Calibrating the Growth Model

Kaldor's "stylized facts"

1. Y_t / L_t (output per worker) exhibits continual growth.

2. K_t / L_t (capital per worker) exhibits continual growth.

3. $r_t - \delta$ (real interest rate) is roughly constant.

4. K_t / Y_t (capital-output ratio) is roughly constant.

5. $r_t K_t / Y_t$, $w_t L_t / Y_t$ (factor shares) are roughly constant.

6. There are wide differences in the rate of growth of productivity across countries.

N. Kaldor (1961), "Capital Accumulation and Economic Growth," in F. A. Lutz and D. C. Hague, editors, *The Theory of Capital*. New York: St. Martin's Press.

The growth model

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t \log C_t \\
&\text{s. t. } C_t + K_{t+1} - K_t \le w_t L_t + (r_t - \delta) K_t \\
&C_t, K_t \ge 0 \\
&K_0 = \bar{K}_0 \\
&L_t = \lambda^t L_0 .
\end{aligned}$$

where $w_t = (1-\alpha)(g^{1-\alpha})^t A K_t^{\alpha} L_t^{-\alpha}$, $r_t = \alpha (g^{1-\alpha})^t A K_t^{\alpha-1} L_t^{1-\alpha}$.

We can solve

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t \log C_t \\
&\text{s. t. } C_t + K_{t+1} - (1-\delta) K_t \le (g^{1-\alpha})^t A K_t^{\alpha} L_t^{1-\alpha} \\
&C_t, K_t \ge 0 \\
&K_0 = \bar{K}_0 \\
&L_t = \lambda^t L_0 .
\end{aligned}$$

First-order conditions:

$$\frac{\beta^t}{C_t} = p_t$$
$$p_{t-1} = p_t\left(\alpha(g^{1-\alpha})^t AK_t^{\alpha-1}L_t^{1-\alpha} + 1 - \delta\right)$$
$$C_t + K_{t+1} - (1-\delta)K_t = (g^{1-\alpha})^t AK_t^{\alpha}L_t^{1-\alpha}.$$

Impose constant growth conditions

$$\frac{C_{t+1}/L_{t+1}}{C_t/L_t} = g_c,\quad \frac{K_{t+1}/L_{t+1}}{K_t/L_t} = g_k.$$

Simple algebra shows that

$$\frac{C_{t+1}/L_{t+1}}{C_t/L_t} = \frac{K_{t+1}/L_{t+1}}{K_t/L_t} = \frac{Y_{t+1}/L_{t+1}}{Y_t/L_t} = g.$$

Redefine variables in terms of effective labor units $\tilde{L}_t = g^t L_t = (g\lambda)^t L_0$:

$$\tilde{c}_t = C_t / \tilde{L}_t = g^{-t}(C_t / L_t)$$
$$\tilde{k}_t = K_t / \tilde{L}_t = g^{-t}(K_t / L_t)$$
$$\log\ C_t / L_t = \log\ g^t \tilde{c}_t = \log\ \tilde{c}_t + t \log\ g.$$

Notice that the balanced growth path is the steady state $\tilde{c}_t = \tilde{c},\ \tilde{k}_t = \tilde{k}$ of the redefined problem

$$\max\ \sum_{t=0}^{\infty} \beta^t \log\ \tilde{c}_t$$
$$\text{s. t. }\ \tilde{c}_t + g\lambda \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t \le A\tilde{k}_t^{\alpha}$$
$$\tilde{c}_t,\ \tilde{k}_t \ge 0$$
$$\tilde{k}_0 = \bar{K}_0 / L_0.$$

The balanced growth path matches Kaldor's stylized facts (although the explanation for fact 6 is not very interesting):

1. $Y_t / L_t = (g^{1-\alpha})^t A(K_t / L_t)^\alpha = g^t A\tilde{k}^\alpha$ grows at rate $g-1$.

2. $K_t / L_t = g^t \tilde{k}$ grows at rate $g-1$.

3. $r_t - \delta = \alpha (g^{1-\alpha})^t A K_t^{\alpha-1} L_t^{1-\alpha} - \delta = \alpha A \tilde{k}^{\alpha-1} - \delta = g\lambda / \beta - 1$ is constant.

4. $K_t / Y_t = \tilde{k}^{1-\alpha} / A$ is constant.

5. $r_t K_t / Y_t = \alpha$, $w_t L_t / Y_t = 1-\alpha$ are constant.

