i,1), \ t = 0, 1, \ldots
\]

\[ \hat{k}_0 = \bar{k}_0. \]

and, since \( \hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots \) and \( \hat{b}_0, \hat{b}_1, \ldots \) solve the consumer’s problem, there exist Lagrange multipliers \( p_t \geq 0, \ t = 0, 1, \ldots, \) such that

\[
\beta' u'(\hat{c}_t) - p_t = 0, \ t = 0, 1, \ldots
\]
\[ -p_t + p_{t+1} (1 + \hat{r}_t^k - \delta) = 0, \ t = 0, 1, \ldots
\]
\[ \lim_{t \to \infty} p_t \hat{k}_{t+1} = 0. \]

From the profit maximization conditions, we know that \( \hat{r}_t^k = F_k(\hat{k}_{t+1},1) \). [We also know other things, like \( \hat{b}_t = 0, \ -p_t + p_{t+1} (1 + \hat{r}_t^b) = 0, \) and \( \lim_{t \to \infty} p_t \hat{b}_{t+1} = 0 \), but we do not need to use these conditions. Our assumptions on \( u \) and \( F \) do not rule out corner solutions, so, to be strictly correct, we should write the first order conditions as inequalities with complementary slackness conditions.]

The necessary and sufficient conditions for \( \hat{c}_i, \hat{k}_i \) to be a Pareto efficient allocation are

\[
\hat{c}_i + \hat{k}_{t+1} - (1-\delta)\hat{k}_i = F(\hat{k}_i,1), \ t = 0, 1, \ldots
\]
\[ \hat{k}_0 = \bar{k}_0. \]
and that there exist some Lagrange multipliers $\pi_t \geq 0$, $t = 0, 1, \ldots$, such that

$$\beta'v'(\tilde{c}_t) - \pi_t = 0, \ t = 0, 1, \ldots$$

$$-\pi_t + \pi_{t+1}\left(1 + F_K(\hat{k}_{t+1}, 1) - \delta\right) = 0, \ t = 0, 1, \ldots$$

$$\lim_{t \to \infty} \pi_t \hat{k}_{t+1} = 0.$$ Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{\omega}_t, \hat{c}_t, \hat{\kappa}_t, \hat{\beta}_t$ is an equilibrium, we can set $\hat{c}_t = c_t$, $\hat{\kappa}_t = \kappa_t$, and $\pi_t = p_t$, and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(d) A sequential markets equilibrium is sequences of rental rates $\hat{r}_t^k$, $\hat{r}_t^b$, $\hat{\omega}_t$, $\hat{c}_t$, $\hat{\kappa}_t$, $\hat{\beta}_t$, $\hat{\omega}_t^i$, $\hat{\kappa}_t^i$, and bond holdings $\hat{b}_t^0, \hat{b}_t^1, \ldots, \ i = 1, 2$, such that

- Given $\hat{r}_t^k, \hat{r}_t^b, \hat{\omega}_t^i, \hat{\kappa}_t^i$, consumer $i$, $i = 1, 2$, chooses $\hat{c}_t^i, \hat{\omega}_t^i, \hat{\kappa}_t^i$, and $\hat{b}_t^0, \hat{b}_t^1$ to solve

  $$\max \sum_{i=0}^{\infty} \beta_i u_t(c_i^t)$$

  s.t. $c_i^t + k_i^{t+1} + b_i^t \leq \hat{\omega}_t + (1 + \hat{r}_t^k - \delta)k_i^t + (1 + \hat{r}_t^b)b_i^t$, $t = 0, 1, \ldots$

  $k_i^t = \hat{k}_0^i$

  $b_i^t \geq -B$

  $c_i^t, k_i^t \geq 0$.

