1. Consider an economy two infinitely lived consumers, each of whom has the same utility function,

\[ u(c_0^i, c_1^i, \ldots) = \sum_{t=0}^{\infty} \beta^t \log c_t^i \]

where \( 0 < \beta < 1 \). Suppose that consumer 1 has the endowments

\[ (w_0^1, w_1^1, w_2^1, w_3^1, \ldots) = (4, 2, 4, 2, \ldots) \]

and consumer 2 has the endowments

\[ (w_0^2, w_1^2, w_2^2, w_3^2, \ldots) = (2, 4, 2, 4, \ldots) \]

a) Describe an Arrow-Debreu market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu equilibrium.

b) Define a Pareto efficient allocation for this economy. Calculate a Pareto efficient allocation by maximizing a weighted sum of utilities, \( \alpha_1 u_1 + \alpha_2 u_2 \).

c) Define an Arrow-Debreu equilibrium with transfers. Find the transfer payments necessary to implement the Pareto efficient allocation in part b as equilibrium with transfers. Demonstrate that the transfer payments are homogeneous of degree one in \( (\alpha_1, \alpha_2) \) and sum to 0.

d) Find the transfer payments necessary to implement the allocation \( (c_1^1, c_1^2) = (3, 3) \) as an equilibrium with transfers.

e) Calculate the (unique) Arrow-Debreu equilibrium of this economy.

f) Define a sequential markets equilibrium. Calculate the unique sequential markets equilibrium of the economy.

2. Consider a simple overlapping generations economy in which the consumer born in period \( t, t = 1, 2, \ldots \), has the utility function
where \( b < 1 \). Suppose that his endowment is \((w', w'_{t+1}) = (w_1, w_2)\).

a) What is the utility function in the case where \( b = 0 \)? [Hint: use l’Hôpital’s rule.]

b) Write down the utility maximization problem in an environment with Arrow-Debreu markets. Derive the excess demand functions \( y(p_t, p_{t+1}) \) and \( y(p_t, p_{t+1}) \). Demonstrate that they are homogeneous of degree zero and that they satisfy Walras’s law.

c) Suppose that the first generation has an excess demand function of the form

\[
z_0(p_t, m) = \frac{m}{p_t}.
\]

Explain the role of \( m \). Define an Arrow-Debreu equilibrium of this model. Write down the equilibrium conditions using the excess demand functions.

d) Find an expression for the offer curve for this model. (Hint: you have to solve for \( y \) as a function of \( z \).)

e) Suppose that \( w_1 = 1 \) and \( w_2 = 0.25 \). Draw the offer curve for the three cases \( b = 0.5 \), \( b = 0 \), and \( b = -1 \).

3. Consider an overlapping generations economy in which the representative consumer in generation \( t \), \( t = 1, 2, \ldots \), has preferences over the consumption of the single good in each of the two periods of her life given by the utility function

\[
u(c'_t, c'_{t+1}) = \log c'_t + \log c'_{t+1}.
\]

This consumer is endowed with quantities of labor \((\ell'_t, \ell'_{t+1}) = (\ell_1, \ell_2)\). In addition there is a generation 0 who representative consumer lives only in period 1 and has the utility function

\[
u^0(c^0_t) = \log c^0_t,
\]

and the endowment of \( \ell_2 \) units of labor and \( k_1 \) units of capital in period 1. In addition, this consumer has an endowment of fiat money \( m \), which can be positive, negative or zero.

The production function is
\[ f(k_t, \ell_t) = \theta k_t^\alpha \ell_t^{1-\alpha}, \]

and capital depreciates at the rate \( \delta \) per period, \( 0 \leq \delta \leq 1 \).

a) Define a sequential market equilibrium for this economy.

b) Define an Arrow-Debreu equilibrium for this economy. State and prove two theorems that establish the equivalence between a sequential market equilibrium and an Arrow-Debreu equilibrium.

c) Reduce the equilibrium conditions to a second-order difference equation in \( k_t \), that is, an equation in \( k_{t+1}, k_t, k_{t-1} \) that includes no other endogenous variables.

d) Suppose that \( m = 0 \). Reduce the equilibrium conditions to a first-order difference equation in \( k_t \). (Hint: in this case you know that the savings of generation \( t \) in period \( t \) in the sequential market equilibrium must equal \( k_{t+1} \).)

4. Suppose that \( \theta = 100, \alpha = 0.4, \delta = 0.8, \ell_1 = 1 \), and \( \ell_2 = 0 \) in question 3.

a) Define a steady state for this economy. Calculate the two steady states.

b) Suppose that \( \bar{k}_1 = 10 \) and \( m = 0 \). Use your answer to question 3d to calculate the equilibrium in the first 10 periods both by hand and on the computer.

c) Suppose now that \( \ell_2 = 0.5 \). Repeat parts a and b, doing your calculations on the computer.

d) Suppose now that \( \ell_2 = 2 \). Repeat parts a and b, doing your calculations on the computer.