Notes on the Diamond-Dybvig Model

DouglasW. Diamond (2007), "Banks and Liquidity Creation: A Simple Exposition of the Diamond-Dybvig Model," *Federal Reserve Bank of Richmond Economic Quarterly*, 93, 189–200.

Douglas W. Diamond and Philip H. Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91, 401–419.

Neil Wallace, "Another Attempt to Explain an Illiquid Banking System: The Diamond and Dybvig Model with Sequential Service Taken Seriously," *Federal Reserve Bank of Minneapolis Quarterly Review*, 12, 3–16.

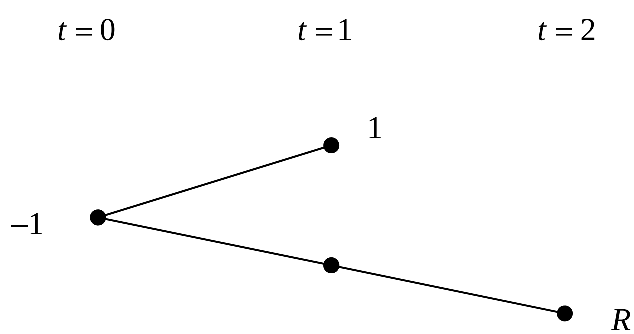
Environment

Three periods: 0, 1, 2

Single, storable good

Consumer's endowment: 1 unit in period 0 only

Production technology:



The good can be invested in a project that pays $R>1$ in $t=2$ for each unit in $t=0$. The project can be shut down in $t=1$ and the investment can be salvaged one-for-one. A project that is shut down cannot be restarted. Consumers can store the good.

Consumer's utility: $v(c_1,c_2,\theta)$, where θ takes on the value 1 or 2 in period 1

$$v(c_1,c_2,1)=u(c_1)$$

$$v(c_1,c_2,2)=\beta u(c_1+c_2)$$

Let $u(c)=\dfrac{c^{1-\sigma}-1}{1-\sigma}$. Notice that, by l'Hôpital's rule, $\lim_{\sigma\to 1}\dfrac{c^{1-\sigma}-1}{1-\sigma}=\log c$. The parameter σ, $\sigma\geq 0$, is called the (Arrow-Pratt) **coefficient of relative risk aversion**. In general, it is defined as

$$\sigma(c) = -\frac{cu'(c)}{u''(c)}.$$

Notice that, with the utility function $u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$, $\sigma(c)$ is just the constant σ.

Individual savings

Suppose that with probability λ, $0 < \lambda < 1$, the consumer has the liquidity shock $\theta = 1$. The consumer invests 1 in the project in $t = 0$. He/she then learns θ in $t = 1$. If $\theta = 1$, the consumer salvages 1 of his/her investment and eats it, $c_1^1 = 1$. If $\theta = 2$, however, the consumer can consume c_1^2 in $t = 1$ and c_2^2 in $t = 2$. The optimal choice of c_1^2 and c_2^2 is the solution to

$$\begin{aligned} &\max\ \beta u(c_1^2 + c_2^2) \\ &\text{s.t. } c_2^2 = R(1 - c_1^2) \\ &c_1^2, c_2^2 \geq 0. \end{aligned}$$

The Lagrangian is

$$L(c_1^2, c_2^2, \mu) = \beta u(c_1^2 + c_2^2) + \mu\left(R(1 - c_1^2) - (1-\lambda)c_2^2\right).$$

There are two ways to see that in the optimal choice is to set $c_1^2 = 0$:

