An Example of a Model with Debt Constraints

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Consider an economy like that in question 2 in problem set 7 where there is a continuum $[0,1]$ of consumers of two symmetric types who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t \log c_i^t.$$ 

Suppose, as in the example in Kehoe and Levine (2001), that $\beta = 0.5$ and that consumers of type 1 have an endowment stream of the single good in each period

$$(w_0^1, w_1^1, w_2^1, w_3^1, ...) = (\omega^g, \omega^b, \omega^g, \omega^b, ...) = (24, 9, 24, 9, ...),$$ 

while consumers of type 2 have

$$(w_0^2, w_1^2, w_2^2, w_3^2, ...) = (\omega^b, \omega^g, \omega^b, \omega^g, ...) = (9, 24, 9, 24, ...).$$ 

In addition there is one unit of trees that produce $r = 1$ units of the good every period. Each consumer of type $i$ owns $\theta_0^i$ of such trees in period 0, $\theta_0^i \geq 0$, $\theta_0^i + \theta_1^i = 1$. Trees do not grow or decay.

In the sequential markets version of this model, consumers solve the problem

$$\max \sum_{t=0}^{\infty} \beta^t \log c_i^t$$

s.t. $c_i^t + v_i \theta_{t+1}^i \leq w_i^t + (v_i + r) \theta_i^t$

$$\theta_i^t \geq 0$$

$$\theta_i^t \geq -\Theta$$

$$\theta_0^i = \bar{\theta}_0^i.$$ 

An equilibrium is sequences $\hat{v}_i, \hat{c}_i^i, \hat{\theta}_i^i$ such that

1. Given $\hat{v}_i$, the consumers choose $\hat{c}_i^i, \hat{\theta}_i^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}_i^1 + \hat{c}_i^2 = w_i^1 + w_i^2 + r = 24 + 9 + 1 = 34.$$ 

3. Asset markets clear:
\[ \hat{\theta}_i + \hat{\theta}_i^2 = 1. \]

A symmetric steady state is \( \nu, (c^g, c^b), (\theta^g, \theta^b) \) such that \( \nu_i = \nu \),

\[ c_i = \begin{cases} c^g & \text{if } w_i = 24 \\ c^b & \text{if } w_i = 9 \end{cases}, \]

and

\[ \theta_i = \begin{cases} \theta^g & \text{if } w_i = 24 \\ \theta^b & \text{if } w_i = 9 \end{cases} \]

satisfy the equilibrium conditions for the right choice of \( (\hat{\theta}_0^1, \hat{\theta}_0^2) \).

Consider the function

\[ f^D(c^g) = \log c^g - \log 24 + \beta(\log(34 - c^g) - \log 9). \]

Let \( \tilde{c} = (w^1_i + w^2_i + r) / 2 = 17 \). Notice that

\[ f^D(c^g) = f^D(24) = \log 24 - \log 24 + \beta(\log(34 - 24) - \log 9) = \beta(\log 10 - \log 9) > 0. \]

Notice too that \( f^D \) is concave:
\[ D^2 f(c) = -\frac{1}{c^2} - \frac{\beta}{(34-c)^2} < 0. \]

Consequently, if \( f^D(\hat{c}) < 0 \) as in the diagram, \( f^D(c) \) can equal 0 only once in the interval \([\hat{c}, \omega^S]\).

There are two possibilities: Either
\[ f^D(\hat{c}) = \log \hat{c} - \log 24 + \beta (\log(34 - \hat{c}) - \log 9) \geq 0 \]
or there exists \( c^S \in [\hat{c}, \omega^S] \) such that
\[ f^D(c^S) = \log c^S - \log 24 + \beta (\log(34 - c^S) - \log 9). \]

Since
\[ f^D(\hat{c}) = \log 17 - \log 24 + 0.5(\log 17 - \log 9) < 0, \]
we look for such a \( c^S \). Setting \( f^D(c^S) = 0 \), we obtain \( c^S = 18 \). We want to find values of \( c^b, \theta^g, \theta^b \), and \( v \) such that these variables constitute a symmetric steady state. Obviously, \( c^b = 34 - 18 = 16 \).

