Calibrating the Growth Model

Kaldor's "stylized facts"

1. \( Y_t / N_t \) (output per worker) exhibits continual growth.

2. \( K_t / N_t \) (capital per worker) exhibits continual growth.

3. \( r_t - \delta \) (real interest rate) is roughly constant.

4. \( K_t / Y_t \) (capital-output ratio) is roughly constant.

5. \( r_t K_t / Y_t \) or \( w_t N_t / Y_t \) (factor shares) are roughly constant.

6. There are wide differences in the rate of growth of productivity across countries.


The growth model

\[
\max \sum_{t=0}^{\infty} \beta^t \log \frac{C_t}{N_t} \\
\text{s. t. } C_t + K_{t+1} - (1 - \delta) K_t \leq (\gamma^{1-\alpha})^{t} \theta K_t \alpha N_{t}^{1-\alpha} \\
C_t, \ K_t \geq 0 \\
K_0 = \bar{K}_0 \\
N_t = \eta^t N_0 
\]

First-order conditions:

\[
\beta^t / C_t = P_t \\
P_{t+1} = P_t (\alpha (\gamma^{1-\alpha})^{t} \theta K_t^{\alpha - 1} N_t^{1-\alpha} + 1 - \delta) \\
C_t + K_{t+1} - (1 - \delta) K_t = (\gamma^{1-\alpha})^{t} \theta K_t^{\alpha} N_t^{1-\alpha} .
\]

Impose balanced growth conditions

\[
\frac{C_{t+1}}{C_t} / N_{t+1} = \gamma_c, \quad \frac{K_{t+1}}{K_t} / N_{t+1} = \gamma_k .
\]
Simple algebra shows that

\[
\frac{C_{t+1}}{N_{t+1}} / \frac{C_t}{N_t} = \frac{K_{t+1}}{N_{t+1}} / \frac{K_t}{N_t} = \frac{Y_{t+1}}{N_{t+1}} / \frac{Y_t}{N_t} = \gamma .
\]

Redefine variables in terms of effective labor units \( \tilde{N}_t = \gamma^t N_t = (\gamma \eta)^t N_0 \):

\[
\begin{align*}
\tilde{c}_t &= C_t / \tilde{N}_t = \gamma^{-i} (C_t / N_t) \\
\tilde{k}_t &= K_t / \tilde{N}_t = \gamma^{-i} (K_t / N_t) \\
\log C_t / N_t &= \log \gamma^i \tilde{c}_t = \log \tilde{c}_t + t \log \gamma .
\end{align*}
\]

Notice that the balanced growth path is the steady state \( \tilde{c}_t = \bar{c} \), \( \tilde{k}_t = \bar{k} \) of the redefined model

\[
\max \sum_{t=0}^{\infty} \beta^t \log \tilde{c}_t \\
\text{s. t. } \tilde{c}_t + \gamma \eta \tilde{k}_{t+1} - (1 - \delta) \tilde{k}_t \leq \theta \tilde{k}_t \]

\[
\tilde{c}_t, \tilde{k}_t \geq 0 \\
\tilde{k}_0 = \bar{k}_0 .
\]

The balanced growth path also matches Kaldor’s stylized facts (although the explanation for fact 6 is not very interesting):

1. \( Y_t / N_t = (\gamma^{1-a})^t \theta (K_t / N_t)^a = \gamma^i \theta \tilde{k}^a \) grows at rate \( \gamma - 1 \).

2. \( K_t / N_t = \gamma^i \tilde{k} \) grows at rate \( \gamma - 1 \).

3. \( r_t - \delta = \alpha (\gamma^{1-a})^t \theta K_t^{a-1} N_t^{1-a} - \delta = \alpha \theta \tilde{k}^{a-1} - \delta = \gamma \eta / \beta - 1 \) is constant.

4. \( K_t / Y_t = \tilde{k}^{1-a} / \theta \) is constant.

5. \( r_t K_t / Y_t = \alpha \), \( w_i N_t / Y_t = 1 - \alpha \) are constant.