6. rate of growth of Y_t / L_t is determined solely by g.

Calibration to the U.S. data

I. First we interpret the data as being observations of a balanced growth path and use employment as the measure of labor input.

All data is from the *Economic Report of the President*, *2004* (http://www.gpoaccess.gov/eop/).

Y_t = Real gross domestic product (Table B-2) (billions of 2000 dollars)

L_t = Civilian employment (B-35) (thousands of persons)

	Y_t	L_t	$\frac{Y_t}{L_t}$	g	λ
1960	2,501.8	65,778	38,034		
1970	3,771.9	78,678	47,941	1.0234	1.0181
1980	5,161.7	99,303	51,979	1.0081	1.0236
1990	7,112.5	118,793	59,873	1.0142	1.0181
2000	9,817.0	136,891	71,714	1.0182	1.0143

$$\log\ Y_{t'}/L_{t'} - \log\ Y_t/L_t = \log g^t A\tilde{k}^{\alpha} - \log g^t A\tilde{k}^{\alpha} = (t'-t)\log g\,.$$

$$g = 1.0160, \qquad \lambda = 1.0185.$$

Y_t = Gross domestic product (B-1) - proprietors' income (B-28) – (taxes on production and imports (B-28) – subsidies (B-28)) (billions of current dollars)

$w_t L_t$ = Compensation of employees (B-28) (billions of current dollars)

(We distribute proprietors' income and indirect business taxes proportionally between labor income and capital income.)

	GDP	proprietors' income	indirect taxes	subsidies	Y_t	$w_t L_t$	$1-\alpha$
1960	526.4	50.8	44.6	1.1	432.1	296.4	0.6860
1970	1,038.5	78.4	91.5	4.8	873.4	617.2	0.7067
1980	2,789.5	174.1	200.7	9.8	2,424.5	1,651.8	0.6813
1990	5,803.1	380.6	425.5	26.8	5,023.8	3,338.2	0.6645
2000	9,817.0	728.4	708.9	44.3	8,424.0	5,782.7	0.6865

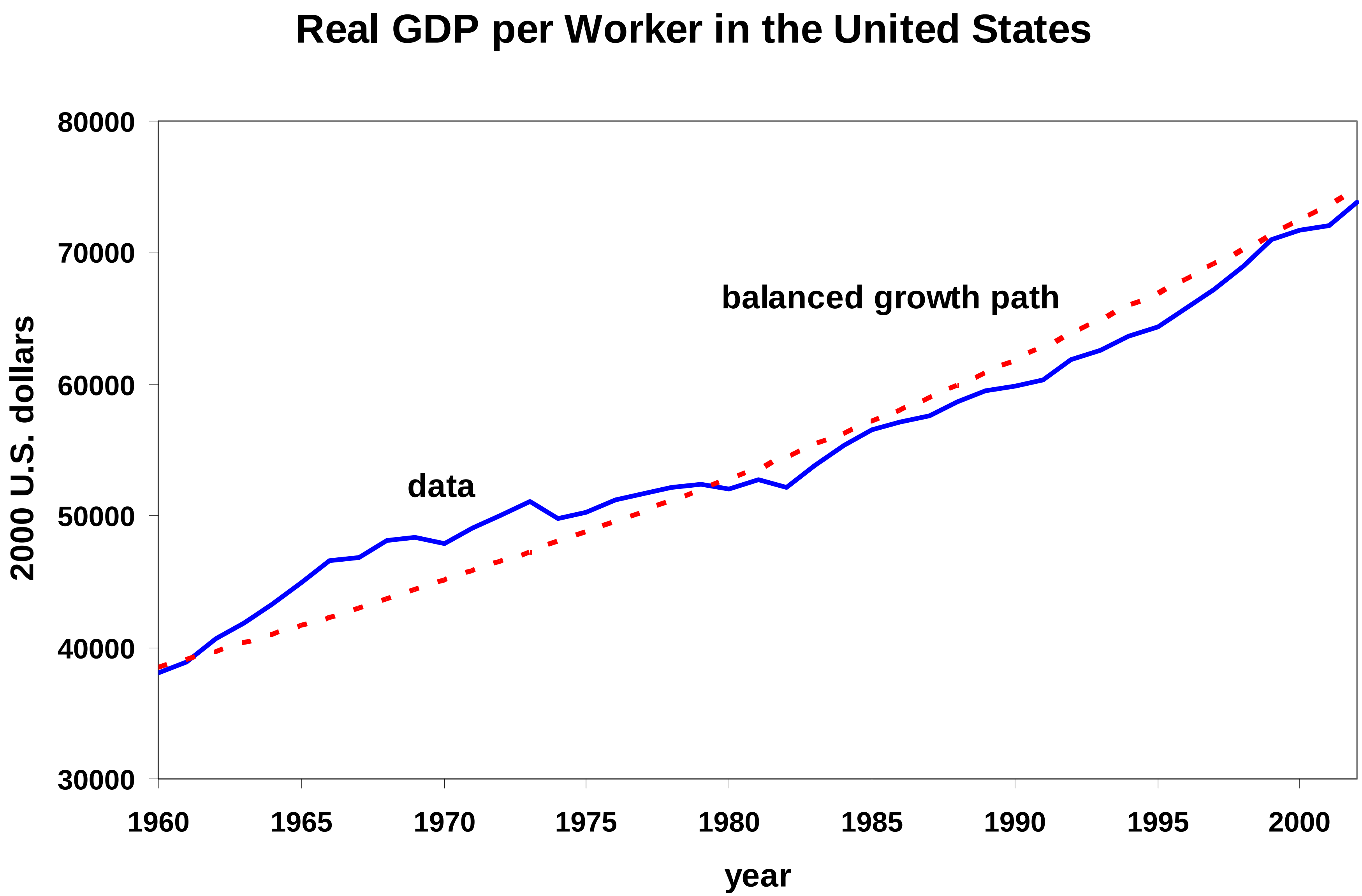
$$1-\alpha = 0.6850, \qquad \alpha = 0.3150.$$