The first order conditions for the consumer’s problem are
\[ \beta' \frac{1}{c_i} - \lambda_i^i + \beta' \frac{1}{c_i} \sum_{i=0}^{t^i} \mu_i^i = 0 \]
\[ -\lambda_i^i v_i + \lambda_{i+1}^i (v_{i+1} + r) = 0. \]

The individual rationality constraint, if it binds at all, can bind only when \( c_i^i = c^S \). (The consumer who receives the bad shock is always happy to receive more that his current income.) In other words, if \( c_i^i = c^b \), then \( \mu_i^i = 0 \). Consider a situation where \( c_i^i = c^S \) and \( c_{i+1}^i = c^b \). Then the first order conditions with respect to \( c_i^i \) and \( c_{i+1}^i \) are
\[ \beta' \frac{1}{c_i^i} - \lambda_i^i + \beta' \frac{1}{c_i^i} \sum_{i=0}^{t^i} \mu_i^i = \beta' \frac{1}{c_i^i} \left(1 + \sum_{i=0}^{t^i} \mu_i^i\right) - \lambda_i^i = 0 \]
\[ \beta' \frac{1}{c_{i+1}^i} - \lambda_{i+1}^i + \beta' \frac{1}{c_{i+1}^i} \sum_{i=0}^{t^i+1} \mu_i^i = \beta' \frac{1}{c_{i+1}^i} \left(1 + \sum_{i=0}^{t^i+1} \mu_i^i\right) - \lambda_{i+1}^i = 0. \]

Letting \( v_i = v \), we can write the first order conditions for the consumer who receives the high consumption level today as
\[
\frac{u'(c^g)}{\beta u'(c^b)} = \frac{c^b}{\beta c^g} = \frac{v + r}{v}
\]

\[
\frac{1}{18} = \frac{v + 1}{v}, \quad \frac{1}{2} = \frac{2}{16}
\]

which can be solved to yield \( v = \frac{9}{7} \). The budget constraint for the consumer who receives the high endowment, \( w' = 24 \), is

\[
c^g + v \theta^g = \omega^g + (v + r) \theta^b
\]

\[
18 + \frac{9}{7} \theta^g = 24 + \left( \frac{9}{7} + 1 \right) \left( 1 - \theta^g \right),
\]

which can be solved to yield \( \theta^g = \frac{58}{25} = 2.32 \). \( \theta^b = 1 - \theta^g = -1.32 \). We can now go back and verify that all of the equilibrium conditions are satisfied for the right choice of \( \theta_0^1 \) and \( \theta_0^2 \).

Now consider the Arrow-Debreu version of this model. The consumers solve

\[
\max \sum_{t=0}^{\infty} \beta_t \log c_t^i,
\]

s.t.

\[
\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t \left( w_t^i + r \theta_0^i \right),
\]

\[
\sum_{t=1}^{\infty} \beta_t \sum_{t=0}^{\infty} \beta_t \log c_t^i \geq \sum_{t=0}^{\infty} \beta_t \log w_t^i,
\]

\[
c_t^i \geq 0.
\]

An equilibrium is sequences \( \hat{p}_t, \hat{e}_t^i \) such that

1. Given \( \hat{p}_t \), the consumers choose \( \hat{e}_t^i \) to solve their maximization problems.

2. Goods markets clear:

\[
\hat{e}_t^1 + \hat{e}_t^2 = w_t^1 + w_t^2 + r = 24 + 9 + 1 = 34.
\]

A symmetric steady state is \( p, (c^g, c^b) \) such that \( p_t = p_t' \) and

\[
c_t^i = \begin{cases} 
    c^g & \text{if } w_t^i = 24 \\
    c^b & \text{if } w_t^i = 9
\end{cases}
\]

satisfy the equilibrium conditions for the right choice of \( (\theta_0^1, \theta_0^2) \).
There are a number of ways that we can find the prices \( p_t = p' \) that, together with the allocation

\[
c'_i = \begin{cases} 
18 & \text{if } w'_i = 24 \\
16 & \text{if } w'_i = 9 
\end{cases},
\]

constitute an Arrow-Debreu equilibrium. Consider the first order conditions

\[
\beta' \frac{1}{c'_i} - \lambda^i p_t + \beta' \frac{1}{c'_i} \sum_{t'=0}^{t'} \mu'_i = 0.
\]