- $\hat{r}_t^k = F_K(\hat{\kappa}_t^i + \hat{\omega}_t^i, \hat{\kappa}_t^i + \hat{\omega}_t^i), \ t = 0, 1, \ldots$

- $\hat{\omega}_t = F_L(\hat{\kappa}_t^i + \hat{\omega}_t^i), \ t = 0, 1, \ldots$

- $(\hat{c}_i^t + \hat{\omega}_t^i) + (\hat{k}_i^{t+1} + \hat{\omega}_t^i) - (1 - \delta)(\hat{k}_i^t + \hat{\omega}_t^i) = F(\hat{k}_i^t + \hat{\omega}_t^i, \hat{\kappa}_t^i + \hat{\omega}_t^i), \ t = 0, 1, \ldots$

- $\hat{b}_t^0 + \hat{b}_t^1 = 0, \ t = 0, 1, \ldots$

(e) A Pareto efficient allocation plan is sequences $\hat{c}_t^i, \hat{\omega}_t^i, \hat{\kappa}_t^i, \hat{\beta}_t^i, \ i = 1, 2$, that are feasible,
and such that there exists no alternative allocation $\vec{c}_i^i, \vec{k}_i^i$ that is also feasible and such that

$$\sum_{t=0}^{\infty} \beta_t^i u_t(\vec{c}_i^i) \geq \sum_{t=0}^{\infty} \beta_t^i u_t(\vec{c}_i^i), \quad i = 1, 2,$$

with strict inequality for some $i$.

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\vec{c}_0^0, (\vec{c}_1^1, \vec{c}_2^1), (\vec{c}_2^2, \vec{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\vec{c}_0^0$ to solve

$$\max \ c_0^0 \quad \text{s.t. } \hat{p}_0 c_0^0 \leq \hat{p}_1 w_2 + m \quad c_0^0 \geq 0.$$ 

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer $t$, $t = 1, 2, \ldots$, chooses $(\vec{c}_t^t, \vec{c}_{t+1}^t)$ to solve

$$\max \ \log c_t^t + c_{t+1}^t \quad \text{s.t. } \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \quad c_t^t, c_{t+1}^t \geq 0.$$ 

- $\vec{c}_t^{t-1} + \vec{c}_t^t = w_2 + w_1, \ t = 1, 2, \ldots$

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t - 1$ and $t$ trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\vec{c}_0^0, (\vec{c}_1^1, \vec{c}_2^1), (\vec{c}_2^2, \vec{c}_3^2), \ldots$, and asset holdings $\vec{s}_1^1, \vec{s}_2^2, \ldots$ such that

- Consumer 0 chooses $\vec{c}_0^0$ to solve

$$\max \ c_0^0 \quad \text{s.t. } c_0^0 \leq w_2 + m \quad c_0^0 \geq 0.$$ 

- Given $\hat{r}_t$, consumer $t$, $t = 1, 2, \ldots$, chooses $(\vec{c}_t^t, \vec{c}_{t+1}^t)$ and $\vec{s}_t^t$ to solve
\[
\begin{align*}
\text{max} & \quad \log c'_i + c'_{i+1} \\
\text{s.t.} & \quad c'_i + s'_i \leq w_i \\
& \quad c'_{i+1} \leq w_2 + (1 + \hat{r}_i)s'_i \\
& \quad c'_i, c'_{i+1} \geq 0.
\end{align*}
\]

- \( \hat{c}^{t-1} + \hat{c}'_t = w_2 + w_i, \ t = 1, 2, \ldots \)

- \( \hat{s}'_1 = m \)
  \[ \hat{s}'_t = \left[ \prod_{i=1}^{t-1} (1 + \hat{r}_i) \right] m, \ t = 2, 3, \ldots \]

(c) **Proposition 1**: Suppose that \( \hat{p}_1, \hat{p}_2, \ldots, \hat{c}_1, \hat{c}_2, (\hat{c}_1^2, \hat{c}_1^3), \ldots \) is an Arrow-Debreu equilibrium. Then \( \hat{r}_1, \hat{r}_2, \ldots, \hat{c}_1, (\hat{c}_1^2, \hat{c}_1^3), \ldots \) is a sequential markets equilibrium where

\[ \hat{r}_t = \frac{\hat{p}_t}{\hat{p}_{t+1}} - 1 \quad \text{and} \quad \hat{s}'_t = w_i - \hat{c}'_i, \ t = 1, 2, \ldots \]

**Proposition 2**: Suppose that \( \hat{p}_1, \hat{p}_2, \ldots, \hat{c}_1, (\hat{c}_1^2, \hat{c}_1^3), \ldots \) is a sequential markets equilibrium. Then \( \hat{p}_1, \hat{p}_2, \ldots, \hat{c}_1, (\hat{c}_1^2, \hat{c}_1^3), \ldots \) is an Arrow-Debreu equilibrium where \( \hat{p}_1 = 1 \) and

\[ \hat{p}_t = \prod_{i=1}^{t-1} \frac{1}{(1 + \hat{r}_i)}, \ t = 2, 3, \ldots \]

(d) **Solving the problem of consumer \( t \) in the Arrow-Debreu world, we obtain**

\[ c'_i = \frac{p_{i+1}}{p_t}, \quad c'_{i+1} = \frac{p_i w_i + p_{i+1} (w_2 - 1)}{p_{i+1}}. \]

[Corner solutions are possible in general, but that is not relevant here.]

