Balanced Growth

Kaldor’s “stylized facts”

1. \( Y_t / N_t \) (output per working age person) exhibits continual growth.
2. \( K_t / N_t \) (capital per working age person) exhibits continual growth.
3. \( r_t - \delta \) (real interest rate) is roughly constant.
4. \( K_t / Y_t \) (capital-output ratio) is roughly constant.
5. \( r, K_t / Y_t, \ w_t L_t / Y_t \) (factor shares) are roughly constant.
6. There are wide differences in the rate of growth of \( Y_t / N_t \) across countries.

I have modified Kaldor’s stylized facts to put things in terms of output per working age person and capital per working age person, rather than output per worker and capital per worker. This is the way Kehoe and Prescott do it.


The growth model

Production function:

\[
Y_t = A_t K_t^\alpha (\gamma^L L_t)^{1-\alpha}.
\]

Representative consumer’s problem:

\[
\max \sum_{t=0}^{\infty} \beta^t \left[ \theta \log C_t + (1 - \theta) \log(N_t h - L_t) \right]
\s.t. \quad C_t + K_{t+1} - K_t + B_{t+1} - B_t \leq w_t L_t + (r_t - \delta)(K_t + B_t) \]
\[
C_t, K_t \geq 0, \quad N_t h \geq L_t \geq 0, \quad B_t \geq -(\eta \gamma) K_t^\alpha \quad K_0 = K_0^0, \quad B_0 = 0 \]
\[
N_t = n_t^t N_0.
\]

where \( w_t = (1-\alpha)(\gamma^{1-\alpha}) A_t K_t^\alpha L_t^{1-\alpha}, \ r_t = \alpha(\gamma^{1-\alpha}) A_t K_t^{\alpha-1} L_t^{1-\alpha}. \)
We can solve

\[
\begin{aligned}
\max & \quad \sum_{t=0}^{\infty} \beta^t \left[ \theta \log C_t + (1-\theta) \log (N_t \bar{h} - L_t) \right] \\
\text{s. t.} & \quad C_t + K_{t+1} - (1-\delta)K_t \leq (\gamma^{t-\alpha})^t A_0 K_t^{\alpha} L_t^{1-\alpha} \\
& \quad C_t, K_t \geq 0, \quad N_t \bar{h} \geq L_t \geq 0 \\
& \quad K_0 = \bar{K}_0 \\
& \quad N_t = \eta^r N_0.
\end{aligned}
\]

First-order conditions:

\[
\frac{\beta^t \theta}{C_t} = p_t
\]

\[
\frac{\beta^t (1-\theta)}{N_t \bar{h} - L_t} = p_t (1-\alpha)(\gamma^{t-\alpha})^t A_0 K_t^{\alpha} L_t^{1-\alpha}
\]

\[
p_{t-1} = p_t \left( \alpha(\gamma^{t-\alpha})^t A_0 K_t^{\alpha-1} L_t^{1-\alpha} + 1-\delta \right).
\]

Feasibility:

\[
C_t + K_{t+1} - (1-\delta)K_t = (\gamma^{t-\alpha})^t A_0 K_t^{\alpha} L_t^{1-\alpha}.
\]

Suppose that \( C_t / N_t \) and \( K_t / N_t \) grow at the same constant rate:

\[
\frac{C_{t+1}}{N_{t+1}} = \frac{K_{t+1}}{N_{t+1}} = g.
\]

Let us argue that \( g = \gamma \) and that \( L_t / N_t = L_{t-1} / N_{t-1} \). From feasibility, we obtain

\[
\frac{C_t}{N_t} + \frac{K_{t+1}}{N_t} - (1-\delta) \frac{K_t}{N_t} = (\gamma^{t-\alpha})^t A_0 \left( \frac{K_t}{N_t} \right)^{\alpha} \left( \frac{L_t}{N_t} \right)^{1-\alpha}
\]

\[
\frac{C_t}{N_t} + (g \eta - 1 + \delta) \frac{K_t}{N_t} = (\gamma^{t-\alpha})^t A_0 \left( \frac{K_t}{N_t} \right)^{\alpha} \left( \frac{L_t}{N_t} \right)^{1-\alpha}
\]

\[
\frac{C_t}{N_t} + (g \eta - 1 + \delta) \frac{K_t}{N_t} = (\gamma^{t-\alpha})^t A_0 \left( \frac{K_t}{N_t} \right)^{\alpha} \left( \frac{L_t}{N_t} \right)^{1-\alpha}
\]

\[
\frac{C_{t-1}}{N_{t-1}} + (g \eta - 1 + \delta) \frac{K_{t-1}}{N_{t-1}} = (\gamma^{t-\alpha})^{t-1} A_0 \left( \frac{K_{t-1}}{N_{t-1}} \right)^{\alpha} \left( \frac{L_{t-1}}{N_{t-1}} \right)^{1-\alpha}
\]

\[
g = \gamma^{t-\alpha} g^\alpha \left( \frac{L_t / N_t}{L_{t-1} / N_{t-1}} \right)^{1-\alpha}
\]
\[ g = \gamma^{1-a} g^a g_t^{1-a}, \]  

(1)

where

\[ g_t = \frac{L_t / N_t}{L_{t-1} / N_{t-1}}, \]

which is necessarily constant.

