1. Consider an economy two infinitely lived consumers, each of whom has the same utility function,

\[ u(c^t_0, c^t_1, \ldots) = \sum_{t=0}^{\infty} \beta^t \log c^t_i, \]

where \( 0 < \beta < 1 \). Suppose that consumer 1 has the endowments

\[ (w^1_0, w^1_1, w^1_2, w^1_3, \ldots) = (5, 1, 5, 1, \ldots), \]

and consumer 2 has the endowments

\[ (w^2_0, w^2_1, w^2_2, w^2_3, \ldots) = (1, 5, 1, 5, \ldots). \]

a) Describe an Arrow-Debreu market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu equilibrium.

b) Define a Pareto efficient allocation for this economy. Calculate a Pareto efficient allocation by maximizing a weighted sum of utilities, \( \alpha_u u_1 + \alpha u_2 \).

c) Define an Arrow-Debreu equilibrium with transfers. Find the transfer payments necessary to implement the Pareto efficient allocation in part b as equilibrium with transfers. Demonstrate that the transfer payments are homogeneous of degree one in \( (\alpha_1, \alpha_2) \) and sum to 0.

d) Find the transfer payments necessary to implement the allocation \( (c^1_t, c^2_t) = (3, 3) \) as an equilibrium with transfers.

e) Calculate the (unique) Arrow-Debreu equilibrium of this economy.

f) Define a sequential markets equilibrium. Calculate the unique sequential markets equilibrium of the economy.
2. Consider a simple overlapping generations economy in which the representative consumer born in period \( t \), \( t = 1, 2, \ldots \), has the utility function

\[
u(c'_t, c'_{t+1}) = c'_t + \left[ \left( \frac{c'_t}{c'_{t+1}} \right)^b - 1 \right] / b,
\]

where \( b < 1 \). Suppose that his endowment is \((w'_t, w'_{t+1}) = (w_1, w_2)\).

a) What is the utility function in the case where \( b = 0 \)? [Hint: use l’Hôpital’s rule.]

b) Write down the utility maximization problem in an environment with Arrow-Debreu markets. Derive the excess demand functions \( y(p_t, p_{t+1}) \) and \( y(p_t, p_{t+1}) \). Demonstrate that they are homogeneous of degree zero and that they satisfy Walras’s law.

c) Suppose that the representative consumer in the first generation has the utility function

\[
u(0^0, c'_{t+1}) = \left[ \left( c_{t+1}^0 \right)^b - 1 \right] / b.
\]

This consumer is endowed with \( w_1^0 = w_2 \) of the good in period 1 as well as \( m \) units of fiat money, where \( m \) can be positive, negative, or 0. Explain the role of \( m \). Define an Arrow-Debreu equilibrium of this model. Write down the equilibrium conditions using the excess demand functions.

d) Find an expression for the offer curve for this model. (Hint: you have to solve for \( y \) as a function of \( z \).)

e) Suppose that \( w_1 = 1 \) and \( w_2 = 0.25 \). Draw the offer curve for the three cases \( b = 0.5 \), \( b = 0 \), and \( b = -1 \).

f) Define a sequential market equilibrium for this economy. Suppose that you have calculated the Arrow-Debreu equilibrium in part d. Explain how you can use the Arrow-Debreu equilibrium to calculate the sequential market equilibrium.

3. Consider an overlapping generations economy in which the representative consumer in generation \( t \), \( t = 1, 2, \ldots \), has preferences over the consumption of the single good in each of the two periods of her life given by the utility function

\[
u(c'_t, c'_{t+1}) = \log c'_t + \log c'_{t+1}.
\]

This consumer is endowed with quantities of labor \((\ell'_t, \ell'_{t+1}) = (\ell_1, \ell_2)\). In addition there is a generation 0 who representative consumer lives only in period 1 and has the utility function
and the endowment of $\ell_2$ units of labor and $k_1$ units of capital in period 1. In addition, this consumer has an endowment of fiat money $m$, which can be positive, negative or zero.

The production function is

$$f(k_t, \ell_t) = \theta k_t^\alpha \ell_t^{1-\alpha},$$

and capital depreciates at the rate $\delta$ per period, $0 \leq \delta \leq 1$.

a) Define a sequential market equilibrium for this economy.

b) Define an Arrow-Debreu equilibrium for this economy. State and prove two theorems that establish the equivalence between a sequential market equilibrium and an Arrow-Debreu equilibrium.

c) Reduce the equilibrium conditions to a second-order difference equation in $k_t$, that is, an equation in $k_{t+1}$, $k_t$, $k_{t-1}$ that includes no other endogenous variables.

d) Suppose that $m = 0$. Reduce the equilibrium conditions to a first-order difference equation in $k_t$. (Hint: in this case you know that the savings of generation $t$ in period $t$ in the sequential market equilibrium must equal $k_{t+1}$.)

4. Suppose that $\theta = 100$, $\alpha = 0.4$, $\delta = 0.8$, $\ell_1 = 1$, and $\ell_2 = 0$ in question 3.

a) Define a steady state for this economy. Calculate the two steady states.

b) Suppose that $k_1 = 10$ and $m = 0$. Use your answer to question 3, part d to calculate the equilibrium in the first 10 periods both by hand and on the computer.

c) Suppose now that $\ell_2 = 0.5$. Repeat parts a and b, doing your calculations on the computer.

d) Suppose now that $\ell_2 = 2$. Repeat parts a and b, doing your calculations on the computer.