How good is the assumption that we are in a balanced growth path?

$Y_t/(g^t L_t) = (Y_t / L_t)/g^t = A\tilde{k}^{\alpha}$ should be constant.

	$\dfrac{Y_t}{g^t L_t}$
1960	38,034
1970	40,912
1980	37,854
1990	37,210
2000	38,034

$$A\tilde{k}^{\alpha} = 38,409$$


Real GDP per Worker in the United States
80000
70000
60000
50000
40000
30000
2000 U.S. dollars
balanced growth path
data
1960
1965
1970
1975
1980
1985
1990
1995
2000
year

$K_{t+1} - (1-\delta)K_t$ = Gross private domestic investment (B-1) + government gross investment (federal defense, federal nondefense, state and local) (B-20) (billions of current dollars)

δK_t = Consumption of fixed capital (B-26) (billions of current dollars)

Y_t = Gross domestic product (B-1) (billions of current dollars)

	private investment	government investment	$K_{t+1} - (1-\delta)K_t$	δK_t	$K_{t+1} - K_t$	Y_t	$\frac{K_{t+1} - K_t}{Y_t}$	$\frac{\delta K_t}{Y_t}$
1960	78.9	28.2	107.1	55.6	51.5	526.4	0.0978	0.1056
1970	152.4	43.7	196.1	106.7	89.4	1,038.5	0.0861	0.1027
1980	479.3	100.3	579.6	343.0	236.6	2,789.5	0.0848	0.1230
1990	861.0	215.7	1,076.7	682.5	394.2	5,803.1	0.0679	0.1176
2000	1,735.5	304.4	2,039.9	1,187.8	852.1	9,817.0	0.0868	0.1210

$$\frac{K_{t+1} - K_t}{Y_t} = 0.0847, \qquad \frac{\delta K_t}{Y_t} = 0.1140.$$

Calculation of parameters:

$$\frac{K_{t+1}-K_t}{Y_t}=\frac{(g\lambda-1)\tilde{k}}{A\tilde{k}^{\alpha}}=0.0847$$

$$A\tilde{k}^{\alpha}=38,409$$

$$\tilde{k}=\frac{0.0847A\tilde{k}^{\alpha}}{g\lambda-1}=\frac{0.0847\times 38,409}{0.0348}=93,561$$

$$\frac{K_t}{Y_t}=\frac{\tilde{k}}{A\tilde{k}^{\alpha}}=\frac{93,561}{38,409}=2.4359$$

$$\frac{\delta K_t}{Y_t}=0.1140,\ \delta=\frac{0.1140}{2.4359}=0.0468$$

$$\frac{r_t K_t}{Y_t}=0.3150,\ r=\frac{0.3150}{2.4359}=0.1293$$

$$A=\frac{A\tilde{k}^{\alpha}}{\tilde{k}^{\alpha}}=\frac{38,409}{93,561^{0.3150}}=1043.23$$

$$r-\delta=0.1293-0.0468=0.0825=\frac{g\lambda}{\beta}-1,\ \beta=\frac{g\lambda}{1+r-\delta}=\frac{1.0348}{1.0825}=0.9559$$