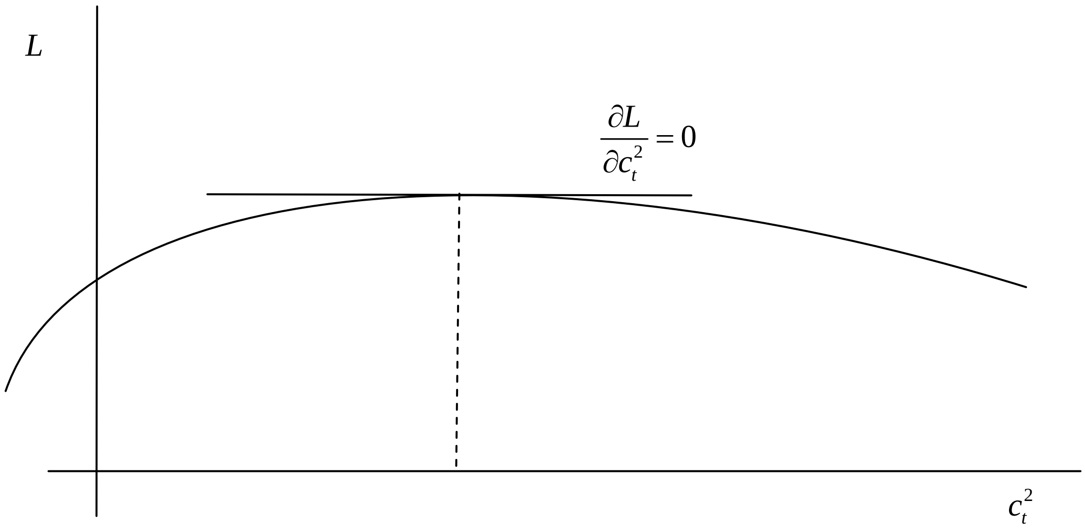
The first is to write out the first order conditions allowing for corner solutions where $c_t^2 = 0$

$$\frac{\partial L(c_1^2, c_2^2, \mu)}{\partial c_1^2} = \beta u'(c_1^2 + c_2^2) - \mu R \leq 0,\ = 0 \text{ if } c_1^2 > 0$$

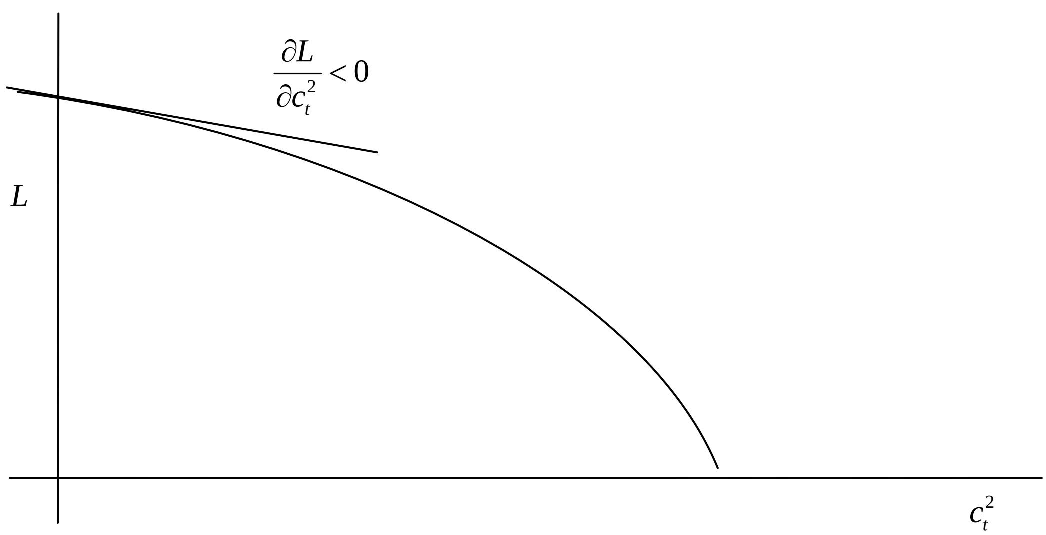
$$\frac{\partial L(c_1^2, c_2^2, \mu)}{\partial c_2^2} = \beta u'(c_1^2 + c_2^2) - \mu \leq 0,\ = 0 \text{ if } c_2^2 > 0.$$

These conditions say that the derivative needs to be equal to zero if we are at an interior solution, where $c_t^2 > 0$, but allow the possibility of a corner solution where $c_t^2 = 0$.

