MONOPOLISTIC COMPETITION WITH HETEROGENOUS FIRMS

There is a continuum of firms that produce differentiated products.

Consumers have utility functions that exhibit love for variety and solve the maximization problem

$$\max \ (1-\alpha)\log c_0 + \frac{\alpha}{\rho}\log\int_0^m c(\omega)^\rho \, d\omega$$

$$\text{s.t. } p_0 c_0 + \int_0^m p(\omega)c(\omega)d\omega = w\bar{\ell} + \pi$$

$$c(\omega) \ge 0 .$$

Here π are profits of the firms, which are owned by the consumers. The solution to this problem is

$$c_0 = (1-\alpha)\frac{w\bar{\ell}+\pi}{p_0}$$

$$c(\omega) = \frac{\alpha(w\bar{\ell}+\pi)}{p(\omega)^{\frac{1}{1-\rho}}\int_0^m p(\omega')^{\frac{-\rho}{1-\rho}}d\omega'} = \frac{\alpha(w\bar{\ell}+\pi)}{p(\omega)^{\frac{1}{1-\rho}}P^{\frac{-\rho}{1-\rho}}},$$

where

$$P = \left(\int_0^m p(\omega)^{\frac{-\rho}{1-\rho}} d\omega \right)^{\frac{-(1-\rho)}{\rho}}.$$

Good 0 is produced with the constant returns production function $y_0 = \ell_0$ and sold in a competitive market.

Firm i has the production function

$$y(\omega) = \max\left[x(\omega)\left(\ell(\omega) - f\right), 0 \right].$$

Notice that firms have potentially different productivity levels $x(i)$. The firm solves the profit maximization problem

$$\max\ p(\omega)c(\omega) - \frac{wc(\omega)}{x(\omega)} - wf = p(\omega)\frac{\alpha(w\overline{\ell} + \pi)}{p(\omega)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} - \frac{\alpha(w\overline{\ell} + \pi)}{x(\omega)p(\omega)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} - wf$$

taking P as given. The solution is

$$p(\omega) = \frac{w}{\rho x(\omega)}.$$

We set $p_0 = w = 1$ as numeraire.

$$p(\omega) = \frac{1}{\rho x(\omega)}.$$

A model with a continuum of productivity levels

Suppose that there is a measure μ of potential firms. Firm productivities are distributed on the interval $x \geq 1$ according to the Pareto distribution with distribution function

$$F(x) = 1 - x^{-\gamma},$$

which has the density function

$$dF(x) = \gamma x^{-\gamma-1}.$$

Notice that the mean of x is

$$E(x) = \int_1^\infty x dF(x) = \int_1^\infty x\gamma x^{-\gamma-1} dx = -\frac{\gamma x^{-(\gamma-1)}}{\gamma - 1}\Bigg|_1^\infty = \frac{\gamma}{\gamma - 1}$$

and the variance is

$$E(x^2) - \left(E(x)\right)^2 = \int_1^\infty \left(x - \frac{\gamma}{\gamma - 1}\right)^2 dF(x) = \int_1^\infty \left(x - \frac{\gamma}{\gamma - 1}\right)^2 \gamma x^{-\gamma-1} dx$$

$$E(x^2)-\left(E(x)\right)^2=-\frac{\gamma x^{-(\gamma-2)}}{\gamma-2}\bigg|_1^\infty-\left(\frac{\gamma}{\gamma-1}\right)^2$$

$$E(x^2)-\left(E(x)\right)^2=\frac{\gamma}{(\gamma-2)(\gamma-1)^2}.$$

For the variance to be finite, we require that $\gamma>2$. We will also require that $\gamma>\rho/(1-\rho)$.

We can think of restricting productivities to satisfy $x\geq 1$ as a normalization of units relating labor to consumption of differentiated goods by fixing the minimum productivity. If we want to normalize units in some other way, we could replace the distribution function with

$$F(x)=1-\underline{x}^{\gamma}x^{-\gamma},$$

for $x\geq\underline{x}$, which has the density function

$$dF(x)=\gamma\underline{x}^{\gamma}x^{-\gamma-1}.$$

There are two possibilities, which depend on parameters:

1. There is a level of productivity $\bar{x} > 1$ for which firms earn 0 profits. The set of firms with productivities $x \geq \bar{x}$ produce. This set has measure $m = \mu \bar{x}^{-\gamma}$.

