Calibrating the Growth Model

Kaldor's “stylized facts”

1. $Y_t / L_t$ (output per worker) exhibits continual growth.
2. $K_t / L_t$ (capital per worker) exhibits continual growth.
3. $r_t - \delta$ (real interest rate) is roughly constant.
4. $K_t / Y_t$ (capital-output ratio) is roughly constant.
5. $r_t K_t / Y_t, w_t L_t / Y_t$ (factor shares) are roughly constant.
6. There are wide differences in the rate of growth of productivity across countries.

The growth model

$$\begin{align*}
& \max \sum_{t=0}^{\infty} \beta^t \log C_t \\
& \text{s. t. } C_t + K_{t+1} - K_t \leq w_t L_t + (r_t - \delta)K_t \\
& \quad C_t, K_t \geq 0 \\
& \quad K_0 = \bar{K}_0 \\
& \quad L_t = \lambda^t L_0.
\end{align*}$$

where $$w_t = (1 - \alpha)(g^{1-\alpha})^t AK_t^\alpha L_t^{-\alpha},$$ $$r_t = \alpha(g^{1-\alpha})^t AK_t^{\alpha-1} L_t^{1-\alpha}.$$ 

We can solve

$$\begin{align*}
& \max \sum_{t=0}^{\infty} \beta^t \log C_t \\
& \text{s. t. } C_t + K_{t+1} - (1 - \delta)K_t \leq (g^{1-\alpha})^t AK_t^\alpha L_t^{-\alpha} \\
& \quad C_t, K_t \geq 0 \\
& \quad K_0 = \bar{K}_0 \\
& \quad L_t = \lambda^t L_0.
\end{align*}$$
First-order conditions:

\[ \frac{\beta^t}{C_t} = p_t \]

\[ p_{t-1} = p_t \left( \alpha (g^{1-\alpha})^t AK_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta \right) \]

\[ C_t + K_{t+1} - (1 - \delta)K_t = (g^{1-\alpha})^t AK_t^{\alpha} L_t^{1-\alpha}. \]

Impose constant growth conditions

\[ \frac{C_{t+1}/L_{t+1}}{C_t/L_t} = g_c, \quad \frac{K_{t+1}/L_{t+1}}{K_t/L_t} = g_k. \]

Simple algebra shows that

\[ \frac{C_{t+1}/L_{t+1}}{C_t/L_t} = \frac{K_{t+1}/L_{t+1}}{K_t/L_t} = \frac{Y_{t+1}/L_{t+1}}{Y_t/L_t} = g. \]
Redefine variables in terms of effective labor units $\tilde{L}_t = g^t L_t = (g^\lambda)^t L_0$:

$$\tilde{c}_t = C_t / \tilde{L}_t = g^{-t} (C_t / L_t)$$
$$\tilde{k}_t = K_t / \tilde{L}_t = g^{-t} (K_t / L_t)$$

$$\log C_t / L_t = \log g^t \tilde{c}_t = \log \tilde{c}_t + t \log g.$$  

Notice that the balanced growth path is the steady state $\tilde{c}_t = \tilde{c}$, $\tilde{k}_t = \tilde{k}$ of the redefined problem

$$\max \sum_{t=0}^{\infty} \beta^t \log \tilde{c}_t$$

s. t. $\tilde{c}_t + g^\lambda \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t \leq A \tilde{k}_t^\alpha$

$\tilde{c}_t, \tilde{k}_t \geq 0$

$\tilde{k}_0 = \tilde{K}_0 / L_0$. 
The balanced growth path matches Kaldor’s stylized facts (although the explanation for fact 6 is not very interesting):

1. \[ \frac{Y_t}{L_t} = (g^{1-\alpha}) A(K_t / L_t)^\alpha = g^\prime \tilde{A}^{\alpha} \] grows at rate \( g - 1 \).

2. \[ \frac{K_t}{L_t} = g^\prime \tilde{k} \] grows at rate \( g - 1 \).