Example of interior solution $c_t^2 > 0$:



Example of corner solution $c_t^2 = 0$:



A second way to see this is to realize that the patient depositor is indifferent between receiving consumption in period 1 and consumption in period 2. The constraint that the consumption plan be feasible,

$$c_2^2 = R(1 - c_1^2)$$

however, implies that consuming ε more in period 1 means consuming $R\varepsilon$ less, $R\varepsilon > \varepsilon$ in period 2. Therefore, we should set c_1^2 as low as possible, 0.

Of course, this is just what the first order conditions say: The marginal benefits of c_1^2 and c_2^2 are the same, $\beta u'(c_1^2+c_2^2)$. The marginal cost of c_1^2, however, is μR, which is larger than the marginal cost of c_2^2, μ. We therefore set $c_1^2=0$ and let

$$\frac{\partial L(c_1^2,c_2^2,\mu)}{\partial c_1^2}= \beta u'(c_1^2+c_2^2)-\mu R<0$$

and set $c_1^2=R$. Consequently, the expected value of utility of a consumer who invests on his own is

$$\lambda u(c_1^1)+(1-\lambda)\beta u(c_1^2+c_2^2)=\lambda u(1)+(1-\lambda)\beta u(R).$$

Notice that we could also introduce c_2^1, the consumption in $t=2$ of the consumer who receives the liquidity shock $\theta=1$ and argue that it is optimal to set $c_2^1=0$, but, since the consumer does not value this consumption at all, this is obvious.

Optimal deposit contract

Suppose that a fixed fraction λ of consumers have the liquidity shock $\theta=1$. Suppose too, for the moment, that the value of θ for each consumer is public information. Later, we examine the more interesting case where the value of θ is private information.

Let c_t^θ be the withdrawal of the consumer in period t who has liquidity shock θ. We argue that by pooling risk in a bank, the consumers can achieve higher utility than they can with individual savings, $\lambda u(1)+(1-\lambda)\beta u(R)$.

The optimal contract that banks offer depositors solves

$$\begin{gathered}\max\ \lambda u(c_1^1)+(1-\lambda)\beta u(c_1^2+c_2^2)\\ \text{s.t. } (1-\lambda)c_2^2=R\left[1-(\lambda c_1^1+(1-\lambda)c_1^2)\right]\\ c_1^1,c_1^2,c_2^2\geq 0\end{gathered}$$

Here $\lambda u(c_1^1)+(1-\lambda)\beta u(c_1^2+c_2^2)$ is the expected utility of the contract for a depositor:

λ is the probability that $\theta=1$ and $u(c_1^1)$ is the utility when $\theta=1$.

$(1-\lambda)$ is the probability that $\theta=2$ and $\beta u(c_1^2+c_2^2)$ is the utility when $\theta=2$.

Here too $(1-\lambda)c_2^2=R\left[1-(\lambda c_1^1+(1-\lambda)c_1^2)\right]$ is the restriction that the contract be feasible:

$\lambda c_1^1+(1-\lambda)c_1^2$ are the withdrawals in $t=1$.

$1-(\lambda c_1^1+(1-\lambda)c_1^2)$ are the deposits not withdrawn in $t=1$.

$(1-\lambda)c_2^2$ are the withdrawals in $t=2$. They are equal to the gross returns on deposits not with drawn in $t=1$, $R\left[1-(\lambda c_1^1+(1-\lambda)c_1^2)\right]$

The Lagrangian is

$$L(c_1^1, c_1^2, c_2^2, \mu) = \lambda u(c_1^1) + (1-\lambda)\beta u(c_1^2 + c_2^2) + \mu\left(R\left[1-(\lambda c_1^1 + (1-\lambda)c_1^2)\right] - (1-\lambda)c_2^2\right),$$

where μ is the Lagrange multiplier.