The first-order conditions also imply that

\[
\frac{(1-\theta)C_t}{\theta(N_t \bar{h} - L_t)} = (1-\alpha)(\gamma^{1-a}) A_0 K_t^\alpha L_t^{-a}
\]

\[
\frac{(1-\theta)C_t / N_t}{\theta(\bar{h} - L_t / N_t)} = (1-\alpha)(\gamma^{1-a}) A_0 \left( \frac{K_t}{N_t} \right)^\alpha \left( \frac{N_t}{L_t} \right)^{-a}
\]

\[
\frac{g(\bar{h} - L_{t-1} / N_{t-1})}{\bar{h} - L_t / N_t} = \gamma^{1-a} g^a g_t^{-a}.
\]

(2)

Dividing the expression (1) that we obtained using the feasibility condition by this expression (2), we obtain

\[
\frac{\bar{h} - L_t / N_t}{\bar{h} - L_{t-1} / N_{t-1}} = g_t,
\]

\[
\bar{h} - L_t / N_t = g_t (\bar{h} - L_{t-1} / N_{t-1})
\]

\[
\bar{h} = g_t \bar{h}
\]

\[
g_t = 1,
\]

which implies that

\[
\frac{C_{t+1} / N_{t+1}}{C_t / N_t} = \frac{K_{t+1} / N_{t+1}}{K_t / N_t} = \frac{Y_{t+1} / N_{t+1}}{Y_t / N_t} = \gamma.
\]

Redefine variables \( C_t, K_t, \) and \( Y_t \) by dividing by effective working age persons \( \bar{N}_t = \gamma' N_t = (\gamma \eta)' N_0 \). Divide \( L_t \) by \( N_t \):

\[
\tilde{c}_t = C_t / \bar{N}_t = \gamma^{-1} (C_t / N_t)
\]
\[ \tilde{k}_i = K_i / \tilde{N}_i = \gamma^{-t} (K_i / N_i) \]
\[ \tilde{y}_i = Y_i / \tilde{N}_i = \gamma^{-t} (Y_i / N_i) \]
\[ \tilde{\ell}_i = L_i / N_i \]

Notice that
\[
\theta \log C_i / N_i + (1 - \theta) \log (\tilde{h}N_i - L_i) / N_i = \theta \log \gamma^t \tilde{c}_i + (1 - \theta) \log (\tilde{h} - \tilde{\ell}_i)
\]
\[
\theta \log C_i / N_i + (1 - \theta) \log (\tilde{h}N_i - L_i) / N_i = \theta \log \tilde{c}_i + (1 - \theta) \log (\tilde{h} - \tilde{\ell}_i) + t \theta \log \gamma,
\]
where \( t \theta \log \gamma \) is a constant that we can ignore in the maximization problem.

(A more consistent notation might be to write \( \ell_i = L_i / N_i \) because we are not dividing by \( \gamma^t \).)

Notice that the balanced growth path is the steady state \( \tilde{c}_i = \tilde{c}, \tilde{k}_i = \tilde{k}, \tilde{\ell}_i = \tilde{\ell} \) of the redefined problem
\[
\max \sum_{t=0}^{\infty} \beta^t \left[ \theta \log \tilde{c}_i + (1 - \theta) \log (\tilde{h} - \tilde{\ell}_i) \right]
\]
\[
s. t. \quad \tilde{c}_i + \gamma \eta \tilde{k}_{i+1} - (1 - \delta) \tilde{k}_i \leq A_0 \tilde{k}_i \tilde{\ell}_i^{1-\alpha} \\
\tilde{c}_i, \tilde{k}_i \geq 0, \quad \tilde{h} \geq \tilde{\ell}_i \geq 0 \\
\tilde{k}_0 = K_0 / N_0.
\]

The balanced growth path matches Kaldor’s stylized facts (although the explanation for fact 6 is not very interesting):

1. \( Y_i / N_i = (\gamma^{1-\alpha})^t A_0 (K_i / N_i)^{\alpha} (L_i / N_i)^{1-\alpha} = \gamma^t A_0 \tilde{k}^a \tilde{\ell}^{1-\alpha} \) grows at rate \( \gamma - 1 \).

2. \( K_i / N_i = \gamma^t \tilde{k} \) grows at rate \( \gamma - 1 \).

3. \( r_i - \delta = \alpha (\gamma^{1-\alpha})^t A_0 K_i^{\alpha-1} L_i^{1-\alpha} - \delta = \alpha A_0 \tilde{k}^{a-1} \tilde{\ell}^{1-\alpha} - \delta = \gamma \eta / \beta - 1 \) is constant.

4. \( K_i / Y_i = \tilde{k} / (A_0 \tilde{k}^{1-\alpha} \tilde{\ell}^{1-\alpha}) \) is constant.

5. \( r_i K_i / Y_i = \alpha, \ w_i L_i / Y_i = 1 - \alpha \) are constant.

6. rate of growth of \( Y_i / N_i \) is determined solely by \( \gamma \).