EQUILIBRIUM AND PARETO EFFICIENCY

Environment:

Pure exchange economy with two infinitely lived consumers and one good per period.

Utility: $\sum_{t=0}^{\infty} \beta_t^i \log c_t^i$ where $0 < \beta_i < 1$, $i = 1, 2$.

Endowments: $(w_t^i, w_t^j, w_t^k, \ldots)$ where $w_t^i > 0$, $i = 1, 2$, $t = 0, 1, 2, \ldots$.

Market structure:

With an Arrow-Debreu markets structure, futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

Equilibrium:

An Arrow-Debreu equilibrium is a sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{c}_0^j, \hat{c}_1^j, \hat{c}_2^j, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer $i$, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta_t^i \log c_t^i$$

s.t. $\sum_{t=0}^{\infty} \hat{p}_t^i c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t^i w_t^i$

$c_t^i \geq 0$.

- $\hat{c}_t^i + \hat{c}_{t}^j \leq w_t^i + w_t^j$, = if $\hat{p}_t > 0$, $t = 0, 1, 2, \ldots$.

Characterization of equilibrium using calculus:

The Kuhn-Tucker theorem says that $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ solves the consumer’s maximization problem if and only if there exists a Lagrange multiplier $\hat{\lambda}_i \geq 0$ such that

$$\beta_t^i \frac{1}{\hat{c}_t^i} - \hat{\lambda}_i \hat{p}_t \leq 0$$

= if $\hat{c}_t^i > 0$.
\[ \sum_{t=0}^{\infty} \hat{p}_t w_t^j - \sum_{t=0}^{\infty} \hat{p}_t c_t^j \geq 0, = 0 \text{ if } \hat{\lambda}^i > 0. \]

For any \( t, t = 0,1,2, \ldots \), \( \lim_{c \to 0} \frac{1}{\beta^i} = \infty \) implies that \( \hat{c}_t^i > 0 \), which implies that \( \hat{\lambda}^i > 0 \). It also implies that \( \hat{p}_t > 0, t = 0,1,2, \ldots \). Consequently, \( \hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots ; \hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots ; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots \) is an equilibrium if and only if there exist Lagrange multipliers \( \hat{\lambda}^1, \hat{\lambda}^2 \), \( \hat{\lambda}^i > 0 \), such that

- \( \beta^i \frac{1}{\hat{c}_t^i} = \hat{\lambda}_i \hat{p}_t, i = 1,2, t = 0,1,2, \ldots \)

- \( \sum_{t=0}^{\infty} \hat{p}_t c_t^j = \sum_{t=0}^{\infty} \hat{p}_t w_t^j, i = 1,2 \)

- \( \hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, t = 0,1,2, \ldots \)

**Pareto efficiency:**

An allocation \( \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots ; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots \) is **Pareto efficient** if it is feasible,

\[ \hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2, t = 0,1,2, \ldots, \]

and there exists no other allocation, \( \overline{c}_0^1, \overline{c}_1^1, \overline{c}_2^1, \ldots ; \overline{c}_0^2, \overline{c}_1^2, \overline{c}_2^2, \ldots \) that is also feasible and is such that

\[ \sum_{t=0}^{\infty} \beta^i \log \overline{c}_t^i > \sum_{t=0}^{\infty} \beta^i \log \hat{c}_t^i, \text{ some } i, i = 1,2, \text{ and} \]

\[ \sum_{t=0}^{\infty} \beta^i \log \overline{c}_t^i \geq \sum_{t=0}^{\infty} \beta^i \log \hat{c}_t^i, \text{ all } i, i = 1,2. \]