2. All firms produce and earn nonnegative profits. In this case $m = \mu$.

Case 1:

We start by supposing that there is a cutoff productivity $\bar{x}$ where firms earn 0 profits and calculate

$$P^{\frac{-\rho}{1-\rho}} = \mu\int_{\bar{x}}^{\infty} p(x)^{\frac{-\rho}{1-\rho}} dF(x) = \mu\int_{\bar{x}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx$$

$$P^{\frac{-\rho}{1-\rho}} = -\left.\frac{\mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}\right|_{\bar{x}}^{\infty} = \frac{\mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}.$$

The demand for goods produced by a firm with productivity x is

$$c(x) = \frac{\alpha(\bar{\ell}+\pi)}{p(x)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)}{(\rho x)^{\frac{-1}{1-\rho}} \mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)x^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}$$

if $\gamma(1-\rho) > \rho$, which we can write as $\gamma > \rho/(1-\rho)$.

We calculate the cutoff productivity $\bar{x}$

$$p(\bar{x})c(\bar{x}) - \frac{c(\bar{x})}{\bar{x}} - f = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)\bar{x}^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}\left(\frac{1}{\rho\bar{x}} - \frac{1}{\bar{x}}\right) - f = 0$$

$$\frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)\bar{x}^{\gamma}}{\mu\gamma} - f = 0$$

$$\bar{x} = \left(\frac{\mu\gamma f}{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)}\right)^{\frac{1}{\gamma}}.$$

Notice that this expression depends on profits π, which we can calculate as

$$\pi = \mu\int_{\bar{x}}^{\infty}\left(p(x)c(x) - \frac{c(x)}{x} - f\right)dF(x) = \mu\int_{\bar{x}}^{\infty}\left(\frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)x^{\frac{\rho}{1-\rho}}}{\mu\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} - f\right)\gamma x^{-\gamma-1}dx$$

$$\pi = \mu\frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)}{\mu\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}\int_{\bar{x}}^{\infty}x^{\frac{\rho}{1-\rho}}\gamma x^{-\gamma-1}dx - \mu f\int_{\bar{x}}^{\infty}\gamma x^{-\gamma-1}dx$$

$$\pi = (1-\rho)\alpha(\overline{\ell} + \pi) - \mu \overline{x}^{-\gamma} f$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell} - \mu \overline{x}^{-\gamma} f}{1-(1-\rho)\alpha}.$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell}}{1-(1-\rho)\alpha} - \frac{\mu f}{1-(1-\rho)\alpha}\frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\overline{\ell}+\pi)}{\mu\gamma f}$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell}}{1-(1-\rho)\alpha} - \frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\overline{\ell}+\pi)}{\gamma\big(1-(1-\rho)\alpha\big)}$$

$$\pi = \frac{\rho\alpha\overline{\ell}}{\gamma - \rho\alpha},$$

which implies that

$$\overline{x}^{-\gamma} = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha\left(\overline{\ell} + \dfrac{\rho\alpha\overline{\ell}}{\gamma-\rho\alpha}\right)}{\mu\gamma f} = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha\overline{\ell}}{(\gamma-\rho\alpha)\mu f}.$$

Case 2:

Notice that we are wrong to guess that there is a cutoff productivity $\bar{x}$ where firms earn 0 profits if the value that we calculate for $\bar{x}$ is less than 1:

$$\bar{x} = \left(\frac{(\gamma - \rho\alpha)\mu f}{\left(\gamma(1-\rho) - \rho\right)\alpha \bar{\ell}} \right)^{\frac{1}{\gamma}} < 1$$

$$\mu f < \frac{\left(\gamma(1-\rho) - \rho\right)\alpha}{\gamma - \rho\alpha} \bar{\ell},$$

that is, if the fixed costs of having all potential firms produce is sufficiently low compared to aggregate labor $\bar{\ell}$. In this case,

$$P^{\frac{-\rho}{1-\rho}} = \frac{\mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma}{\gamma(1-\rho) - \rho}$$

$$c(x) = \frac{\alpha(\bar{\ell} + \pi)}{p(x)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{\left(\gamma(1-\rho) - \rho\right)\alpha(\bar{\ell} + \pi)}{(\rho x)^{\frac{-1}{1-\rho}} \mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma} = \frac{\rho\left(\gamma(1-\rho) - \rho\right)\alpha(\bar{\ell} + \pi) x^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma}$$

The calculation of total profits becomes

$$\pi = \mu \int_1^{\infty} \left(p(x)c(x) - \frac{c(x)}{x} - f \right) dF(x) = \mu \int_1^{\infty} \left(\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)x^{\frac{\rho}{1-\rho}}}{\mu\gamma} - f \right) \gamma x^{-\gamma-1} dx$$

$$\pi = \mu \frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)}{\mu\gamma} \int_1^{\infty} x^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx - \mu f \int_1^{\infty} \gamma x^{-\gamma-1} dx$$

$$\pi = (1-\rho)\alpha(\overline{\ell}+\pi) - \mu f$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell} - \mu f}{1-(1-\rho)\alpha}.$$