The first order conditions, again allowing for corner solutions, are

$$\frac{\partial L(c_1^1, c_1^2, c_2^2, \mu)}{\partial c_1^1} = \lambda u'(c_1^1) - \mu\lambda R \le 0, \ = 0 \text{ if } c_1^1 > 0$$

$$\frac{\partial L(c_1^1, c_1^2, c_2^2, \mu)}{\partial c_1^2} = (1-\lambda)\beta u'(c_1^2 + c_2^2) - \mu(1-\lambda)R \le 0, \ = 0 \text{ if } c_1^2 > 0$$

$$\frac{\partial L(c_1^1, c_1^2, c_2^2, \mu)}{\partial c_2^2} = (1-\lambda)\beta u'(c_1^2 + c_2^2) - \mu(1-\lambda) \le 0, \ = 0 \text{ if } c_2^2 > 0.$$

Once again, it is optimal to set $c_1^2 = 0$ because increasing c_1^2 provides the same marginal increase in expected utility as does increasing c_2^2, $(1-\lambda)\beta u'(c_1^2 + c_2^2)$, but it has a large marginal cost, $\mu(1-\lambda)R$, compared to $\mu(1-\lambda)$.

To find the optimal deposit contract, we solve

$$\frac{\partial L(c_1^1, c_1^2, c_2^2, \mu)}{\partial c_1^1} = \lambda u'(c_1^1) - \mu\lambda R = 0$$

$$\frac{\partial L(c_1^1, c_1^2, c_2^2, \mu)}{\partial c_2^2} = (1-\lambda)\beta u'(c_2^2) - \mu(1-\lambda) = 0$$

$$(1-\lambda)c_2^2 = R(1-\lambda c_1^1).$$

We rewrite the first two conditions as

$$u'(c_1^1) = \mu R$$

$$\beta u'(c_2^2) = \mu.$$

Dividing the first condition by the second, we obtain

$$\frac{u'(c_1^1)}{u'(c_2^2)} = \beta R.$$

With our choice of utility function $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$, this can be rewritten as

$$\frac{(c_1^1)^{-\sigma}}{(c_2^2)^{-\sigma}} = \beta R$$

$$(c_2^2)^{\sigma} = \beta R (c_1^1)^{\sigma}$$

$$c_2^2 = (\beta R)^{\frac{1}{\sigma}} c_1^1 .$$

Plugging this into the restriction that the contract be feasible,

$$(1-\lambda)c_2^2 = R(1-\lambda c_1^1),$$

we can solve for the optimal deposit contract:

$$(1-\lambda)(\beta R)^{\frac{1}{\sigma}} c_1^1 = R\left(1-\lambda c_1^1\right)$$

$$(1-\lambda)(\beta R)^{\frac{1}{\sigma}} c_1^1 = R - \lambda R c_1^1$$

$$(1-\lambda)(\beta R)^{\frac{1}{\sigma}} c_1^1 + \lambda R c_1^1 = R$$

$$c_1^1 = \frac{R}{(1-\lambda)(\beta R)^{\frac{1}{\sigma}} + \lambda R},$$

which implies that

$$c_2^2 = \frac{(\beta R)^{\frac{1}{\sigma}} R}{(1-\lambda)(\beta R)^{\frac{1}{\sigma}} + \lambda R}.$$

Proposition. Suppose that $R > 1/\beta > 1$ and that $\sigma \geq 1$. Then

$$1 < c_1^1 < c_2^2 < R .$$

Proof. $R > 1/\beta$ implies that $\beta R > 1$. Since $c_2^2 = (\beta R)^{\frac{1}{\sigma}} c_1^1$ and $\sigma > 1 > 0$, this implies that $c_2^2 > c_1^1$.

