Calculating $\bar{b}$:

Utility of repaying even if bankers do not lend:

$$u((1-\theta)\bar{y}, \theta\bar{y} - B) + \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y})}{1 - \beta}$$

Utility of defaulting if bankers do not lend:

$$\frac{u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1 - \beta}.$$

$\bar{b}$ is determined by

$$u((1-\theta)\bar{y}, \theta\bar{y} - \bar{b}) + \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y})}{1 - \beta} = \frac{u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1 - \beta}$$
\[
\log((1 - \theta) \bar{y}) + \gamma \log(\theta \bar{y} - \bar{b}) + \frac{\beta \log((1 - \theta) \bar{y} + \beta \gamma \log(\theta \bar{y}))}{1 - \beta} \\
= \log((1 - \theta) \bar{y}) + \beta \gamma \log(\theta \bar{y}) \\
= \frac{1 - \beta}{1 - \beta}
\]

This has a simple analytical solution for \( \bar{b} \).

It is easiest to work with a grid of debt levels \([0, \tilde{B}]\), where \( \tilde{B} \) is a number large enough so that the government would always want to default if it had debt equal to \( \tilde{B} \). In the example, \( \tilde{B} = 150 \) is large enough.
For every grid point $B \in (\bar{b}, \hat{B}]$, we can calculate the expected utility of the government if it reduces decides the debt to $\bar{b}$ in $T$ periods, $T = 1, 2, ..., 6$. First-order conditions imply that $g_i = g^T(B)$ is constant as long as $B > \bar{b}$. We can solve for $g^T(B)$:

$$g^1(B) = \theta \bar{y} - B + \beta \bar{b}$$

$$g^T(B) = \theta \bar{y} - \frac{1 - \beta (1 - \pi)}{1 - (\beta (1 - \pi))^T}(B - (\beta (1 - \pi))^{T-1} \beta \bar{b}).$$
Compute $V^T(B)$:

$$
V^1(B) = u((1 - \theta)\bar{y}, g^1(B)) \\
+ \frac{\beta u((1 - \theta)\bar{y}, \theta\bar{y} - (1 - \beta)b)}{1 - \beta}
$$

$$
V^T(B) = \frac{1 - (\beta(1 - \pi))^T}{1 + \beta(1 - \pi)} u((1 - \theta)\bar{y}, g^T(B)) \\
+ \frac{1 - (\beta(1 - \pi))^{T-1}}{1 + \beta(1 - \pi)} \beta \pi u((1 - \theta)Z\bar{y}, \theta Z\bar{y}) \\
+ \frac{1 - (\beta(1 - \pi))^{T-2}}{1 - \beta} \beta u((1 - \theta)\bar{y}, \theta\bar{y} - (1 - \beta)b)
$$
For every $B \in (\overline{b}, \overline{\tilde{B}}]$, we can calculate the optimal number of periods $T(B)$ to run down the debt finding what $T$ maximizes $\left[ V^1(B), V^2(B), \ldots, V^6(B) \right]$. That is,

$$T(B) = \arg \max_T V^T(B)$$

$$\tilde{V}(B) = \max_T V^T(B).$$

To find $\overline{B}$, we solve

$$\max \left[ V^1(\overline{B}), V^2(\overline{B}), \ldots, V^6(\overline{B}) \right]$$

$$= u((1 - \theta)\overline{Zy}, \theta\overline{Zy} + \beta(1 - \pi)\overline{B}) + \frac{\beta u((1 - \theta)\overline{Zy}, \theta\overline{Zy})}{1 - \beta}.$$
Now

\[ V(B) = \begin{cases} 
\tilde{V}(B) & \text{if } B \leq \bar{B} \\
 u((1-\theta)Z\bar{y}, \theta Z\bar{y}) + \frac{\beta u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} & \text{if } B > \bar{B}.
\end{cases} \]

Notice that, if \( B > \bar{B} \), the value function is not

\[ u((1-\theta)Z\bar{y}, \theta Z\bar{y} + \beta(1-\pi)\bar{B}) + \frac{\beta u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \]

because the bankers realize that, if they lend to the government, the government would default. Therefore they do not lend. Consequently, \( V(B) \) is discontinuous at \( \bar{B} \).
$g(B)$