1. Consider an overlapping generations economy in which there is one good in each period and each generation, except the initial one, lives for two periods. The representative consumer in generation $t$, $t = 1, 2, \ldots$, has the utility function

$$\log c'_t + \log c'_{t+1}$$

and the endowment $(w'_t, w'_{t+1}) = (w_1, w_2)$. The representative consumer in generation 0 lives only in period 1, prefers more consumption to less, and has the endowment $w'^0_1 = w_2$. There is no fiat money.

a) Define an Arrow-Debreu equilibrium for this economy. Calculate the unique Arrow-Debreu equilibrium.

b) Define a sequential markets equilibrium for this economy. Calculate the unique sequential markets equilibrium.

c) Define a Pareto efficient allocation. Letting the endowment process $w'_1, w'_2$ vary, where $w'^0_1 = w_2$, $(w'_t, w'_{t+1}) = (w_t, w_{t+1})$, produce an example of an economy for which you can prove the equilibrium allocation is Pareto efficient. Sketch out the proof.

d) Suppose now that there are two types of consumers born in each period $t$, $t = 1, 2, \ldots$. All consumers have the same utility function

$$\log c''_t + \log c''_{t+1}, \quad i = 1, 2.$$

The representative consumer of type 1 has endowments $(w'_t, w'_{t+1}) = (6, 1)$. The representative consumer of type 2 has endowments $(w'_t, w'_{t+1}) = (2, 3)$. In addition to these consumers, there is an initial generation 0, made up of two types of consumers who live only in period 1. All consumers in this initial generation have the utility function

$$\log c''^0_i, \quad i = 1, 2.$$

The endowments of the representative consumers of each type are $w'^0_1 = 1$ and $w'^0_2 = 3$. Once again there is no fiat money. Define an Arrow-Debreu equilibrium for this economy. Calculate this equilibrium.
2. Consider an economy in which there is a representative consumer who lives forever. In every period, \( t = 0, 1, \ldots \), one of three random events occurs, \( \eta_t = 1, \eta_t = 2, \) or \( \eta_t = 3 \). At \( t = 0 \), the initial state is \( \eta_0 \) and a stationary Markov process given by a \( 3 \times 3 \) matrix with elements \( \pi_{ij} = \text{prob}(\eta_{t+1} = j | \eta_t = i) \) governs the probability of future states. Let \( \pi(s) \) be the induced probability distribution over states. The representative consumer has the utility function

\[
\sum_{s \in S} \beta^{t(s)} \pi(s) u(c_s, 1 - \ell_s).
\]

Here \( S \) is the set of all states, \( 0 < \beta < 1 \), \( t(s) \) is the date that state \( s \) occurs in, \( c_s \) is the consumption of the single good in that state, and \( 1 - \ell_s \) is the consumption of leisure. Assume that \( u \) is strictly concave, increasing, and continuously differentiable. The consumer is endowed with one unit of labor in each state and \( \bar{k}_0 \) units of capital in the initial state. Feasible consumption/investment plans satisfy the restrictions

\[
c_s + k_{s+1} - (1 - \delta)k_s \leq \theta_s k_s \rho^{1-\alpha} \]

at all \( s \in S \). Here \( k_{s+1} \) denotes the value of \( k \) in each of the three states that immediately follow \( k_s \), which are \( k_{(s,1)}, k_{(s,2)}, \) and \( k_{(s,3)} \); \( 0 \leq \delta \leq 1 \); and \( 0 < \alpha < 1 \). In addition, \( \theta_s \) takes on one of three values as determined by the current event, \( \theta_s = \theta_i \) if \( \eta_s = i \), where \( \theta_1 < \theta_2 < \theta_3 \).

da) Define an Arrow-Debreu equilibrium for this economy.

b) Define a sequential markets equilibrium for this economy. Carefully state a proposition or propositions that establish the essential equivalence between this equilibrium concept and that in part a. Be sure to specify the relationships between the objects in the Arrow-Debreu equilibrium and those in the sequential markets equilibrium.