To see that $c_1^1 > 1$, notice that

$$R^{1-\frac{1}{\sigma}} \geq 1$$

if $\sigma \geq 1$. This implies that

$$R \geq R^{\frac{1}{\sigma}} .$$

Since $\beta < 1$, $1 > \beta^{\frac{1}{\sigma}}$, which implies

$$R > (\beta R)^{\frac{1}{\sigma}}$$

$$(1-\lambda)R > (1-\lambda)(\beta R)^{\frac{1}{\sigma}}$$

$$R > (1-\lambda)(\beta R)^{\frac{1}{\sigma}} + \lambda R$$

$$\frac{R}{(1-\lambda)(\beta R)^{\frac{1}{\sigma}} + \lambda R} > 1$$

$$c_1^1 > 1 .$$

Plugging this into the restriction that the contract be feasible, we see that

$$(1-\lambda)c_2^2 = R(1-\lambda c_1^1) < R(1-\lambda)$$

$$c_2^2 < R .$$

We now have the complete characterization of the optimal contract:

$$1 < c_1^1 < c_2^2 < R ,$$

which concludes the proof.

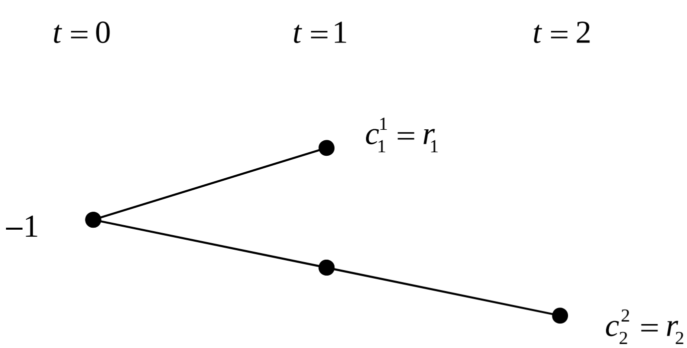
Let $r_1 = c_1^1$ be the gross return on deposits withdrawn in $t=1$ and $r_2 = c_2^2$ be the gross return on deposits withdrawn in $t=2$. Notice that

$$1 < r_1 < r_2 < R .$$

Notice too for $1 < r_1 < r_2 < R$, all we require is that $R > (\beta R)^{\frac{1}{\sigma}}$. This is true whenever $\sigma \geq 1$, but, depending on R and β, can even be true if $\sigma < 1$.

The higher σ, the more risk averse are depositors, the more the depositors are willing to give up a high return r_2 in $t=2$ to get a return r_1 in $t=1$.

We can represent the optimal deposit contract diagrammatically:



Bank runs

A problem with the contract arises if the type of a depositor in $t=1$ is not verifiable, that is, it is private information.

Suppose now that, in $t=1$, that a patient depositor predicts that other patient depositors will withdraw their deposits. What should he or she do?

If all depositors try to withdraw, there are not enough funds to pay off their withdrawals because

$$r_1 > 1 .$$

If $f = 1/r_1 < 1$ of depositors withdraw in $t=1$, then the bank will fail and have nothing to pay depositors who try to withdraw in $t=2$. Notice that with the utility functions $u(c) = \log c$ consumers will receive utility $-\infty$ if they withdraw 0. This is also true for the utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ where $\sigma > 1$.

There are several ways to model this. Here are two:

All withdrawers treated equally

The bank realizes that there is a run as soon as it starts. If $f \geq 1/r_1$ depositors try to withdraw at $t=1$, then the bank pays out all its deposits to withdrawers. Once all of these deposits are paid out, no other depositor receives anything. Each withdrawer receives

$$\frac{1}{f} \leq r_1 .$$

In period $t=1$ all depositors who receive the liquidity shock $\theta = 1$ try to with draw. It is the consumers who receive $\theta = 2$ in whom we are interested. Each consumer of this type can choose on of two actions: W to try to withdraw, that is to run on the bank, or N to not try to withdraw.

Let $V(a, a_{-1})$ be the payoff to a consumers who chooses action $a \in \{W, N\}$ when all other consumers choose action a_{-1}. We assume the each consumer is a very small part of the total population, so that aggregate outcomes depend only on a_{-1}.