3. \[ r_t - \delta = \alpha (g^{1-\alpha}) A K_t^{\alpha-1} L_t^{1-\alpha} - \delta = \alpha \tilde{A}^{\alpha-1} - \delta = g \lambda / \beta - 1 \] is constant.

4. \[ \frac{K_t}{Y_t} = \tilde{k}^{1-\alpha} / A \] is constant.

5. \[ r_t K_t / Y_t = \alpha, \ w_t L_t / Y_t = 1 - \alpha \] are constant.

6. rate of growth of \( Y_t / L_t \) is determined solely by \( g \).
Calibration to the U.S. data

I. First we interpret the data as being observations of a balanced growth path and use employment as the measure of labor input.

All data is from the *Economic Report of the President, 2004* (http://www.gpoaccess.gov/eop/).

\[ Y_t = \text{Real gross domestic product (Table B-2) (billions of 2000 dollars)} \]

\[ L_t = \text{Civilian employment (B-35) (thousands of persons)} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & Y_t & L_t & \frac{Y_t}{L_t} & g & \lambda \\
\hline
1960 & 2,501.8 & 65,778 & 38,034 & & \\
1970 & 3,771.9 & 78,678 & 47,941 & 1.0234 & 1.0181 \\
1980 & 5,161.7 & 99,303 & 51,979 & 1.0081 & 1.0236 \\
1990 & 7,112.5 & 118,793 & 59,873 & 1.0142 & 1.0181 \\
2000 & 9,817.0 & 136,891 & 71,714 & 1.0182 & 1.0143 \\
\hline
\end{array}
\]

\[
\log Y_t / L_t - \log Y_t / L_t = \log g^\alpha \tilde{k}^\alpha - \log g^\alpha \tilde{k}^\alpha = (t' - t) \log g.
\]

\[ g = 1.0160, \quad \lambda = 1.0185. \]
\( Y_t = \text{Gross domestic product (B-1)} - \text{proprietors’ income (B-28)} - (\text{taxes on production and imports (B-28)} - \text{subsidies (B-28)}) \) (billions of current dollars)

\( w_t L_t = \text{Compensation of employees (B-28)} \) (billions of current dollars)

(We distribute proprietors’ income and indirect business taxes proportionally between labor income and capital income.)

<table>
<thead>
<tr>
<th>Year</th>
<th>GDP</th>
<th>Proprietors’ Income</th>
<th>Indirect Taxes</th>
<th>Subsidies</th>
<th>( Y_t )</th>
<th>( w_t L_t )</th>
<th>( 1 - \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>526.4</td>
<td>50.8</td>
<td>44.6</td>
<td>1.1</td>
<td>432.1</td>
<td>296.4</td>
<td>0.6860</td>
</tr>
<tr>
<td>1970</td>
<td>1,038.5</td>
<td>78.4</td>
<td>91.5</td>
<td>4.8</td>
<td>873.4</td>
<td>617.2</td>
<td>0.7067</td>
</tr>
<tr>
<td>1980</td>
<td>2,789.5</td>
<td>174.1</td>
<td>200.7</td>
<td>9.8</td>
<td>2,424.5</td>
<td>1,651.8</td>
<td>0.6813</td>
</tr>
<tr>
<td>1990</td>
<td>5,803.1</td>
<td>380.6</td>
<td>425.5</td>
<td>26.8</td>
<td>5,023.8</td>
<td>3,338.2</td>
<td>0.6645</td>
</tr>
<tr>
<td>2000</td>
<td>9,817.0</td>
<td>728.4</td>
<td>708.9</td>
<td>44.3</td>
<td>8,424.0</td>
<td>5,782.7</td>
<td>0.6865</td>
</tr>
</tbody>
</table>

\[ 1 - \alpha = 0.6850, \quad \alpha = 0.3150. \]
How good is the assumption that we are in a balanced growth path?

\[ \frac{Y_t}{(g^tL_t)} = \frac{Y_t}{L_t}/g^t = A\tilde{k}^\alpha \] should be constant.