Once again we can show that, if \( c'_i = c^b \), then \( \mu'_i = 0 \). Consider a situation where \( c'_i = c^g \) and \( c'_{t+1} = c^b \). Then the first order conditions with respect to \( c'_i \) and \( c'_{t+1} \) are

\[
\beta' \frac{1}{c'_i} - \lambda^i p_t + \beta' \frac{1}{c'_i} \sum_{t'=0}^{t'} \mu'_i = \beta' \frac{1}{c'_i} \left(1 + \sum_{t'=0}^{t'} \mu'_i\right) - \lambda^i p_t = 0
\]

\[
\beta' \frac{1}{c'_{t+1}} - \lambda^i p_{t+1} + \beta' \frac{1}{c'_{t+1}} \sum_{t'=0}^{t+1} \mu'_{t+1} = \beta' \frac{1}{c'_{t+1}} \left(1 + \sum_{t'=0}^{t'} \mu'_{t+1}\right) - \lambda^i p_{t+1} = 0.
\]

These imply that

\[
\frac{\beta' \frac{1}{c^g}}{\frac{1}{c^g}} = \frac{p_{t+1}}{p_t}.
\]

Consequently, \( p_t = (9/16)' \).

Consider now a stochastic version of this economy. Let \( \eta_t \in \{1, 2\} \) be the event that occurs in period \( t \). Assume that

\[
\text{prob}(\eta_{t+1} = 1|\eta_t = 2) = \text{prob}(\eta_{t+1} = 2|\eta_t = 1) = \pi = 1/2.
\]

and that

\[
w'_i = \begin{cases} 
24 & \text{if } \eta_t = i \\
9 & \text{if } \eta_t \neq i 
\end{cases}
\]

for \( i = 1, 2 \). Once again, there is one unit of trees that produce \( r = 1 \) units of the good every period.
In the sequential markets version of this model, consumers solve the problem

\[
\max \sum_{s=5} \beta^{(s)} \pi_s \log c^i_s \\
\text{s.t. } c^j_s + q_{(s,j)} \theta^j_{(s,j)} + q_{(s,j)} \theta^j_{(s,j)} \leq w^j_s + (v^j_s + r) \theta^j_s \\
\sum_{\sigma=2s} \beta^{(\sigma)} \pi_{\sigma} \log c^j_\sigma \geq \sum_{\sigma=2s} \beta^{(\sigma)} \pi_{\sigma} \log w^j_\sigma \\
c^j_s \geq 0 \\
\theta^j_s \geq -\Theta \\
\theta^j_{n^j} = \bar{\theta}^j_{n^j}.
\]

An equilibrium is sequences \( \hat{v}_s, \hat{q}_{(s,1)}, \hat{q}_{(s,2)} \hat{c}_s^j, \hat{\theta}_{(s,1)}^j, \hat{\theta}_{(s,2)}^j \) such that

1. Given \( \hat{v}_s, \hat{q}_{(s,1)}, \hat{q}_{(s,2)} \), the consumers choose \( \hat{c}_s^j, \hat{\theta}_{(s,1)}^j, \hat{\theta}_{(s,2)}^j \) to solve their maximization problems.