Summary:

$\beta = 0.9559$, $\delta = 0.0468$, $g = 1.0160$, $A = 1043.23$, $\alpha = 0.3150$, $\lambda = 1.0185$.

Kaldor's stylized facts (again):

1. $Y_t / L_t = (1.0160)^{t-1960} 38,409$

2. $K_t / L_t = (1.0160)^{t-1960} 93,561$

3. $r_t - \delta = 0.0825$

4. $K_t / Y_t = 2.4359$

5. $r_t K_t / Y_t = 0.3150$, $w_t L_t / Y_t = 0.6850$

6. $g = 1.0160$

A puzzle: Interest rates on bonds

i_t = Corporate bond yield (Moody's Aaa) (percent per year) (B-73)

π_t = Change in implicit GNP deflator (percent per year) (B-3)

	i_t	π_t	$i_t - \pi_t$
1960-1969	2.35	5.01	2.66
1970-1979	6.99	8.62	1.63
1980-1989	4.75	11.34	6.59
1990-1999	2.22	7.72	5.50
2000-2002	2.03	7.06	5.03

Arbitrage implies that

$$r_t - \delta = 0.0825 \approx i_t - \pi_t = 0.0416$$

There is an equity premium. Until the 1980s, it was very large. See

R. Mehra and E. C. Prescott (1985), "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145-161.

E. R. McGrattan and E. C. Prescott (2000), "Is the Stock Market Overvalued?" *Federal Reserve Bank of Minneapolis Quarterly Review*, 24(4), 20–40.

II. Now we interpret the data as being observations of a balanced growth path, but we use total hours worked as the measure of labor input and we put leisure into the utility function.

The utility function is now

$$\max \sum_{t=0}^{\infty} \beta^t \left(\gamma \log C_t + (1-\gamma)\log(N_t \bar{h} - L_t)\right)$$

where N_t is the working-age (16-64) population and $\bar{h}$ is the maximum number for hours available for work per person, taken to be 5200 per year (100 hours per week × 52 weeks per year).

There is a new first-order condition:

$$\frac{1-\gamma}{N_t \bar{h} - L_t} = \frac{\gamma w_t}{C_t} = \frac{\gamma}{C_t}(1-\alpha)(g^{1-\alpha})^t A K_t^{\alpha} L_t^{-\alpha} = \frac{\gamma}{C_t}(1-\alpha)\frac{Y_t}{L_t}$$

$$\frac{1-\gamma}{\gamma} = (1-\alpha)\frac{Y_t}{C_t}\frac{N_t \bar{h} - L_t}{L_t}.$$

N_t = Population 14-64 (B-34) (thousands of persons)

L_t = 52 × average total private weekly hours (B-47, spliced with average total manufacturing weekly hours at 1963) × civilian employment (B-35) (thousands of persons) (L_t is expressed in billions of hours)

Y_t = Gross domestic product (B-1) (billions of current dollars)

C_t = Y_t - $K_{t+1} - (1-\delta)K_t$ (billions of current dollars)

	N_t	hours	employment	L_t	$\frac{N_t\bar{h} - L_t}{L_t}$	C_t	Y_t	$\frac{C_t}{Y_t}$	γ
1960	105,160	37.6	65,778	128.5	3.2568	419.3	526.4	0.7965	0.2631
1970	122,963	37.0	78,678	151.4	3.2240	842.4	1,038.5	0.8112	0.2686
1980	146,731	35.2	99,303	181.8	3.1978	2,209.9	2,789.5	0.7922	0.2656
1990	161,396	34.3	118,793	211.9	2.9610	4,726.4	5,803.1	0.8145	0.2865
2000	183,034	34.3	136,891	244.2	2.8982	7,777.1	9,817.0	0.7922	0.2852

$$\gamma = 0.2738.$$

We need to recalibrate g and λ:

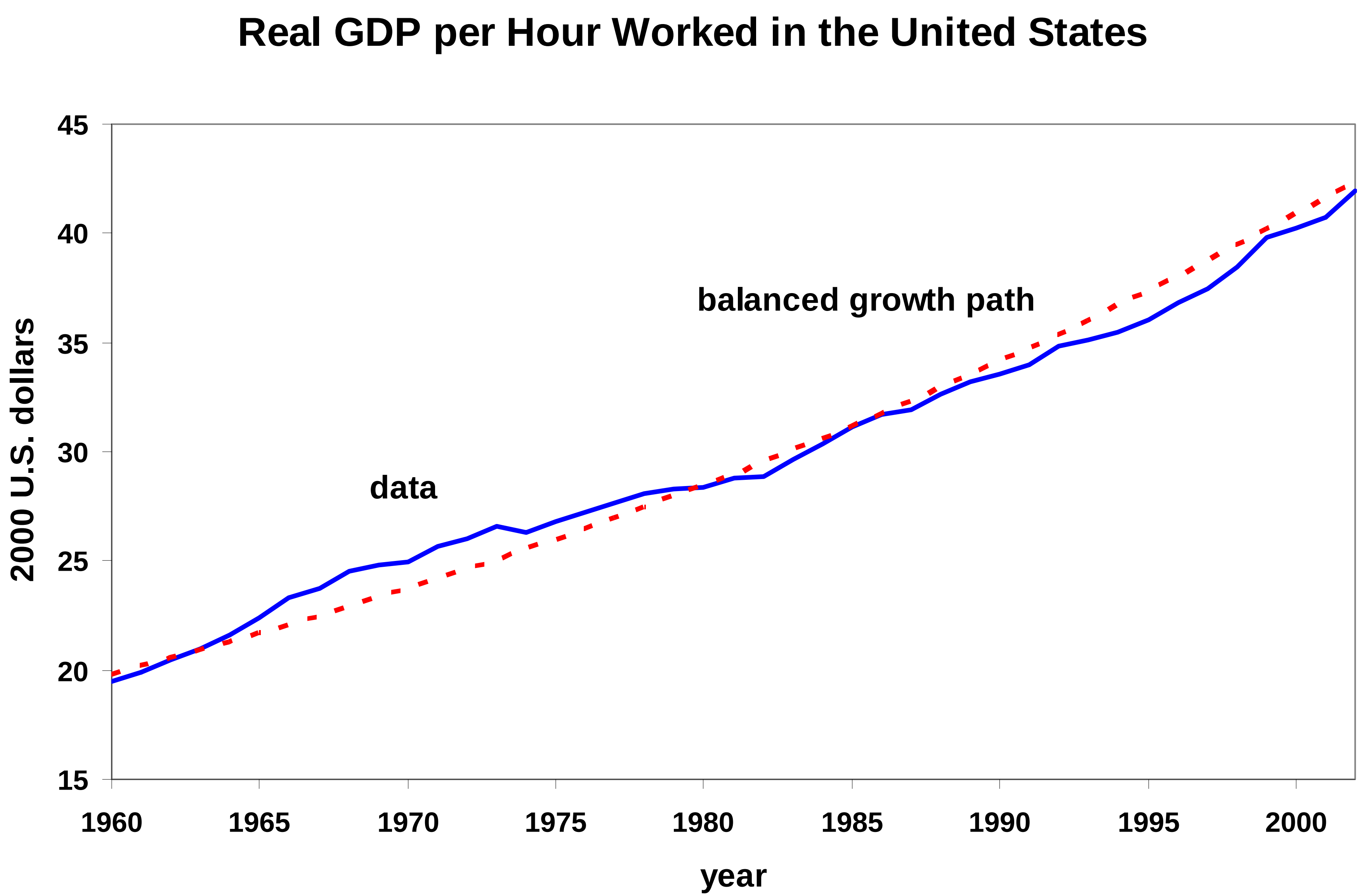
	Y_t	L_t	$\frac{Y_t}{L_t}$	g	λ
1960	2,501.8	128.5	19.48		
1970	3,771.9	151.4	24.92	1.0249	1.0166
1980	5,161.7	181.8	28.40	1.0132	1.0185
1990	7,112.5	211.9	33.57	1.0169	1.0154
2000	9,817.0	244.2	40.21	1.0182	1.0143

$$g = 1.0183, \qquad \lambda = 1.0162.$$

How good is the assumption that we are in a balanced growth path?