For an optimal deposit contract to be susceptible to a bank run, we require that

$$V(W,W) > V(N,W) ,$$

which says that a depositor prefers to with draw if other depositors with $\theta = 2$ withdraw, and that

$$V(N,N) > V(W,N) ,$$

which says that a depositor does not withdraw if other depositors with $\theta = 2$ do not withdraw.

Suppose that all other depositors with $\theta = 2$ try to withdraw, that is, $f = 1$. Then, for an individual depositors with $\theta = 2$,

$$V(W,W) = \beta u(1) > \beta u(0) = V(N,W) .$$

Suppose, however, that all other depositors with $\theta = 2$ do not withdraw. Then, for an individual depositors with $\theta = 2$,

$$V(N,N) = \beta u(r_2) > \beta u(r_1) = V(W,N).$$

The question is whether consumers will want to deposit in $t = 0$ t=0 knowing that there can be a bank run. We have seen that, on their own, consumers can obtain utility $\lambda u(1) + (1-\lambda)\beta u(R)$. Suppose that what touches off a bank run is the realization of a sunspot variable ζ that takes on two values:

$$\text{prob}(\zeta = 1) = \pi$$

$$\text{prob}(\zeta = 0) = 1 - \pi.$$

Depositors run on the bank if $\zeta = 1$ but not if $\zeta = 0$. The expected utility of depositing in the bank is

$$(1-\pi)\left[\lambda u(r_1) + (1-\lambda)\beta u(r_2)\right] + \pi\left[\lambda u(1) + (1-\lambda)\beta u(1)\right].$$

The utility of not depositing in the bank is

$$\lambda u(1) + (1-\lambda)\beta u(R).$$

Since the optimal deposit contract ensures that

$$\lambda u(r_1) + (1-\lambda)\beta u(r_2) > \lambda u(1) + (1-\lambda)\beta u(R),$$

we know that

$$(1-\pi)\left[\lambda u(r_1) + (1-\lambda)\beta u(r_2)\right] + \pi\left[\lambda u(1) + (1-\lambda)\beta u(1)\right] > \lambda u(1) + (1-\lambda)\beta u(R)$$

for $\pi > 0$ small enough.

Notice, however, that, since

$$\lambda u(1) + (1-\lambda)\beta u(1) < \lambda u(1) + (1-\lambda)\beta u(R),$$

if π is close to 1, then

$$(1-\pi)\left[\lambda u(r_1) + (1-\lambda)\beta u(r_2)\right] + \pi\left[\lambda u(1) + (1-\lambda)\beta u(1)\right] < \lambda u(1) + (1-\lambda)\beta u(R),$$

and no one will want to deposit money in the bank.

A **sunspot equilibrium with bank runs** is an optimal deposit contract, $c_1^1 = r_1$, $c_2^2 = r_2$, $c_2^1 = c_2^2 = 0$, and a probability $\pi > 0$ that satisfy

$$V(W,W) > V(N,W)$$

$$V(N,N) > V(W,N)$$

$$(1-\pi)V(N,N) + \pi V(W,W) > V(0)$$

where $V(0) = \lambda u(1) + (1-\lambda)\beta u(R)$ is the expected utility of not depositing anything in the bank.

We have shown that there is a sunspot equilibrium for all $\pi > 0$ sufficiently small. Since we can vary π, as long as it remains sufficiently small, and still have an equilibrium, there is a continuum of sunspot equilibria.

Sequential service constraint

Let us impose what Neil Wallace refers to as a sequential service constraint. For the first $f > \lambda$ withdrawers, the bank pays r_1. Afterwards it pays a fixed amount δ, $0 < \delta < 1$ in $t = 1$ and $R\delta$ in $t = 2$. Here δ is the minimum reserve requirement as a fraction of deposits.