<table>
<thead>
<tr>
<th>Year</th>
<th>( \frac{Y_t}{g^tL_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>38,034</td>
</tr>
<tr>
<td>1970</td>
<td>40,912</td>
</tr>
<tr>
<td>1980</td>
<td>37,854</td>
</tr>
<tr>
<td>1990</td>
<td>37,210</td>
</tr>
<tr>
<td>2000</td>
<td>38,034</td>
</tr>
</tbody>
</table>

\[ A\tilde{k}^\alpha = 38,409 \]
Real GDP per Worker in the United States

2000 U.S. dollars

balanced growth path

data

year

\( K_{t+1} - (1 - \delta)K_t = \) Gross private domestic investment (B-1) + government gross investment (federal defense, federal nondefense, state and local) (B-20) (billions of current dollars)

\( \delta K_t = \) Consumption of fixed capital (B-26) (billions of current dollars)

\( Y_t = \) Gross domestic product (B-1) (billions of current dollars)

<table>
<thead>
<tr>
<th>Year</th>
<th>Private investment</th>
<th>Government investment</th>
<th>( K_{t+1} - (1 - \delta)K_t )</th>
<th>( \delta K_t )</th>
<th>( K_{t+1} - K_t )</th>
<th>( Y_t )</th>
<th>( \frac{K_{t+1} - K_t}{Y_t} )</th>
<th>( \frac{\delta K_t}{Y_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>78.9</td>
<td>28.2</td>
<td>107.1</td>
<td>55.6</td>
<td>51.5</td>
<td>526.4</td>
<td>0.0978</td>
<td>0.1056</td>
</tr>
<tr>
<td>1970</td>
<td>152.4</td>
<td>43.7</td>
<td>196.1</td>
<td>106.7</td>
<td>89.4</td>
<td>1,038.5</td>
<td>0.0861</td>
<td>0.1027</td>
</tr>
<tr>
<td>1980</td>
<td>479.3</td>
<td>100.3</td>
<td>579.6</td>
<td>343.0</td>
<td>236.6</td>
<td>2,789.5</td>
<td>0.0848</td>
<td>0.1230</td>
</tr>
<tr>
<td>1990</td>
<td>861.0</td>
<td>215.7</td>
<td>1,076.7</td>
<td>682.5</td>
<td>394.2</td>
<td>5,803.1</td>
<td>0.0679</td>
<td>0.1176</td>
</tr>
<tr>
<td>2000</td>
<td>1,735.5</td>
<td>304.4</td>
<td>2,039.9</td>
<td>1,187.8</td>
<td>852.1</td>
<td>9,817.0</td>
<td>0.0868</td>
<td>0.1210</td>
</tr>
</tbody>
</table>

\[ \frac{K_{t+1} - K_t}{Y_t} = 0.0847, \quad \frac{\delta K_t}{Y_t} = 0.1140. \]
Calculation of parameters:

\[
\frac{K_{t+1} - K_t}{Y_t} = \frac{(g\lambda - 1)\tilde{k}}{A\tilde{k}^\alpha} = 0.0847
\]

\[A\tilde{k}^\alpha = 38,409\]

\[
\tilde{k} = \frac{0.0847 A\tilde{k}^\alpha}{g\lambda - 1} = \frac{0.0847 \times 38,409}{0.0348} = 93,561
\]

\[
\frac{K_t}{Y_t} = \frac{\tilde{k}}{A\tilde{k}^\alpha} = \frac{93,561}{38,409} = 2.4359
\]

\[
\delta K_t = 0.1140, \quad \delta = \frac{0.1140}{2.4359} = 0.0468
\]

\[
r_t K_t = 0.3150, \quad r = \frac{0.3150}{2.4359} = 0.1293
\]

\[
A = \frac{A\tilde{k}^\alpha}{\tilde{k}^\alpha} = \frac{38,409}{93,561^{0.3150}} = 1043.23
\]

\[
r - \delta = 0.1293 - 0.0468 = 0.0825 = \frac{g\lambda}{\beta} - 1, \quad \beta = \frac{g\lambda}{1 + r - \delta} = \frac{1.0348}{1.0825} = 0.9559
\]
Summary:
\[ \beta = 0.9559, \; \delta = 0.0468, \; g = 1.0160, \; A = 1043.23, \; \alpha = 0.3150, \; \lambda = 1.0185. \]