Notice that the profits of a firm with productivity $x = 1$ are

$$p(1)c(1) - c(1) - f = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\overline{\ell}+\pi)}{\mu(1-\rho)\gamma}\left(\frac{1}{\rho}-1\right) - f$$

$$p(1)c(1) - c(1) - f = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha\left(\overline{\ell}+\dfrac{(1-\rho)\alpha\overline{\ell}-\mu f}{1-(1-\rho)\alpha}\right)}{\mu\gamma} - f$$

$$p(1)c(1) - c(1) - f = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha\overline{\ell}}{\mu\gamma\big(1-(1-\rho)\alpha\big)} - \frac{(\gamma-\rho\alpha)f}{\gamma\big(1-(1-\rho)\alpha\big)}$$

$$p(1)c(1) - c(1) - f = \frac{\gamma-\rho\alpha}{\mu\gamma\big(1-(1-\rho)\mu\big)\alpha}\left(\frac{\big(\gamma(1-\rho)-\rho\big)\alpha\overline{\ell}}{\gamma-\rho\alpha} - \mu f\right) > 0.$$

A two-country model with a continuum of productivity levels

Suppose now that there are two countries, $i = 1, 2$. Let each country have a population of $\bar{\ell}_i$ and a measure of potential firms of μ_i. Firms' productivities are distributed according to the Pareto distribution, $F(x) = 1 - x^{-\gamma}$.

A firm in country i faces a fixed cost of exporting to country j, $j \neq i$, of f_e where $f_e > f_d = f$ and an iceberg transportation cost of $\tau_i^j - 1 \geq 0$. The solution to the firm's profit maximization problem is to set

$$p_i^j(x) = \frac{w_i \tau_i^j}{\rho x}.$$

In each country there are three possibilities:

1. There are two cutoff levels of productivity $\bar{x}_{ie} > \bar{x}_{id} > 1$. Firms with $\bar{x}_{ie}$ earn 0 profits exporting. Firms with $\bar{x}_{id}$ earn 0 profits producing for the domestic market. The set of firms with $x \geq \bar{x}_{ie}$ produce for the domestic market and for export. The set of firms with $\bar{x}_{ie} \geq x \geq \bar{x}_{id}$ produce for the domestic market only. The set of firms with $x < \bar{x}_{id}$ cannot earn nonnegative profits and do not produce.

2. There is one cutoff level of productivity $\bar{x}_{ie} > 1$. Firms with $\bar{x}_{e}$ earn 0 profits exporting. The set of firms with $x \geq \bar{x}_{ie}$ produce for the domestic market and for export. The set of firms with $\bar{x}_{ie} \geq x \geq 1$ produce for the domestic market only and earn nonnegative profits.

3. All firms produce for the domestic market and for export. They earn nonnegative profits doing both.

Suppose that we are in case 1. We calculate the price index in country 1:

$$\left(P^1\right)^{\frac{-\rho}{1-\rho}} = \mu_1 \int_{\overline{x}_{1d}}^{\infty} p_1^1(x)dF(x) + \mu_2 \int_{\overline{x}_{2e}}^{\infty} p_2^1(x)dF(x)$$

$$\left(P^1\right)^{\frac{-\rho}{1-\rho}} = \mu_1 \int_{\overline{x}_{1d}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1}dx + \mu_2 \int_{\overline{x}_{2e}}^{\infty} \left(\frac{\tau_2^1}{\rho x}\right)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1}dx$$

$$\left(P^1\right)^{\frac{-\rho}{1-\rho}} = -\left.\frac{w_1\mu_1\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}\right|_{\overline{x}_{1d}}^{\infty} + \left.\frac{w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}\right|_{\overline{x}_{2e}}^{\infty}$$

$$\left(P^1\right)^{\frac{-\rho}{1-\rho}} = \frac{w_1\mu_1\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma \overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} + \frac{w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma \overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}$$

$$\left(P^1\right)^{\frac{-\rho}{1-\rho}} = \frac{\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\left(w_1\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}{\gamma(1-\rho)-\rho}$$

The demand in country 1 for goods produced by a firm in country 1 with productivity $x \geq x_{1d}$ is

$$c_1^1(x) = \frac{\alpha(w_1\overline{\ell}_1 + \pi_1)}{p_1^1(x)^{\frac{1}{1-\rho}}\left(P^1\right)^{\frac{-\rho}{1-\rho}}} = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(w_1\overline{\ell}_1 + \pi_1)x^{\frac{1}{1-\rho}}}{(1-\rho)\gamma\left(w_1\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}.$$