Since there is no fiat money, the unique equilibrium is autarky:

\[ \hat{c}'_i = w_i, \quad \hat{c}'_{i+1} = w_2. \]

Consequently,

\[ \hat{p}_t = w_i^{t-1}. \]
Using proposition 1 from part c, we obtain
\[
\hat{r}_t = \frac{1}{w_t} - 1, \hat{s}_t = 0, t = 1, 2, \ldots
\]
in the sequential markets world.

(e) An **Arrow-Debreu equilibrium** is a sequence of prices \(\hat{p}_1, \hat{p}_2, \ldots\) and an allocation \(\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots\) such that

- Given \(\hat{p}_1\), consumer \(-1\) chooses \(\hat{c}_1^{-1}\) to solve
  
  \[
  \begin{align*}
  \max & \quad c_1^{-1} \\
  \text{s.t.} & \quad \hat{p}_1 c_1^{-1} \leq \hat{p}_1 w_3 + m^{-1} \\
  & \quad c_1^{-1} \geq 0.
  \end{align*}
  \]

- Given \(\hat{p}_1, \hat{p}_2\), consumer \(0\) chooses \((\hat{c}_1^0, \hat{c}_2^0)\) to solve
  
  \[
  \begin{align*}
  \max & \quad \gamma \log c_1^0 + c_2^0 \\
  \text{s.t.} & \quad \hat{p}_1 c_1^0 + \hat{p}_2 c_2^0 \leq \hat{p}_1 w_2 + \hat{p}_2 w_3 \\
  & \quad c_1^0, c_2^0 \geq 0.
  \end{align*}
  \]

- Given \(\hat{p}_1, \hat{p}_{t+1}, \hat{p}_{t+2}\), consumer \(t\), \(t = 1, 2, \ldots\), chooses \((\hat{c}_1^t, \hat{c}_2^t, \hat{c}_3^t)\) to solve
  
  \[
  \begin{align*}
  \max & \quad \log c_1^t + \gamma \log c_{t+1}^t + c_{t+2}^t \\
  \text{s.t.} & \quad \hat{p}_1 c_1^t + \hat{p}_{t+1} c_{t+1}^t + \hat{p}_{t+2} c_{t+2}^t \leq \hat{p}_1 w_1 + \hat{p}_{t+1} w_2 + \hat{p}_{t+2} w_3 \\
  & \quad c_1^t, c_{t+1}^t, c_{t+2}^t \geq 0.
  \end{align*}
  \]

- \(\hat{c}_1^{t-2} + \hat{c}_1^{t-1} + \hat{c}_1^t = w_3 + w_2 + w_1, t = 1, 2, \ldots\)

A **sequential markets equilibrium** is a sequence of interest rates \(\hat{r}_1, \hat{r}_2, \ldots\), an allocation \(\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots\), and asset holdings \(\hat{s}_1^0, (\hat{s}_1^1, \hat{s}_2^1), (\hat{s}_2^2, \hat{s}_3^2), \ldots\) such that

- Consumer \(-1\) chooses \(\hat{c}_1^{-1}\) to solve
\[ \begin{align*}
\max & \quad c_i^{-1} \\
\text{s.t.} & \quad c_i^{-1} \leq w^i + m^{-1} \\
& \quad c_i^{-1} \geq 0.
\end{align*} \]

- Consumer 0 chooses \((\hat{c}_i^0, \hat{s}_i^0)\) and \(s_i^0\) to solve

\[ \begin{align*}
\max & \quad \gamma \log c_i^0 + c_2^0 \\
\text{s.t.} & \quad c_i^0 + s_i^0 \leq w_2 + m^0 \\
& \quad c_2^0 \leq w_3 + (1 + \hat{r}_s) s_i^0 \\
& \quad c_i^0, c_2^0 \geq 0.
\end{align*} \]