Kaldor’s stylized facts (again):

1. \[ \frac{Y_t}{L_t} = (1.0160)^{t-1960} 38,409 \]
2. \[ \frac{K_t}{L_t} = (1.0160)^{t-1960} 93,561 \]
3. \[ r_t - \delta = 0.0825 \]
4. \[ \frac{K_t}{Y_t} = 2.4359 \]
5. \[ r_t K_t / Y_t = 0.3150, \; w_t L_t / Y_t = 0.6850 \]
6. \[ g = 1.0160 \]
A puzzle: Interest rates on bonds

\[ i_t = \text{Corporate bond yield (Moody's Aaa) (percent per year)} \] (B-73)

\[ \pi_t = \text{Change in implicit GNP deflator (percent per year)} \] (B-3)

<table>
<thead>
<tr>
<th></th>
<th>(i_t)</th>
<th>(\pi_t)</th>
<th>(i_t - \pi_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960-1969</td>
<td>2.35</td>
<td>5.01</td>
<td>2.66</td>
</tr>
<tr>
<td>1980-1989</td>
<td>4.75</td>
<td>11.34</td>
<td>6.59</td>
</tr>
<tr>
<td>1990-1999</td>
<td>2.22</td>
<td>7.72</td>
<td>5.50</td>
</tr>
<tr>
<td>2000-2002</td>
<td>2.03</td>
<td>7.06</td>
<td>5.03</td>
</tr>
</tbody>
</table>

Arbitrage implies that

\[ r_t - \delta = 0.0825 \approx i_t - \pi_t = 0.0416 \]

There is an equity premium. Until the 1980s, it was very large. See


II. Now we interpret the data as being observations of a balanced growth path, but we use total hours worked as the measure of labor input and we put leisure into the utility function.

The utility function is now

$$\max \sum_{t=0}^{\infty} \beta^t \left( \gamma \log C_t + (1 - \gamma) \log(N_t \bar{h} - L_t) \right)$$

where $N_t$ is the working-age (16-64) population and $\bar{h}$ is the maximum number for hours available for work per person, taken to be 5200 per year (100 hours per week $\times$ 52 weeks per year).

There is a new first-order condition:

$$\frac{1 - \gamma}{N_t \bar{h} - L_t} = \frac{\gamma w_t}{C_t} = \frac{\gamma}{C_t} (1 - \alpha)(g^{1-\alpha})^t AK_t^\alpha L_t^{-\alpha} = \frac{\gamma}{C_t} (1 - \alpha) \frac{Y_t}{L_t}$$

$$\frac{1 - \gamma}{\gamma} = (1 - \alpha) \frac{Y_t}{C_t} \frac{N_t \bar{h} - L_t}{L_t}.$$
\( N_t = \text{Population 14-64 (B-34) (thousands of persons)} \)

\( L_t = 52 \times \text{average total private weekly hours (B-47, spliced with average total manufacturing weekly hours at 1963)} \times \text{civilian employment (B-35) (thousands of persons)} \) \((L_t \text{ is expressed in billions of hours})\)

\( Y_t = \text{Gross domestic product (B-1) (billions of current dollars)} \)

\( C_t = Y_t - K_{t+1} - (1 - \delta)K_t \) \((\text{billions of current dollars})\)