We calculate an expression for the cutoff productivity $\overline{x}_{1d}$:

$$p_1^1(\overline{x}_{1d})c_1^1(\overline{x}_{1d}) - \frac{w_1c_1^1(\overline{x}_{1d})}{\overline{x}_{1d}} - w_1f_d = 0$$

$$\frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(w_1\overline{\ell}_1 + \pi_1)\overline{x}_{1d}}{(1-\rho)\gamma\left(w_1\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}\left(\frac{w_1}{\rho\overline{x}_{1d}} - \frac{w_1}{\overline{x}_{1d}}\right) - w_1f_d = 0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(w_1\overline{\ell}_1+\pi_1)\overline{x}_{1d}^{\frac{\rho}{1-\rho}}}{\gamma\left(w_1\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-w_1f_d=0.$$

Similarly, we calculate an expression for the cutoff productivity $\overline{x}_{1e}$:

$$p_1^2(\overline{x}_{1e})c_1^2(\overline{x}_{1e})-\frac{w_1\tau_1^2c_1^2(\overline{x}_{1e})}{\overline{x}_{1e}}-w_1f_e$$

$$=\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(w_2\overline{\ell}_2+\pi_2)(\tau_1^2)^{\frac{-\rho}{1-\rho}}\overline{x}_{1e}^{\frac{\rho}{1-\rho}}}{\gamma\left(w_2\mu_2\overline{x}_{2d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+w_1\mu_1(\tau_1^2)^{\frac{-\rho}{1-\rho}}\overline{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-w_1f_e=0^{\cdot}$$

The expression for π_1 is

$$\pi_1 = \mu_1 \int_{\bar{x}_{1d}}^{\infty} \left(p_1^1(x) c_1^1(x) - \frac{w_1 c_1^1(x)}{x} - w_1 f_d \right) dF(x)$$

$$+ \mu_1 \int_{\bar{x}_{1e}}^{\infty} \left(p_1^2(x) c_1^2(x) - \frac{w_1 c_1^2(x)}{x} - w_1 f_e \right) dF(x)$$

$$\pi_1 = \mu_1 \int_{\bar{x}_{1d}}^{\infty} \left(\frac{w_1 \big(\gamma(1-\rho)-\rho\big) \alpha (w_1 \bar{\ell}_1 + \pi_1) x^{\frac{\rho}{1-\rho}}}{\gamma \left(w_1 \mu_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_2 \mu_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - w_1 f_d \right) \gamma x^{-\gamma-1} dx$$

$$+ \mu_1 \int_{\bar{x}_{1e}}^{\infty} \left(\frac{w_1 \big(\gamma(1-\rho)-\rho\big) \alpha (w_2 \bar{\ell}_2 + \pi_2) (\tau_1^2)^{\frac{-\rho}{1-\rho}} x^{\frac{\rho}{1-\rho}}}{\gamma \left(w_2 \mu_2 \bar{x}_{2d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_1 \mu_1 (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - w_1 f_e \right) \gamma x^{-\gamma-1} dx$$

$$\pi_1 = \frac{w_1\alpha(1-\rho)(w_1\overline{\ell}_1+\pi_1)\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{w_1\mu_1\overline{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_2\mu_2(\tau_2^1)^{\frac{-\rho}{1-\rho}}\overline{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} + \frac{w_1\alpha(1-\rho)(w_2\overline{\ell}_2+\pi_2)\mu_2(\tau_1^2)^{\frac{-\rho}{1-\rho}}\overline{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{w_2\mu_2\overline{x}_{2d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + w_1\mu_1(\tau_1^2)^{\frac{-\rho}{1-\rho}}\overline{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}$$
$$- w_1\mu_1\left(\overline{x}_{1d}^{-\gamma} f_d + \overline{x}_{1e}^{-\gamma} f_e\right)$$

There are analogous expressions for $\overline{x}_{2d}$, $\overline{x}_{2e}$, and π_2. In addition, there are labor market clearing conditions in two countries and two wage rates w_1 and w_2. This provides us with a system of 8 equations in 8 unknowns to be solved for $\overline{x}_{1d}$, $\overline{x}_{1e}$, π_1, w_1, $\overline{x}_{2d}$, $\overline{x}_{2e}$, π_2, and w_2.

As is usual in general equilibrium models, we can normalize one of the wages as numeraire, setting $w_1 = 1$. We can also use Walras's law to ignore labor market clearing in one of the countries. This leaves us with a system of 7 equations in 7 unknowns.