	$\frac{Y_t}{g^t L_t}$
1960	19.48
1970	20.79
1980	19.76
1990	19.49
2000	19.48

$$A\tilde{k}^{\alpha} = 19.80$$


Real GDP per Hour Worked in the United States
45
40
35
30
25
20
15
2000 U.S. dollars
balanced growth path
data
1960
1965
1970
1975
1980
1985
1990
1995
2000
year

$$\tilde{k} = \frac{0.0847 A\tilde{k}^{\alpha}}{g\lambda - 1} = \frac{0.0847 \times 19.80}{0.0348} = 48.23$$

$$A = \frac{A\tilde{k}^{\alpha}}{\tilde{k}^{\alpha}} = \frac{19.80}{48.23^{0.3150}} = 5.8390$$

The calibration of all of the other parameters stays the same.

Summary:

$\beta = 0.9559$, $\gamma = 0.2738$, $\delta = 0.0468$, $g = 1.0183$, $A = 5.8390$, $\alpha = 0.3150$, $\lambda = 1.0162$.

III. Now we interpret the data as being observations, not of a balanced growth path, but of a perfect foresight equilibrium.

We calculate a capital stock series using investment data 1959-2001 and the cumulation equation

$$K_{t+1} = (1-\delta)K_t + I_t.$$

We need to choose a value for K_{1959}. We do so by requiring, more or less arbitrarily, that

$$\frac{K_{1959}}{Y_{1959}} = \frac{1}{11}\left(\sum_{t=1960}^{1970} \frac{K_t}{Y_t}\right).$$

We choose δ so that $\delta K_t / Y_t = 0.1168$ over the period 1970-2002, its average value in the data over this period.

Iterating on guesses for K_{1959} and δ, we obtain $K_{1959} = 5{,}632.2$ and $\delta = 0.0469$.

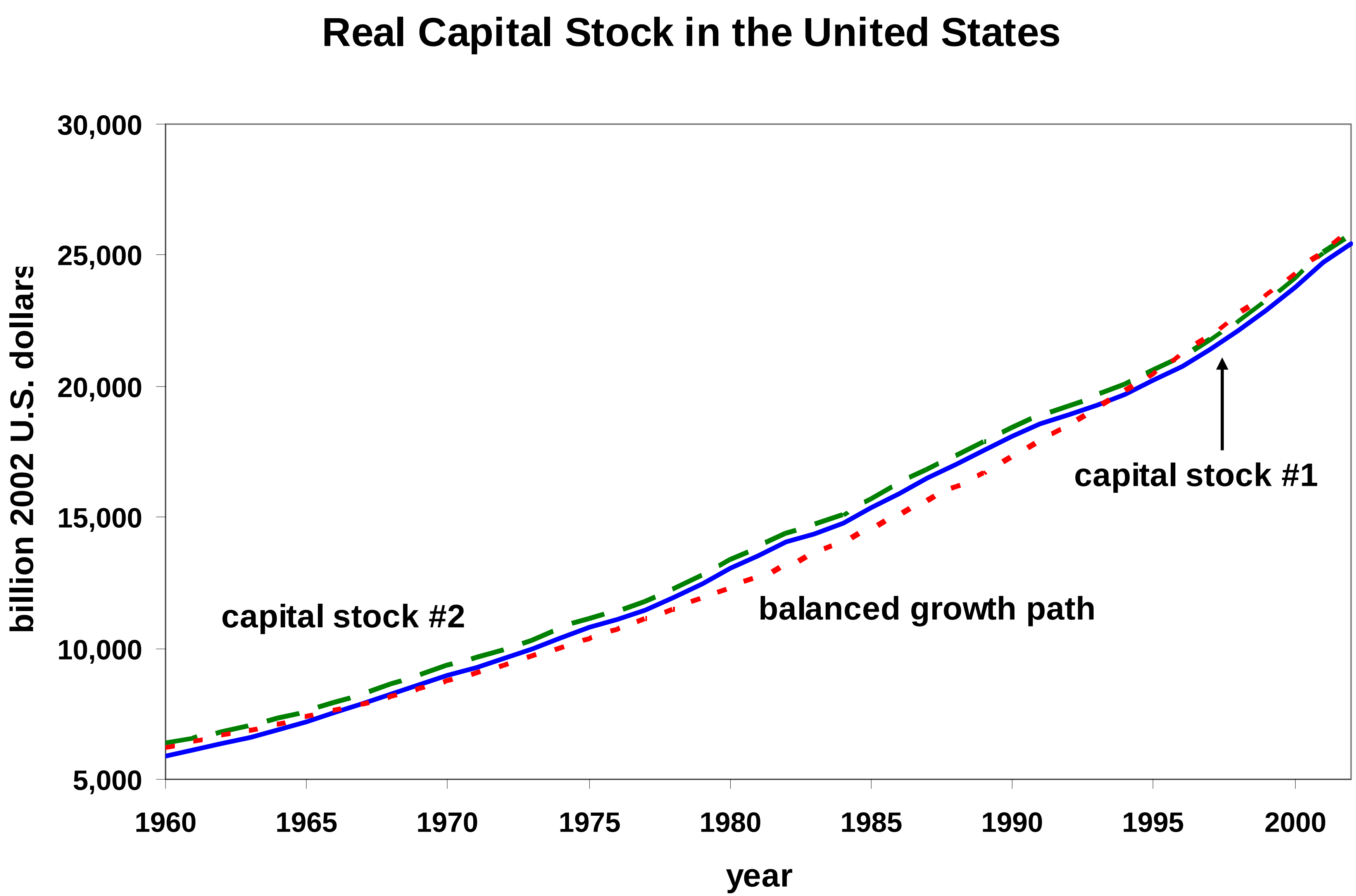
Suppose instead we choose K_{1959} so that