2. Goods markets clear:

\[
\hat{c}_s^1 + \hat{c}_s^2 = w_s^1 + w_s^2 + r = 24 + 9 + 1 = 34.
\]

3. Asset markets clear:

\[
\hat{\theta}_{(s,1)}^j + \hat{\theta}_{(s,2)}^j = 1.
\]

A symmetric stochastic steady state is \( v, (q_s, q_r) (c^x, c^b), (\theta^x, \theta^b) \) such that \( v_i = v, q_{(s,\eta)} = q_s \) if \( \eta^i = \eta_s \), that is, if there is no reversal, \( q_{(s,\eta')} = q_r \) if \( \eta^i = \eta_r \), that is, if there is a reversal,

\[
c^j_s = \begin{cases} 
  c^x & \text{if } w^j_s = 24 \\
  c^b & \text{if } w^j_s = 9
\end{cases}
\]

and

\[
\theta^j_i = \begin{cases} 
  \theta^x & \text{if } w^j_s = 24 \\
  \theta^b & \text{if } w^j_s = 9
\end{cases}
\]

satisfy the equilibrium conditions for the right choice of \( (\bar{\theta}_0^1, \bar{\theta}_0^2) \).

Let us begin by observing that, if we add together the budget constraints of the two consumer types, we obtain

\[
c^1_s + c^2_s + q_{(s,1)} (\theta^1_{(s,1)} + \theta^2_{(s,1)}) + q_{(s,2)} (\theta^1_{(s,2)} + \theta^2_{(s,2)}) = w^1_s + w^2_s + (v_s + r)(\theta^1_s + \theta^2_s).
\]
Feasibility implies that
\[ q_{(s,1)} + q_{(s,2)} = v_s. \]

This can be thought of as an arbitrage condition: The price that a consumer receives for selling one unit of trees in state \( s \) is \( v_s \). The price that a consumer pays to receive this tree if event 1 occurs is \( q_{(s,1)} \), and the price that a consumer pays to receive this tree if no reversal occurs is \( q_{(s,2)} \). Since these two events are exhaustive and mutually exclusive, the total amount paid for the tree is \( q_{(s,1)} + q_{(s,2)} \).

The first order conditions for the consumer’s problem are
\[
\beta^{(s)} \pi_s \frac{1}{c_s^i} - \lambda_s^i + \beta^{(s)} \pi_s \frac{1}{c_s^j} \sum_{\sigma \leq s} \mu_{\sigma}^i = 0
\]
\[
-\lambda_s^i q_{(s,\eta)} + \lambda_s^i \left( v_{(s,\eta)} + r \right) = 0.
\]

Once again we can show that, if \( c_{(s,\eta)}^i = c^b \), then \( \mu_{(s,\eta)} = 0 \). First, consider the case where \( c_s^i = c^g \) and \( c_{(s,\eta)}^i \) is \( c^b \). The, since \( \mu_{(s,\eta)} = 0 \), we can write out the first order condition for \( c_{(s,\eta)}^i \) as
\[
\beta^{(s+1)} \pi_s \pi \frac{1}{c^b} - \lambda_{(s,\eta)}^i + \beta^{(s+1)} \pi_s \pi \frac{1}{c^b} \sum_{\sigma \leq s} \mu_{\sigma}^i = 0.
\]

Combining this with the first order condition for \( c_s^i \), as in the deterministic case, we obtain
\[
\frac{u'(c^g)}{\beta \pi u'(c^b)} = \frac{v_{(s,\eta)} + r}{q_{(s,\eta)}}.
\]

Imposing \( v_{(s,\eta)} = v \), this becomes
\[
q_{(s,\eta)} = q_r = \frac{\beta \pi u'(c^b)}{u'(c^g)} (v + r).
\]

Here \( q_r \) is the price paid for an Arrow security to purchase one unit of the tree in the case of reversal — where \( \eta_r = 1 \), for example, but \( \eta' = 2 \). Consider now the case where \( c_s^i = c^b \) and \( c_{(s,\eta)}^i \) is \( c^b \). (We think of this as the same state \( s \); we are just looking at the other consumer type’s first order conditions.) We obtain
\[
\frac{u'(c^b)}{\beta(1-\pi)u'(c^b)} = \frac{v+r}{q_n}
\]

\[
q_n = \beta(1-\pi)(v+1).
\]

Here \( q_n \) is the price paid for an Arrow security to purchase one unit of the tree in the case of no reversal.

Consider now the function

\[
f^D(x^g) = \left(1 - \beta(1-\pi)\right) \left(\log c^g - \log 24\right) + \beta\pi \left(\log(34-c^g) - \log 9\right).
\]

Setting \( f^D(c^g) = 0 \) in the case where \( \beta = 0.5 \) and \( \pi = 0.5 \), we obtain \( c^g = 21.5252 \).