Let us calculate how many depositors f can withdraw in $t = 1$ before the bank hits the minimum reserve requirement δ:

$$1 - fr_1 = (1 - f)\delta .$$

Here $1 - fr_1$ is the amount of deposits left in the bank after the fraction f of depositors withdraw r_1.

$$f(r_1 - \delta) = 1 - \delta$$

$$f = \frac{1-\delta}{r_1 - \delta} .$$

Notice that $f < 1$. During a bank run a fraction f of withdrawers receive r_1. In the remaining fraction $1 - f$, the fraction $\lambda(1 - f)$ who have the liquidity shock $\theta = 1$ receives δ in $t = 1$ and the fraction $(1-\lambda)(1 - f)$ who have the shock $\theta = 2$ receives $R\delta$ in $t = 2$.

Suppose that all depositors run. The expected utility of a depositor with $\theta = 2$ who tries to withdraw is

$$f\beta u(r_1) + (1 - f)\beta u(R\delta) .$$

The utility of a depositor with $\theta = 2$ who does not try to withdraw is

$$\beta u(R\delta) .$$

If $r_1 > R\delta$, then

$$V(W,W) = f\beta u(r_1) + (1 - f)\beta u(R\delta) > \beta u(R\delta) = V(N,W) ,$$

and all depositors with $\theta = 2$ prefer to run on the bank. Notice that

$$V(N,N) = \beta u(r_2) > \beta u(r_1) = V(W,N) ,$$

which implies that patient depositors do not try to withdraw unless others do.

Again, we can model a sunspot such that $\zeta = 1$ with probability π. The expressions now are more complicated. Then expected value of depositing in the bank is

$$(1-\pi)V(N,N)+(1-\lambda)V(W,W)=$$
$$(1-\pi)\left[\lambda u(r_1)+(1-\lambda)\beta u(r_2)\right]+\pi\left[\lambda\left(fu(r_1)+(1-f)u(\delta)\right)+(1-\lambda)\beta\left(fu(r_1)+(1-f)u(R\delta)\right)\right]$$

Once again, for $\pi > 0$ small enough, this expression is great than the utility of not depositing in the bank,

$$\lambda u(1)+(1-\lambda)\beta u(R),$$

and consumers want to deposit in the bank.

Why we need positive reserves

Notice that, if we set $\delta = 0$, then no one will want to deposit in the bank at $t = 0$ if $\sigma \geq 1$. The problem is that, when $\sigma \geq 1$, $u(\delta) = u(0) = -\infty$. Consequently,

$$(1-\pi)V(N,N)+(1-\lambda)V(W,W)=$$
$$(1-\pi)\left[\lambda u(r_1)+(1-\lambda)\beta u(r_2)\right]+\pi\left[\lambda\left(fu(r_1)+(1-f)u(0)\right)+(1-\lambda)\beta\left(fu(r_1)+(1-f)u(0)\right)\right]=-\infty$$

As long as a bank run occurs with a positive probability π, no matter how small, and the consumer has utility $-\infty$ when this happens, the consumer will no take the risk of depositing in the bank.

Stopping bank runs

The government can intervene in a number of ways to stop a bank run. It can guarantee deposits. Alternatively, it can serve as lender of last resort. Suppose that payoffs are such that

$$V(N,W) > V(W,W),$$

that is, such that a depositor with $\theta = 2$ does not want to withdraw in $t = 1$ even if all other depositors are withdrawing. Then there cannot be a bank run in equilibrium. Any government policy that ensures that this condition holds stops bank runs. One policy that the bank can enact on its own to prevent bank runs is a partial suspension of payments.

Suppose that the bank sets the minimum reserves δ so that

$$R\delta > r_1$$

$$\delta > \frac{r_1}{R}.$$

Then no depositor with $\theta = 2$ wants to withdraw in $t = 1$. Since $r_1 < R$, we can set $\delta < 1$. Depositors with $\theta = 2$ do not run on the bank because they know that the bank will keep enough deposits invested in the projects that pay R in period $t = 2$ so that they would receive more in $t = 2$, $R\delta$, than they would even if they were at the front of the line during the bank run in $t = 1$, r_1.