$$\frac{K_{1960}}{K_{1959}} = \left(\frac{K_{1970}}{K_{1960}}\right)^{\frac{1}{10}}$$

and that we choose δ so that $\delta K_t / Y_t = 0.1168$ over the period 1970-2002. We obtain $K_{1959} = 6{,}104.1$ and $\delta = 0.0469$.

The two series generated for the capital stocks are very similar, especially after 10 years or so, when the values chosen for K_{1959} make less and less difference.

The two series are also similar to the series for the capital stock generated by the balanced growth path in the previous calibration.


Real Capital Stock in the United States
30,000
25,000
20,000
15,000
10,000
5,000
billion 2002 U.S. dollars
capital stock #1
capital stock #2
balanced growth path
1960
1965
1970
1975
1980
1985
1990
1995
2000
year

To calibrate γ, we continue to use the first order condition

$$\frac{1-\gamma}{N_t\bar{h}-L_t}=\frac{\gamma}{C_t}(1-\alpha)(g^{1-\alpha})^t AK_t^{\alpha}L_t^{-\alpha}=\frac{\gamma}{C_t}(1-\alpha)\frac{Y_t}{L_t}=\frac{\gamma w_t}{C_t}$$

$$\gamma=\frac{C_tL_t}{C_tL_t+(1-\alpha)Y_t(N_t\bar{h}-L_t)}.$$

To calibrate β, we use the first order condition

$$\frac{\beta^{t-1}}{C_{t-1}}=\frac{\beta^t}{C_t}\left(r_t+1-\delta\right)$$

$$\beta=\frac{C_t}{C_{t-1}\left(r_t+1-\delta\right)}=\frac{C_t}{C_{t-1}\left(\alpha Y_t/K_t+1-\delta\right)}.$$

Using 1970-2002 data, we estimate $\gamma=0.2741$ and $\beta=0.9550$.

Summary:

$\beta=0.9550$, $\gamma=0.2741$, $\delta=0.0469$, $\alpha=0.3150$.

A note on real investment

We have cumulated investment to generate a capital stock, where real investment is nominal investment divided by the implicit GDP deflator.

It makes less sense, in the context of the one-sector growth model, to cumulate a real investment series, say that in Table B2, where real investment is nominal investment divided by an investment deflator. If we want to model the impact of changes in the relative price of investment to consumption (in particular, the fall in this price) over the period 1960-2002, we could use a two-sector model in which the budget constraint is

$$C_t + q_t\left(K_{t+1} - (1-\delta)K_t\right) \leq w_t L_t + r_t K_t$$

where q_t is the price of investment relative to consumption. Depending on the choice of the production technologies of the consumption good and the investment good, this model can produce results similar to those produced by the one-sector model that we are studying. In this two-sector model, however, we would attribute some technical progress to improvements in technology in the consumption good sector and some to improvements in the investment good sector.