- Given \(\hat{r}_s, \hat{r}_{s+1},\) consumer \(t, t = 1, 2, \ldots\), chooses \((\hat{c}_t^i, \hat{c}_{t+1}^i, \hat{c}_{t+2}^i)\) and \((\hat{s}_t^i, \hat{s}_{t+1}^i)\) to solve

\[ \begin{align*}
\max & \quad \log c_t^i + \gamma \log c_{t+1}^i + c_{t+2}^i \\
\text{s.t.} & \quad c_t^i + s_t^i \leq w_1 \\
& \quad c_{t+1}^i + s_{t+1}^i \leq w_2 + (1 + \hat{r}_s) s_t^i \\
& \quad c_{t+2}^i \leq w_3 + (1 + \hat{r}_{s+1}) s_{t+1}^i \\
& \quad c_t^i, c_{t+1}^i, c_{t+2}^i \geq 0.
\end{align*} \]

- \(\hat{c}_{t-2}^i + \hat{c}_{t-1}^i + \hat{c}_t^i = w_3 + w_2 + w_1, t = 1, 2, \ldots\)

- \(s_1^0 + s_1^1 = m^{-1} + m^0\)

- \(s_{t-1}^i + s_t^i = \left[ \prod_{k=1}^{t-1} (1 + \hat{r}_s) \right] (m^{-1} + m^0), t = 2, 3, \ldots.\)

3. (a) With sequential market structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A sequential markets equilibrium is sequences of rental rates \(\hat{r}_0^b, \hat{r}_1^b, \ldots\), interest rates \(\hat{r}_0^b, \hat{r}_1^b, \ldots\), wages \(\hat{w}_0, \hat{w}_1, \ldots\), consumption levels \(\hat{c}_0, \hat{c}_1, \ldots\), capital stocks \(\hat{k}_0, \hat{k}_1, \ldots\), and bond holdings \(\hat{b}_0, \hat{b}_1, \ldots\), such that

- Given \(\hat{r}_0^k, \hat{r}_1^k, \ldots, \hat{r}_0^b, \hat{r}_1^b, \ldots\), and \(\hat{w}_0, \hat{w}_1, \ldots\), the consumer chooses \(\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots\), and \(\hat{b}_0, \hat{b}_1, \ldots\) to solve
\[
\begin{align*}
\max & \sum_{t=0}^{\infty} \beta^t \log c_t \\
\text{s.t.} & \quad c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t + \hat{r}_t^b k_t + (1 + \hat{r}_t^b) b_t, \quad t = 0, 1, \ldots \\
& \quad k_0 = \overline{k}_0 \\
& \quad b_t \geq -B \\
& \quad c_t, k_t \geq 0.
\end{align*}
\]

- \( \hat{r}_t^k = \alpha \theta \hat{k}_t^{a-1} \), \( t = 0, 1, \ldots \) \\
  \( \hat{w}_t = (1 - \alpha) \theta \hat{k}_t^a \), \( t = 0, 1, \ldots \)

- \( \hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^a \), \( t = 0, 1, \ldots \)

- \( \hat{b}_t = 0 \), \( t = 0, 1, \ldots \)

(b) A Pareto efficient allocation is sequences \( \hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots \) that are feasible,

\[
\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^a, \quad t = 0, 1, \ldots \\
\hat{k}_0 \leq \overline{k}_0,
\]

and such that there exists no alternative allocation \( \hat{{\hat{c}}}_t, \hat{{\hat{k}}}_t \) that is also feasible and such that

\[
\sum_{t=0}^{\infty} \beta^t \log \hat{c}_t > \sum_{t=0}^{\infty} \beta^t \log \hat{\hat{c}}_t.
\]

In other words, the allocation \( \hat{c}_t, \hat{k}_t \) solves

\[
\begin{align*}
\max & \sum_{t=0}^{\infty} \beta^t \log c_t \\
\text{s.t.} & \quad c_t + k_{t+1} \leq \theta \hat{k}_t^a, \quad t = 0, 1, \ldots \\
& \quad k_0 \leq \overline{k}_0 \\
& \quad c_t, k_t \geq 0.
\end{align*}
\]

