Answer **two** of the following four questions.

1. Consider an overlapping generations economy in which the representative consumer born in period $t$, $t = 1, 2, \ldots$, has the utility function over consumption of the single good in periods $t$ and $t + 1$

   \[ u(c'_t, c'_{t+1}) = \log c'_t + \gamma \log c'_{t+1} \]

   and endowments $(w'_t, w'_{t+1}) = (w_t, w_{t+1})$. Suppose that the representative consumer in the initial old generation has the utility function

   \[ u^0(c^0_t) = \gamma \log c^0_t \]

   and endowment $w^0_t = w_2$ of the good in period 1 and endowment $m$ of fiat money.

   (a) Describe an Arrow-Debreu market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu equilibrium.

   (b) Describe a sequential market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define a sequential markets equilibrium.

   (c) Suppose that $m = 0$. Calculate both the Arrow-Debreu equilibrium and the sequential markets equilibrium.

   (d) Define a Pareto efficient allocation. Suppose that $\gamma = 2$ and $(w_t, w_{t+1}) = (4, 5)$. Is the equilibrium allocation in part c Pareto efficient? Explain carefully why or why not.

   (e) Suppose now that, rather than endowments of consumption goods, the consumers have endowments of labor $(\ell'_t, \ell'_{t+1}) = (\bar{\ell}_t, \bar{\ell}_2)$ and $\ell^0_t = \bar{\ell}_2$. The representative consumer in the initial old generation has an endowment of capital $\bar{k}^0_1$ and an endowment $m$ of fiat money. Final output, which can be consumed or invested is produced using the production function

   \[ \theta k_t^\alpha \ell_t^{1-\alpha}, \]

   $\theta > 0$, $0 < \alpha < 1$, and a fraction $\delta$, $0 \leq \delta \leq 1$, of capital depreciates every period. Define a sequential markets equilibrium for this economy.
2. Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer’s utility function is
\[ \sum_{t=0}^{\infty} \beta^t c_t^\rho. \]
Here \( 0 < \beta < 1 \) and \( 0 < \rho < 1 \). The consumer is endowed with 1 unit of labor in each period and with \( \bar{k}_0 \) units of capital in period 0. Feasible allocations satisfy
\[ c_t + k_{t+1} - (1-\delta)k_t \leq \theta k_t^\alpha k_{t+1}^{-\alpha} \]
\[ c_t, k_t \geq 0. \]
Here \( \theta > 0, \ 0 < \alpha < 1, \) and \( 0 \leq \delta \leq 1. \)

(a) Formulate the problem of maximizing the representative consumer’s utility subject to feasibility conditions as a dynamic programming problem. Write down the appropriate Bellman’s equation.

(b) Let \( K = [0, \bar{k}] \). Explain how you can use the feasibility condition to choose \( \bar{k} \) to be the maximum sustainable capital stock. Let \( C(K) \) be the space of continuous bounded functions on \( K \). Endow \( C(K) \) with the topology induced by the sup norm
\[ d(V,W) = \sup_{k \in K} |V(k) - W(k)| \text{ for any } V,W \in C(K). \]
Define a contraction mapping \( T : C(K) \to C(K) \).

(c) State Blackwell’s sufficient conditions for \( T \) to be a contraction. (You do not need to prove that these conditions are sufficient for \( T \) to be a contraction.)

(d) Using the Bellman’s equation from part a, define the mapping for the value function iteration algorithm,
\[ V_{n+1} = T(V_n), \]
where \( T : C(K) \to C(K) ; \) that is \( V = T(V) \) is the Bellman’s equation. (You do not need to prove that \( T(V) \in C(K) \) for all \( V \in C(K) \).) Prove that \( T \) satisfies Blackwell’s sufficient conditions to be a contraction.

(e) Specify an economic environment for which the solution to the social planner’s problem in part a is a Pareto efficient allocation/production plan. Define a sequential markets equilibrium for this environment. Explain how you could use the value function iteration algorithm \( V_{n+1} = T(V_n) \) to calculate the unique sequential markets equilibrium. (You do not have to prove that this equilibrium is unique.)
3. Consider the social planner’s problem of choosing sequences of $c_t$, $\ell_t$, and $k_t$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \left[ \log c_t + \gamma \log (1 - \ell_t) \right]$$

subject to:

$$c_t + k_{t+1} \leq \theta k_t^{\gamma} \ell_t^{1-\gamma}$$

$$c_t, k_t \geq 0, 1 \geq \ell_t \geq 0$$

$$k_0 \leq \bar{k}_0.$$  

(a) Write down the Euler conditions and the transversality condition for this problem.

(b) Formulate this social planner’s problem as a dynamic programming problem by writing down the relevant Bellman’s equation. Guessing that the value function takes the form

$$V(k) = a_0 + a_1 \log k,$$

solve for the policy functions $c = c(k)$, $\ell = \ell(k)$, $k' = k'(k)$. (Hint: the optimal value of $\ell$ does not vary with $k$.)

(c) Verify that the solution to the social planner’s generated by the policy functions in part b satisfy the Euler conditions and transversality condition in part a.

(d) Specify an economic environment for which the solution to this social planning problem is a Pareto efficient allocation. Define a sequential markets equilibrium for this economy. Explain how you can use the policy functions from part b to calculate his equilibrium.

(e) Define an Arrow-Debreu equilibrium for the economy in part d. Explain how you can use the policy functions from part b to calculate this equilibrium.
4. Consider the problem faced by an unemployed worker searching for a job. Every period that the worker searches, she receives a job offer with the wage \( w \) drawn independently from the time invariant probability distribution \( F(v) = \text{prob}(w \leq v), \ v \in [0, B], \ B > 0 \). After receiving the wage offer \( w \) the worker faces the choice (1) to accept it or (2) to reject it, receive unemployment benefit \( b \), and search again next period. That is,

\[
y_t = \begin{cases} 
  w & \text{if job offer has been accepted} \\
  b & \text{if searching}
\end{cases}
\]

The worker solves

\[
\max \ E \sum_{t=0}^{\infty} \beta^t y_t,
\]

where \( 1 > \beta > 0 \). Once a job offer has been accepted, there are no fires or quits.

(a) Formulate the worker’s problem as a dynamic programming problem by writing down Bellman’s equation.

(b) Using Bellman’s equation from part a, characterize the value function \( V(w) \) in a graph and argue that the worker’s problem reduces to determining a reservation wage \( \bar{w} \) such that she accepts any wage offer \( w \geq \bar{w} \) and rejects any wage offer \( w < \bar{w} \).

(c) Consider two economies with different unemployment benefits \( b_1 \) and \( b_2 \) but otherwise identical. Let \( \bar{w}_1 \) and \( \bar{w}_2 \) be the reservation wages in these two economies. Suppose that \( b_2 > b_1 \). Prove that \( \bar{w}_2 > \bar{w}_1 \). Provide some intuition for this result.

(d) Consider two economies with different wage distributions \( F_1 \) and \( F_2 \) but otherwise identical. Define what it means for \( F_2 \) to be a mean preserving spread of \( F_1 \).

(e) Suppose that \( F_2 \) is a mean preserving spread of \( F_1 \). Let \( \bar{w}_1 \) be the reservation wage in the economy with wage distribution \( F_1 \) and \( \bar{w}_2 \) be the reservation wage in economy with wage distribution \( F_2 \). Prove that \( \bar{w}_2 > \bar{w}_1 \). Provide some intuition for this result.