<table>
<thead>
<tr>
<th>Year</th>
<th>N_t</th>
<th>hours</th>
<th>employment</th>
<th>L_t</th>
<th>( \frac{N_t \bar{h} - L_t}{L_t} )</th>
<th>C_t</th>
<th>Y_t</th>
<th>( C_t / Y_t )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>105,160</td>
<td>37.6</td>
<td>65,778</td>
<td>128.5</td>
<td>3.2568</td>
<td>419.3</td>
<td>526.4</td>
<td>0.7965</td>
<td>0.2631</td>
</tr>
<tr>
<td>1970</td>
<td>122,963</td>
<td>37.0</td>
<td>78,678</td>
<td>151.4</td>
<td>3.2240</td>
<td>842.4</td>
<td>1,038.5</td>
<td>0.8112</td>
<td>0.2686</td>
</tr>
<tr>
<td>1980</td>
<td>146,731</td>
<td>35.2</td>
<td>99,303</td>
<td>181.8</td>
<td>3.1978</td>
<td>2,209.9</td>
<td>2,789.5</td>
<td>0.7922</td>
<td>0.2656</td>
</tr>
<tr>
<td>1990</td>
<td>161,396</td>
<td>34.3</td>
<td>118,793</td>
<td>211.9</td>
<td>2.9610</td>
<td>4,726.4</td>
<td>5,803.1</td>
<td>0.8145</td>
<td>0.2865</td>
</tr>
<tr>
<td>2000</td>
<td>183,034</td>
<td>34.3</td>
<td>136,891</td>
<td>244.2</td>
<td>2.8982</td>
<td>7,777.1</td>
<td>9,817.0</td>
<td>0.7922</td>
<td>0.2852</td>
</tr>
</tbody>
</table>

\( \gamma = 0.2738 \).
We need to recalibrate $g$ and $\lambda$:

<table>
<thead>
<tr>
<th>Year</th>
<th>$Y_t$</th>
<th>$L_t$</th>
<th>$\frac{Y_t}{L_t}$</th>
<th>$g$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>2,501.8</td>
<td>128.5</td>
<td>19.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970</td>
<td>3,771.9</td>
<td>151.4</td>
<td>24.92</td>
<td>1.0249</td>
<td>1.0166</td>
</tr>
<tr>
<td>1980</td>
<td>5,161.7</td>
<td>181.8</td>
<td>28.40</td>
<td>1.0132</td>
<td>1.0185</td>
</tr>
<tr>
<td>1990</td>
<td>7,112.5</td>
<td>211.9</td>
<td>33.57</td>
<td>1.0169</td>
<td>1.0154</td>
</tr>
<tr>
<td>2000</td>
<td>9,817.0</td>
<td>244.2</td>
<td>40.21</td>
<td>1.0182</td>
<td>1.0143</td>
</tr>
</tbody>
</table>

$g = 1.0183$, $\lambda = 1.0162$.

How good is the assumption that we are in a balanced growth path?

\[
\frac{Y_t}{g' L_t}
\]

<table>
<thead>
<tr>
<th>Year</th>
<th>$\frac{Y_t}{g' L_t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>19.48</td>
</tr>
<tr>
<td>1970</td>
<td>20.79</td>
</tr>
<tr>
<td>1980</td>
<td>19.76</td>
</tr>
<tr>
<td>1990</td>
<td>19.49</td>
</tr>
<tr>
<td>2000</td>
<td>19.48</td>
</tr>
</tbody>
</table>

$A\hat{k}^\alpha = 19.80$
Real GDP per Hour Worked in the United States

2000 U.S. dollars

Year


Data

Balanced growth path
\[\tilde{k} = \frac{0.0847A\tilde{k}^\alpha}{g\lambda - 1} = \frac{0.0847 \times 19.80}{0.0348} = 48.23\]

\[A = \frac{A\tilde{k}^\alpha}{\tilde{k}^\alpha} = \frac{19.80}{48.23^{0.3150}} = 5.8390\]

The calibration of all of the other parameters stays the same.