c) Define a Pareto efficient allocation for this economy. Specify the dynamic programming problem that such an allocation must satisfy by writing down its Bellman’s equation. Suppose that you have a solution to this dynamic programming problem. Explain carefully how you could calculate the values of the variables in the definition of an Arrow-Debreu equilibrium in any state \( s = (\eta_0, \eta_1, \ldots, \eta_t) \). Explain carefully how you could calculate the values of the variables in the definition of a sequential markets equilibrium in any state \( s = (\eta_0, \eta_1, \ldots, \eta_t) \).

d) Consider now an economy with two types of consumers, rather than one, but otherwise identical to that in parts a, b, and c. The representative consumer of type \( i \), \( i = 1, 2 \), has the utility function

\[
\sum_{s \in S} \beta^{t(s)} \pi(s) u_i(c_i, \ell_i - \ell_i),
\]

an endowment of \( \bar{\ell}^i \) units of labor in every state and an endowment of \( \bar{k}_0^i \) units of capital in the initial state. Define a sequential markets equilibrium.
3. Consider an economy with a continuum $[0,1]$ of consumers of two symmetric types who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t u(c^i_t)$$

where $0 < \beta < 1$ and $u$ is strictly concave, increasing, and continuously differentiable. Consumers of type $i$ have an endowment stream of the single good in each period

$$(w^i_0, w^i_1, w^i_2, w^i_3, \ldots) = (\omega^b, \omega^b, \omega^b, \omega^b, \ldots),$$

while consumers of type 2 have

$$(w^2_0, w^2_1, w^2_2, w^2_3, \ldots) = (\omega^s, \omega^s, \omega^s, \omega^s, \ldots)$$

where $\omega^s > \omega^b$. In addition there is one unit of trees that produce $r$ units of the good every period. Each consumer of type $i$ owns $\tilde{\theta}^i_0$ of such trees in period 0, $\tilde{\theta}^i_0 \geq 0$, $\tilde{\theta}^0_0 + \tilde{\theta}^2_0 = 1$.

a) Define an Arrow-Debreu equilibrium for this economy. Define the corresponding sequential markets equilibrium. Find initial asset holdings $\tilde{\theta}^i_0$ and $\tilde{\theta}^2_0$ such that, in equilibrium, $\hat{c}^i_t = \hat{c}^2_t$, $t = 0, 1, \ldots$

b) Suppose now that contracts can be enforced only by the threat of exclusion from future trades and the seizure of any tree holdings. That is, at any date, consumers of either type can opt to renege on any debts and to subsist in autarky where $c^i_t = w^i_t$ forever after. Define an Arrow-Debreu equilibrium for this economy with debt constraints. Define the corresponding sequential markets equilibrium.

c) Define a symmetric steady state in which consumption depends only on the value of $w^i_t$, but not on $i$ or $t$. (You can base this definition either on the Arrow-Debreu equilibrium or the sequential markets equilibrium.) Consider the function

$$f^D(c) = u(c) - u(\omega^s) + \beta \left( u(\omega^s + \omega^b + r - c) - u(\omega^b) \right).$$

Let $\hat{c}$ be the symmetric consumption in the equilibrium in part a. Explain carefully why, if $f^D(\hat{c}) > 0$, then the symmetric steady state consumption allocation in the economy with debt constraints is the same as in the economy without debt constraints. Explain why, if $f^D(\hat{c}) < 0$, however, the symmetric steady state consumption allocation must give higher consumption to the consumer when $w^i_t = \omega^s$ than when $w^i_t = \omega^b$. Also explain how to calculate the values of all of the variables in the definition of a symmetric steady state in this case.

d) Suppose now that the endowments $w^i_t$ are random. Specifically, letting $\eta_t \in \{1, 2\}$ be the event that occurs in period $t$, assume that
The events $\eta_t$ follow a symmetric, stationary Markov process in which

$$\text{prob}(\eta_{t+1} = 1|\eta_t = 2) = \text{prob}(\eta_{t+1} = 2|\eta_t = 1) = \pi.$$ 

Generalize your definitions of an equilibrium (either Arrow-Debreu or sequential markets) and a symmetric steady state to this environment with uncertainty. What is the generalization of the function $f^D(c)$ in part c? Explain briefly why this function works in determining consumption in the symmetric stochastic steady state.