1. Consider an economy with two infinitely lived consumers. There is one good in each period. Consumer \( i, i = 1, 2 \), has the utility function

\[
\sum_{t=0}^{\infty} \beta^t \log c_i^t.
\]

Here \( \beta, 0 < \beta < 1 \), is the common discount factor. Each of the consumers is endowed with a sequence of goods:

\[
(w_0^1, w_1^1, w_2^1, w_3^1, \ldots) = (2, 2, 2, 2, \ldots)
\]

\[
(w_0^2, w_1^2, w_2^2, w_3^2, \ldots) = (1, 4, 1, 4, \ldots).
\]

There is no production or storage.

(a) Describe an Arrow-Debreu market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu equilibrium for this economy.

(b) Describe a sequential market structures for this economy, explaining when markets are open, who trades with whom, and so on. Define a sequential markets equilibrium for this economy.

(c) Carefully state a proposition or propositions that establish the essential equivalence of the equilibrium concept in part a with that in part b. Be sure to specify the relationships between the objects in the Arrow-Debreu equilibrium and those in the sequential markets equilibrium. (You do not have to prove these propositions.)

(d) Calculate the Arrow-Debreu equilibrium for this economy. (This equilibrium is unique up to a normalization of prices, but you do not have to prove this fact.) Use this answer and the answer to part c to calculate the sequential markets equilibrium.

(e) Suppose now that there is a third consumer. Consumer 3 has the same utility function

\[
\sum_{t=0}^{\infty} \beta^t \log c_3^t,
\]

and has the endowment sequence

\[
(w_0^3, w_1^3, w_2^3, w_3^3, \ldots) = (4, 4, 1, 4, \ldots).
\]

Define a sequential markets equilibrium.
2. Consider an overlapping generations economy in which the representative consumer born in period \( t, t = 1, 2, \ldots \), has the utility function over consumption of the single good in periods \( t \) and \( t + 1 \)

\[
u(c_t, c_{t+1}) = c_t^{\prime} + \log c_{t+1}^{\prime}
\]

and endowments \((w_t', w_{t+1}') = (w_1, w_2)\). \( (\text{Notice that the utility function is not } \log c_t^{\prime} + \log c_{t+1}^{\prime}.\)\)

Suppose that the representative consumer in the initial old generation has the utility function

\[
u^0(c_0) = \log c_0
\]

and endowment \( w_0^0 = w_2 \) of the good in period 1 and endowment \( m \) of fiat money.

(a) Describe an Arrow-Debreu market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu equilibrium for this economy.

(b) Describe a sequential market structures for this economy, explaining when markets are open, who trades with whom, and so on. Define a sequential markets equilibrium for this economy.

(c) Suppose that \( m = 0 \). Calculate both the Arrow-Debreu equilibrium and the sequential markets equilibrium.

(d) Define a Pareto efficient allocation. Suppose that \( w_2 > 1 \). Is the equilibrium allocation in part c Pareto efficient? Explain carefully why or why not.

(e) Suppose now that there are two types of consumers of equal measure in each generation. The representative consumer of type 1 born in period \( t, t = 1, 2, \ldots \), has the utility function over consumption of the single good in periods \( t \) and \( t + 1 \)

\[
u_1(c_t^{\prime}, c_{t+1}^{\prime}) = c_t^{\prime} + \log c_{t+1}^{\prime},
\]

while the representative consumer of type 2 has the utility function

\[
u_2(c_{t+1}^{\prime}, c_{t+1}^{\prime}) = \log c_t^{2t} + c_{t+1}^{2t}.
\]

The endowments of these consumers are \((w_1^{it}, w_2^{it}) = (w_1^1, w_2^1), i = 1, 2 \). The representative consumers of type 1 and 2 who live only in period 1 have utility functions \( \log c_1^{10} \) and \( c_1^{20} \), endowments \( w_1^{10} = w_2^1 \) and \( w_1^{20} = w_2^2 \) of the good in period 1, and endowments \( m^1 \) and \( m^2 \) of fiat money. Define an Arrow-Debreu equilibrium for this economy.
3. Consider an overlapping generations economy in which the representative consumer in generation \( t, t = 1, 2, \ldots \), has preferences over the consumption of the single good in each of the two periods of her life given by the utility function

\[
   u(c_t', c_{t+1}') = \log c_t' + \log c_{t+1}' .
\]

This consumer is endowed with quantities of labor \((\ell_t', \ell_{t+1}') = (\ell_1', \ell_2')\). In addition, there is a generation 0 who representative consumer lives only in period 1 and has the utility function

\[
   u^0(c_1^0) = \log c_1^0 ,
\]

and the endowment of \(\bar{\ell}_2\) units of labor and \(\bar{k}_1^0\) units of capital in period 1. In addition, this consumer has an endowment of fiat money \(m\), which can be positive, negative or zero.

The production function is

\[
   f(k_t, \ell_t) = \theta k^{\alpha} \ell^{1-\alpha} ,
\]

where \(\theta > 1\) and \(1 > \alpha > 0\). Capital depreciates at the rate \(\delta\) per period, \(0 \leq \delta \leq 1\).

(a) Define a sequential market equilibrium for this economy.

(b) Assume that consumers own capital and rent it to the firms. Define an Arrow-Debreu equilibrium for this economy. State and prove two theorems that establish the equivalence between a sequential market equilibrium and an Arrow-Debreu equilibrium.

(c) Suppose that the generation of consumers born in period \(t, t = 1, 2, \ldots\), live for three periods and have the utility function

\[
   u(c_t', c_{t+1}', c_{t+2}') = \log c_t' + \log c_{t+1}' + \log c_{t+2}'
\]

and labor endowments \((\ell_t', \ell_{t+1}', \ell_{t+2}') = (\ell_1', \ell_2', \ell_3')\). In addition there is an old generation, generation \(-1\), that lives only in period 0 and a middle aged generation, generation 0, that lives in period 1 and 2. The consumers in these generations have utility functions

\[
   u(c_1^{-1}) = \log c_1^{-1}
\]

\[
   u(c_1^0, c_2^0) = \log c_1^0 + \log c_2^0 ,
\]

endowments of labor \(\ell_1^{-1} = \ell_3\) and \((\ell_1^0, \ell_2^0) = (\ell_2, \ell_3)\), endowments of capital \(\bar{k}_1^{-1}\) and \(\bar{k}_1^0\), and endowments of fiat money \(m^{-1}\) and \(m^0\). Define a sequential market equilibrium for this economy.

(d) Assume that consumers own capital and rent it to the firms. Define an Arrow-Debreu equilibrium for the economy in part c.