To simplify the calculations, we consider the symmetric case where $\overline{\ell}_1 = \overline{\ell}_2 = \overline{\ell}$, $\mu_1 = \mu_2 = \mu$, and $\tau_2^1 = \tau_1^2 = \tau$. Notice that this implies that $\overline{x}_{1d} = \overline{x}_{2d} = \overline{x}_d$, $\overline{x}_{1e} = \overline{x}_{2e} = \overline{x}_e$, $\pi_1 = \pi_2 = \pi$, $w_2 = w_1 = 1$ and that we can ignore labor market clearing. We now have a system of 3 equations in 3 unknowns, $\overline{x}_d$, $\overline{x}_e$, and π:

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_d=0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\tau^{\frac{-1}{1-\rho}}\overline{x}_e^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_e=0$$

$$\pi=(1-\rho)\alpha(\overline{\ell}+\pi)-\mu\left(\overline{x}_d^{-\gamma}f_d+\overline{x}_e^{-\gamma}f_e\right).$$

Notice that

$$\frac{\overline{x}_e}{\overline{x}_d} = \tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}}$$

The equation for π can be rewritten as

$$\pi = (1-\rho)\alpha(\overline{\ell}+\pi) - \mu\left(\overline{x}_d^{-\gamma} f_d + \left(\tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}}\overline{x}_d\right)^{-\gamma} f_e\right)$$

$$\pi = (1-\rho)\alpha(\overline{\ell}+\pi) - \mu\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right) f_d^{\frac{\gamma(1-\rho)}{\rho}}\overline{x}_d^{-\gamma}.$$

The equation for $\overline{x}_d$ can be rewritten as

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\left(\tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}}\overline{x}_d\right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_d=0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d^{\gamma}}{\mu\gamma\left(1+\tau^{-\gamma}\left(\frac{f_e}{f_d}\right)^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}-f_d=0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d^{\gamma}}{\mu\gamma\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}}+\tau^{-\gamma}f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}-f_d^{\frac{\gamma(1-\rho)}{\rho}}=0$$

$$\bar{x}_d^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu\gamma\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}.$$

Notice the similarity between this expression and the analogous expression for the closed economy model. Plugging this expression into the expression for π, we obtain

$$\pi = \frac{\rho\alpha(\bar{\ell}+\pi)}{\gamma}$$

$$\pi = \frac{\rho\alpha\bar{\ell}}{\gamma-\rho\alpha},$$

which implies that

$$\bar{x}_d^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha\bar{\ell}f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu(\gamma-\rho\alpha)\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}.$$

Therefore

$$\overline{x}_e = \tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}} \overline{x}_d$$

$$\overline{x}_e^{-\gamma} = \tau^{-\gamma} \left(\frac{f_e}{f_d} \right)^{\frac{-\gamma(1-\rho)}{\rho}} \overline{x}_d^{-\gamma}$$

$$\overline{x}_e^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha \overline{\ell} \tau^{-\gamma} f_e^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu(\gamma-\rho\alpha)\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right)}.$$

A model with costly entry

Consider the closed economy case. The entry condition is

$$\int_{\bar{x}}^{\infty}\left(p(x)c(x)-\frac{c(x)}{x}-f\right)dF(x)=\phi,$$

where ϕ is the entry cost.

In this formulation,

$$\pi = 0,$$

but now the measure of potential firms μ is a variable,

As before, we can obtain an expression for the cutoff productivity $\bar{x}$:

$$\bar{x} = \left(\frac{\mu\gamma f}{\left(\gamma(1-\rho)-\rho\right)\alpha\overline{\ell}}\right)^{\frac{1}{\gamma}}.$$

We now calculate an expression for μ:

$$\int_{\bar{x}}^{\infty}\left(\frac{\rho\left(\gamma(1-\rho)-\rho\right)\alpha\bar{\ell}x^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}\left(\frac{1}{\rho x}-\frac{1}{x}\right)-f\right)\gamma x^{-\gamma-1}dx=\phi$$

$$\frac{\left(1-\rho\right)\alpha\bar{\ell}\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\mu\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}-f\bar{x}^{-\gamma}=\phi$$

$$\bar{x}^{-\gamma}=\frac{\left(1-\rho\right)\alpha\bar{\ell}-\mu\phi}{\mu f}=\frac{\left(\gamma(1-\rho)-\rho\right)\alpha\bar{\ell}}{\mu\gamma f}$$

$$\mu=\frac{\rho\alpha\bar{\ell}}{\gamma\phi}.$$

This implies that

$$\bar{x}=\left(\frac{\rho f}{\left(\gamma(1-\rho)-\rho\right)\phi}\right)^{\frac{1}{\gamma}}.$$