**Summary:**

\[\beta = 0.9559, \ \gamma = 0.2738, \ \delta = 0.0468, \ g = 1.0183, \ A = 5.8390, \ \alpha = 0.3150, \ \lambda = 1.0162.\]
III. Now we interpret the data as being observations, not of a balanced growth path, but of a perfect foresight equilibrium.

We calculate a capital stock series using investment data 1959-2001 and the cumulation equation

\[ K_{t+1} = (1 - \delta)K_t + I_t. \]

We need to choose a value for \( K_{1959} \). We do so by requiring, more or less arbitrarily, that

\[ \frac{K_{1959}}{Y_{1959}} = \frac{1}{11} \left( \sum_{t=1960}^{1970} \frac{K_t}{Y_t} \right). \]

We choose \( \delta \) so that \( \delta K_t / Y_t = 0.1168 \) over the period 1970-2002, its average value in the data over this period.

Iterating on guesses for \( K_{1959} \) and \( \delta \), we obtain \( K_{1959} = 5,632.2 \) and \( \delta = 0.0469 \).
Suppose instead we choose $K_{1959}$ so that

$$\frac{K_{1960}}{K_{1959}} = \left( \frac{K_{1970}}{K_{1960}} \right)^{\frac{1}{10}}$$

and that we choose $\delta$ so that $\delta K_t / Y_t = 0.1168$ over the period 1970-2002. We obtain $K_{1959} = 6,104.1$ and $\delta = 0.0469$.

The two series generated for the capital stocks are very similar, especially after 10 years or so, when the values chosen for $K_{1959}$ make less and less difference.

The two series are also similar to the series for the capital stock generated by the balanced growth path in the previous calibration.
Real Capital Stock in the United States

- Capital Stock #1
- Capital Stock #2
- Balanced Growth Path

Year: 1960 to 2000

Billion 2002 U.S. dollars
To calibrate $\gamma$, we continue to use the first order condition

$$
\frac{1 - \gamma}{N_t h - L_t} = \frac{\gamma}{C_t} (1 - \alpha) (g^{1-\alpha})^t AK_t^\alpha L_t^{-\alpha} = \frac{\gamma}{C_t} (1 - \alpha) \frac{Y_t}{L_t} = \frac{\gamma w_t}{C_t}
$$

$$
\gamma = \frac{C_t L_t}{C_t L_t + (1 - \alpha) Y_t (N_t h - L_t)}.
$$

To calibrate $\beta$, we use the first order condition

$$
\frac{\beta^{t-1}}{C_{t-1}} = \frac{\beta^t}{C_t} (r_t + 1 - \delta)
$$

$$
\beta = \frac{C_t}{C_{t-1} (r_t + 1 - \delta)} = \frac{C_t}{C_{t-1} (\alpha Y_t / K_t + 1 - \delta)}.
$$

Using 1970-2002 data, we estimate $\gamma = 0.2741$ and $\beta = 0.9550$.

**Summary:**

$\beta = 0.9550$, $\gamma = 0.2741$, $\delta = 0.0469$, $\alpha = 0.3150$. 
A note on real investment

We have cumulated investment to generate a capital stock, where real investment is nominal investment divided by the implicit GDP deflator.

It makes less sense, in the context of the one-sector growth model, to cumulate a real investment series, say that in Table B2, where real investment is nominal investment divided by an investment deflator. If we want to model the impact of changes in the relative price of investment to consumption (in particular, the fall in this price) over the period 1960-2002, we could use a two-sector model in which the budget constraint is

\[ C_t + q_t (K_{t+1} - (1 - \delta)K_t) \leq w_t L_t + r_t K_t \]

where \( q_t \) is the price of investment relative to consumption. Depending on the choice of the production technologies of the consumption good and the investment good, this model can produce results similar to those produced by the one-sector model that we are studying. In this two-sector model, however, we would attribute some technical progress to improvements in technology in the consumption good sector and some to improvements in the investment good sector.