We want to find values of \( c^b \), \( \theta^g \), \( \theta^b \), \( q_r \), \( q_n \), and \( v \) such that these variables constitute a symmetric steady state. Obviously, \( c^b = 12.4748 \).

Plugging these values into the first order conditions that we obtained above, we find that

\[
q_r = \frac{1}{4} \frac{12.4748}{21.5252} (v+1) = 0.4314(v+1)
\]

\[
q_n = 0.25(v+1).
\]

Notice that we can combine these two conditions to obtain

\[
q_r + q_n = 0.6814(v+1)
\]

\[
v = 0.6814(v+1),
\]

which implies that \( v = 2.1385 \), \( q_r = 1.3539 \), \( q_n = 0.7846 \). We can plug this into the budget constraint for the consumer with the high endowment,

\[
c^g + q_n\theta^g + q_r\theta^b = \omega^g + (v+1)\theta^g
\]

\[
21.5252 + 0.7846\theta^g + 1.3539(1-\theta^g) = 24 + 3.1385\theta^g,
\]

to solve for \( \theta^g = -0.3023 \), \( \theta^b = 1.3023 \). We can now go back and verify that all of the equilibrium conditions are satisfied for the right choice of \( \theta_0^1 \) and \( \theta_0^2 \).
Now consider the Arrow-Debreu version of this model. The consumers solve

$$\max \sum_{s \in S} \beta^{i(s)} \pi_s \log c^i_s$$

subject to

$$\sum_{s \in S} p_s c^i_s \leq \sum_{s \in S} p_s \left( w^i_s + r \theta^i_0 \right)$$

$$\sum_{\sigma \in S} \beta^{i(\sigma)} \pi_{\sigma} \log c^i_{\sigma} \geq \sum_{\sigma \in S} \beta^{i(\sigma)} \pi_{\sigma} \log w^i_{\sigma}$$

$$c_s \geq 0.$$

An **equilibrium** is sequences $\hat{p}_s$, $\hat{c}_s^i$, such that

1. Given $\hat{p}_s$, the consumers choose $\hat{c}_s^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}^1_s + \hat{c}^2_s = w^1_s + w^2_s + r = 24 + 9 + 1 = 34.$$

A **symmetric stochastic steady state** is $(p_n, p_r)$, $(c^s, c^b)$ such that $p_{(s, \eta)} = p_n p_s$ if $\eta' = \eta_s$, that is, if there is no reversal, $p_{(s, \eta)} = p_r p_s$ if $\eta' \neq \eta_s$, that is, if there is a reversal and

$$c^i_s = \begin{cases} c^s & \text{if } w^i_s = 24 \\ c^b & \text{if } w^i_s = 9 \end{cases}$$

satisfy the equilibrium conditions for the right choice of $(\overline{\theta}_0^1, \overline{\theta}_0^2)$.

As above, the first order conditions become

$$p_r = \frac{p_{(s, \eta)}}{p_s} = \frac{\beta \pi u'(c^b)}{u'(c^s)}$$

if there is a reversal and

$$p_n = \frac{p_{(s, \eta)}}{p_s} = \beta (1 - \pi)$$

if not.

Letting $p_{0h} = 1$, we can therefore construct Arrow-Debreu prices using the rule

$$p_{(s, \eta)} = \begin{cases} \beta (1 - \pi) p_s & \text{if } \eta' = \eta_s \\ \beta \pi \left( u'(c^b) / u'(c^s) \right) p_s & \text{if } \eta' \neq \eta_s \end{cases}$$
In particular,

\[ p_{(s,\eta)} = \begin{cases} 
\frac{1}{2} \frac{1}{2} p_s = 0.25 p_s & \text{if } \eta' = \eta_s \\
\frac{1}{2} \frac{1}{2} \left( \frac{21.5252}{12.4748} \right) p_s = 0.4314 p_s & \text{if } \eta' \neq \eta_s
\end{cases} \]