(c) Suppose that \( \hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t \) is an equilibrium. Then

\[
\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^a, \quad t = 0, 1, \ldots \\
\hat{k}_0 \leq \overline{k}_0,
\]

and there exist Lagrange multipliers \( p_t \geq 0 \), \( t = 0, 1, \ldots \), such that
The necessary and sufficient conditions for \( \hat{c}_t, \hat{k}_t \) to be a Pareto efficient allocation are

\[
\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^\alpha, \quad t = 0, 1, \ldots \\
\hat{k}_0 = \bar{k}_0,
\]

and that there exist some Lagrange multipliers \( \pi_t \geq 0, \ t = 0, 1, \ldots \), such that

\[
\frac{\beta^t}{\hat{c}_t} - \pi_t = 0, \quad t = 0, 1, \ldots \\
-\pi_t + \pi_{t+1} \alpha \hat{k}_{t+1}^{\alpha-1} = 0, \quad t = 0, 1, \ldots \\
\lim_{t \to \infty} \pi_t \hat{k}_{t+1} = 0.
\]

Given that \( \hat{r}_i^k, \hat{r}_i^b, \hat{w}_i, \hat{c}_i, \hat{k}_i, \hat{h}_i \) is an equilibrium, we can set \( \hat{c}_i = \hat{c}_i, \hat{k}_i = \hat{k}_i, \) and \( \pi_i = p_i \) and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(d) Bellman’s equation is

\[
V(k) = \max \log c + \beta V(k') \\
\text{s.t. } c + k' \leq \theta k^\alpha \\
c, k' \geq 0.
\]

Guessing that \( V(k) \) has the form \( a_0 + a_i \log k \), we can solve for \( c \) and \( k' \):

\[
c = \frac{1}{1 + \beta a_i} \theta k^\alpha, \quad k' = \frac{\beta a_i}{1 + \beta a_i} \theta k^\alpha.
\]

We can plug these solutions back into Bellman’s equation to obtain

\[
a_0 + a_i \log k = \log \left( \frac{1}{1 + \beta a_i} \theta k^\alpha \right) + \beta \left[ a_0 + a_i \log \left( \frac{\beta a_i}{1 + \beta a_i} \theta k^\alpha \right) \right].
\]
Collecting all the terms on the right-hand side that involve $\log k$, we can solve for $a_i$:

$$a_i = \alpha + \alpha \beta a_i$$

$$a_i = \frac{\alpha}{1 - \alpha \beta},$$

which implies that

$$c = (1 - \alpha \beta)\theta k^\alpha, \; k' = \alpha \beta \theta k^\alpha.$$

[We can also solve for $a_0$:

$$a_0 = \frac{1}{1 - \beta} \left[ \log \left( \frac{\theta}{1 + \beta a_i} \right) + \beta a_i \log \left( \frac{\beta a_i \theta}{1 + \beta a_i} \right) \right],$$

$$a_0 = \frac{1}{1 - \beta} \left[ \log \left( (1 - \alpha \beta)\theta \right) + \frac{\alpha \beta}{1 - \alpha \beta} \log \left( \alpha \beta \theta \right) \right],$$

but this is tedious, and, besides, the question does not ask us to do it.]

(e) To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{i+1} = \alpha \beta \theta k^\alpha_i$$

forward, starting at $k_0 = \bar{k}_0$. We set

$$c_i = (1 - \alpha \beta)\theta k^\alpha_i$$

$$b_i = 0$$

$$r^k_i = \alpha \theta k^{\alpha-1}_i$$

$$r^b_i = \alpha \theta k_i^{\alpha-1} - 1$$

$$w_i = (1 - \alpha)\theta k^\alpha_i.$$

Notice that this problem actually has an analytical solution:

$$k_i = \alpha \beta \theta k^\alpha_{i-1} = \alpha \beta \theta \left( \alpha \beta \theta k^\alpha_{i-2} \right)^\alpha = \left( \alpha \beta \theta \right)^{\sum_{t=0}^{i-1} \alpha^t} \bar{k}_0^{\alpha^i} = \left( \alpha \beta \theta \right)^{\frac{1 - \alpha^i}{1 - \alpha}} \bar{k}_0^{\alpha^i}.$$