Equilibrium of the General Equilibrium Growth Model

Consider a model with an infinitely-lived, representative consumer. The production function is

\[ Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \]

The consumer solves the problem

\[
\max \sum_{t=t_0}^\infty \beta^{t-t_0} \left[ \gamma \log C_t + (1-\gamma) \log(N_t \bar{h} - L_t) \right] \\
\text{s.t. } C_t + K_{t+1} - K_t = w_t L_t + (r_t - \delta) K_t, \quad t = t_0, t_0 + 1, t_0 + 2, \ldots \\
K_{t_0} = \bar{K}_{t_0}.
\]

The sequences of total factor productivities \( A_t, A_{t+1}, A_{t+2}, \ldots \) and of working age populations \( N_t, N_{t+1}, N_{t+2}, \ldots \) are exogenous.

An equilibrium is sequences of consumption \( \hat{C}_t, \hat{C}_{t+1}, \hat{C}_{t+2}, \ldots \), labor \( \hat{L}_t, \hat{L}_{t+1}, \hat{L}_{t+2}, \ldots \), capital \( \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \ldots \), wages \( \hat{w}_t, \hat{w}_{t+1}, \hat{w}_{t+2}, \ldots \), and rental rates \( \hat{r}_t, \hat{r}_{t+1}, \hat{r}_{t+2}, \ldots \) such that

1. **Consumer maximization:** Given \( \hat{w}_t, \hat{w}_{t+1}, \hat{w}_{t+2}, \ldots \) and \( \hat{r}_t, \hat{r}_{t+1}, \hat{r}_{t+2}, \ldots \), the consumer chooses \( \hat{C}_t, \hat{C}_{t+1}, \hat{C}_{t+2}, \ldots, \hat{L}_t, \hat{L}_{t+1}, \hat{L}_{t+2}, \ldots, \) and \( \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \ldots \) to solve

\[
\max \sum_{t=t_0}^\infty \beta^{t-t_0} \left[ \gamma \log C_t + (1-\gamma) \log(N_t \bar{h} - L_t) \right] \\
\text{s.t. } C_t + K_{t+1} - K_t = w_t L_t + (r_t - \delta) K_t, \quad t = t_0, t_0 + 1, t_0 + 2, \ldots \\
K_{t_0} = \bar{K}_{t_0}.
\]

2. **Firm maximization:** Given \( \hat{w}_t, \hat{w}_{t+1}, \hat{w}_{t+2}, \ldots \) and \( \hat{r}_t, \hat{r}_{t+1}, \hat{r}_{t+2}, \ldots \), the firm chooses \( \hat{L}_t, \hat{L}_{t+1}, \hat{L}_{t+2}, \ldots, \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \ldots \) to minimize costs and to earn zero profit. This results in the conditions

\[
\hat{r}_t = \alpha A \hat{K}_t^{\alpha-1} \hat{L}_t^{-\alpha} \\
\hat{w}_t = (1-\alpha) A \hat{K}_t^{\alpha} \hat{L}_t^{-1}, \quad t = t_0, t_0 + 1, t_0 + 2, \ldots
\]

3. **Feasibility:** \( \hat{C}_t, \hat{C}_{t+1}, \hat{C}_{t+2}, \ldots, \hat{L}_t, \hat{L}_{t+1}, \hat{L}_{t+2}, \ldots, \) and \( \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \ldots \) are feasible:

\[
C_t + K_{t+1} - (1-\delta) K_t = A K_t^{\alpha} L_t^{1-\alpha}, \quad t = t_0, t_0 + 1, t_0 + 2, \ldots
\]
Notice that it is possible to define separate $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$ and $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$ for the consumer and for the firm and then require that they be equal as part of the feasibility conditions:

\[
\hat{L}_t = \hat{L}_t' \\
\hat{K}_t^c = \hat{K}_t'^f, \ t = t_0, t_0 + 1, t_0 + 2, \ldots
\]