**Alternatively,**

An allocation \( \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots ; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots \) is **Pareto efficient** if and only if there exist numbers \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_i \geq 0 \) and not both 0, such that \( \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots ; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots \) solves

\[
\begin{align*}
\max \ & \hat{\alpha}_1 \sum_{t=0}^{\infty} \beta^i \log c_t^1 + \hat{\alpha}_2 \sum_{t=0}^{\infty} \beta^i \log c_t^2 \\
\text{s.t.} \ & c_t^1 + c_t^2 \leq w_t^1 + w_t^2, t = 0,1,2, \ldots \\
& c_t^i \geq 0.
\end{align*}
\]
(Note: It is easy to show that, if an allocation solves the above social planner’s problem, it satisfies the first definition of Pareto efficiency. It is a little more difficult to show that, if an allocation satisfies the first definition of Pareto efficiency, there exist welfare weights $\tilde{\alpha}_1, \tilde{\alpha}_2$ such that the allocation solves the social planner’s problem.)

Characterization of Pareto efficiency using calculus:

The Kuhn-Tucker theorem says that $c_{01}, c_{11}, c_{21}, ..., c_{02}, c_{12}, c_{22}, ...$ solves the social planner’s problem if and only if there exists a Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, ..., \hat{\pi}_t \geq 0$, such that

$$\hat{\alpha}_t \pi_t \frac{1}{c_{ii}} - \hat{\pi}_t \leq 0, \text{ if } \hat{c}_i > 0,$$

$$w^1_t + w^2_t - \hat{c}_i + \hat{c}_i^2 \geq 0, \text{ if } \hat{\pi}_t > 0.$$

For any $t, t = 0, 1, 2, ..., \lim_{c \to 0} \beta_t \frac{1}{c} = \infty$ implies that $\hat{c}_i > 0$, which implies that $\hat{\pi}_t > 0$.

Consequently, $c_{01}, c_{11}, c_{21}, ..., c_{02}, c_{12}, c_{22}, ...$ is a Pareto efficient allocation if and only if there exist Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, ..., \hat{\pi}_t > 0$, such that

- $\hat{\alpha}_t \pi_t \frac{1}{c_{ii}} = \hat{\pi}_t, \quad i = 1, 2, \quad t = 0, 1, 2, ...$

- $\hat{c}_i = w^1_t + w^2_t, \quad i = 0, 1, 2, ....$

(Note: Since $\hat{\alpha}_i > 0$ for at least one $i, \ i = 1, 2$, we know that, for that consumer $i$, $\hat{c}_i > 0$ for all $t, \ t = 0, 1, 2, ...,$ and, consequently, that $\hat{\pi}_t > 0$. If one of the welfare weights $\hat{\alpha}_i$ equals 0, then $\hat{c}_i = 0$. We can imagine the first order conditions for that consumer $i$ as being satisfied in the limit or we can simply ignore them. In what follows, we avoid the case where one of the welfare weights equals 0.)

First welfare theorem:

Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, ..., c_{01}, c_{11}, c_{21}, ..., c_{02}, c_{12}, c_{22}, ...$ is an equilibrium. Then the allocation $c_{01}, c_{11}, c_{21}, ..., c_{02}, c_{12}, c_{22}, ...$ is Pareto efficient.
Proof:

Since $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an equilibrium, we know that there exist Lagrange multipliers $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_i > 0$, such that

$$\beta_i \frac{1}{\hat{c}_i} = \hat{\lambda}_i \hat{p}_i$$

$$\hat{c}_i^1 + \hat{c}_i^2 = w_i^1 + w_i^2$$

We also know that, if there exist welfare weights $\hat{\alpha}_i, \hat{\alpha}_2, \hat{\alpha}_i > 0$, and Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_i > 0$, such that

$$\hat{\alpha}_i \beta_i \frac{1}{\hat{c}_i} = \hat{\pi}_i$$

$$\hat{c}_i^1 + \hat{c}_i^2 = w_i^1 + w_i^2,$$

then $\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation. (In other words, we are given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ and $\hat{\lambda}_1, \hat{\lambda}_2$ that satisfy certain properties, and we want to construct $\hat{\alpha}_i, \hat{\alpha}_2$ and $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots$ that, together with $\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, satisfy certain other properties.) To prove the theorem, we set

$$\hat{\alpha}_i = \frac{1}{\hat{\lambda}_i}$$

$$\hat{\pi}_i = \hat{p}_i.$$
Equilibrium with transfers:

An Arrow-Debreu equilibrium with transfers is a sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, an allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and transfers $\hat{t}_1, \hat{t}_2$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer $i$, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve
  \[
  \max \sum_{t=0}^\infty \beta^t \log c_t^i \\
  \text{s.t. } \sum_{t=0}^\infty \hat{p}_t c_t^i \leq \sum_{t=0}^\infty \hat{p}_t w_t^i + \hat{t}_i \\
  c_t^i \geq 0.
  \]

- \[
  \hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2, \text{ if } \hat{p}_t > 0, \ t = 0, 1, 2, \ldots
  \]

Characterization of equilibrium with transfers using calculus:

Once again, we use the Kuhn-Tucker theorem to show that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{t}_1, \hat{t}_2$ is an equilibrium with transfers if and only if there exist Lagrange multipliers $\hat{\lambda}^1, \hat{\lambda}^2, \hat{\lambda}^i > 0$, such that

- \[
  \beta^t \frac{1}{c_t^i} = \hat{\lambda}_t \hat{p}_t, \ i = 1, 2, \ t = 0, 1, 2, \ldots
  \]

- \[
  \sum_{t=0}^\infty \hat{p}_t c_t^i = \sum_{t=0}^\infty \hat{p}_t w_t^i + \hat{t}_i, \ i = 1, 2
  \]

- \[
  \hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \ t = 0, 1, 2, \ldots
  \]

Second welfare theorem:

Suppose that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation where each consumer receives strictly positive consumption. Then there exist prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and transfers $\hat{t}_1, \hat{t}_2$ such that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{t}_1, \hat{t}_2$ is an equilibrium.
Proof:

Since \( \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots \) is a Pareto efficient allocation equilibrium, we know that there exist welfare weights \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_i \geq 0 \), and Lagrange multipliers \( \hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_i > 0 \), such that

\[
\hat{\alpha}_i \beta_i \frac{1}{c_i} = \hat{\pi}_i,
\]

\[
\hat{c}_i^1 + \hat{c}_i^2 = w_i^1 + w_i^2.
\]

Since \( \hat{c}_i > 0 \), we know that \( \hat{\alpha}_i > 0 \), \( i = 1, 2 \). We also know that, if there exist prices \( \hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots \), transfers \( \hat{i}_1, \hat{i}_2 \), and Lagrange multipliers \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_i > 0 \), such that

\[
\beta_i \frac{1}{c_i} = \hat{\lambda}_i \hat{p}_i,
\]

\[
\sum_{i=0}^{\infty} \hat{p}_i c_i' = \sum_{i=0}^{\infty} \hat{p}_i w_i' + \hat{i}_i
\]

\[
\hat{c}_i^1 + \hat{c}_i^2 = w_i^1 + w_i^2
\]

then \( \hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{i}_1, \hat{i}_2 \) is an equilibrium with transfers. (In other words, we are given \( \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{\alpha}_1, \hat{\alpha}_2 \); and \( \hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots \) that satisfy certain properties, and we want to construct \( \hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{i}_1, \hat{i}_2 \); and \( \hat{\lambda}_1, \hat{\lambda}_2 \) that, together with \( \hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots \), satisfy certain other properties.) To prove the theorem, we set

\[
\hat{p}_i = \hat{\pi}_i
\]

\[
\hat{\lambda}_i = \frac{1}{\hat{\alpha}_i}
\]

\[
\hat{i}_i = \sum_{i=0}^{\infty} \hat{p}_i c_i' - \sum_{i=0}^{\infty} \hat{p}_i w_i'
\]

\[\blacksquare\]