6. rate of growth of \( Y_t / N_t \) is determined solely by \( \gamma \).
Calibration to the U.S. data

\( Y_t = \) Real gross domestic product (Economic Report of the President, 2002, Table B-2) (Billions of 1996 dollars)

\( N_t = \) Civilian Employment, Table (B-32) (Thousands of Persons)

<table>
<thead>
<tr>
<th>Year</th>
<th>( Y_t ) (Billions of 1996 dollars)</th>
<th>( N_t ) (Thousands of Persons)</th>
<th>( Y_t / N_t )</th>
<th>( \gamma )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>2,376.7</td>
<td>65,778</td>
<td>36,132</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970</td>
<td>3,578.0</td>
<td>78,678</td>
<td>45,476</td>
<td>1.0233</td>
<td>1.0181</td>
</tr>
<tr>
<td>1980</td>
<td>4,900.9</td>
<td>99,303</td>
<td>49,353</td>
<td>1.0082</td>
<td>1.0236</td>
</tr>
<tr>
<td>1990</td>
<td>6,707.9</td>
<td>118,793</td>
<td>56,467</td>
<td>1.0136</td>
<td>1.0181</td>
</tr>
<tr>
<td>2000</td>
<td>9,224.0</td>
<td>140,863</td>
<td>65,482</td>
<td>1.0149</td>
<td>1.0172</td>
</tr>
</tbody>
</table>

\[
\log Y_t / N_t - \log Y_t / N_t = \log \gamma \hat{k}^\alpha - \log \gamma' \hat{k}^\alpha = (t' - t) \log \gamma.
\]

\( \gamma = 1.0150, \quad \eta = 1.0192 \)

\( Y_t = \) Gross domestic product (B-26) - proprietors' income (B-28) - indirect business tax (B-26) (Billions of current dollars)

\( w_t N_t = \) Compensation of employees (B-28) (Billions of current dollars)

(We distribute proprietors’ income and indirect business taxes proportionally between labor income and capital income.)

<table>
<thead>
<tr>
<th>Year</th>
<th>( Y_t ) (Billions of current dollars)</th>
<th>( w_t N_t ) (Billions of current dollars)</th>
<th>( 1 - \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>430.0</td>
<td>296.4</td>
<td>0.6893</td>
</tr>
<tr>
<td>1970</td>
<td>865.6</td>
<td>617.2</td>
<td>0.7130</td>
</tr>
<tr>
<td>1980</td>
<td>2,406.0</td>
<td>1,651.7</td>
<td>0.6865</td>
</tr>
<tr>
<td>1990</td>
<td>4,974.9</td>
<td>3,351.0</td>
<td>0.6736</td>
</tr>
<tr>
<td>2000</td>
<td>8,395.2</td>
<td>5,715.2</td>
<td>0.6808</td>
</tr>
</tbody>
</table>

\( 1 - \alpha = 0.6886, \quad \alpha = 0.3114 \)

How good is the assumption that we are in a balanced growth path? Let 1960 be \( t = 0 \):

\[
Y_t / (\gamma' N_t) = (Y_t / N_t) / \gamma' = \theta \hat{k}^\alpha \]

should be constant.

<table>
<thead>
<tr>
<th>Year</th>
<th>( \gamma' N_t )</th>
<th>( Y_t )</th>
<th>( \gamma' N_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>36,132</td>
<td>36,132</td>
<td></td>
</tr>
<tr>
<td>1970</td>
<td>39,195</td>
<td>39,195</td>
<td></td>
</tr>
<tr>
<td>1980</td>
<td>36,661</td>
<td>36,661</td>
<td></td>
</tr>
<tr>
<td>1990</td>
<td>36,151</td>
<td>36,151</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>36,132</td>
<td>36,132</td>
<td></td>
</tr>
</tbody>
</table>

\( \theta \hat{k}^\alpha = 36,854 \)
\[ K_{t+1} - (1 - \delta)K_t = \text{Gross private domestic investment (B-1) + government gross investment (B-20)} \] (Billions of current dollars)

\[ \delta K_t = \text{Capital consumption allowances (B-26)} \] (Billions of current dollars)

\[ Y_t = \text{Gross domestic product (B-1)} \] (Billions of current dollars)

\[
\begin{array}{ccccccc}
K_{t+1} - (1 - \delta)K_t & \delta K_t & K_{t+1} - K_t & Y_t & \frac{K_{t+1} - K_t}{Y_t} & \frac{\delta K_t}{Y_t} \\
1960 & 107.2 & 56.9 & 50.3 & 527.4 & 0.0954 & 0.1079 \\
1970 & 197.1 & 152.4 & 88.0 & 1,039.7 & 0.0846 & 0.1049 \\
1980 & 578.2 & 345.2 & 233.0 & 2,795.6 & 0.0833 & 0.1235 \\
1990 & 1,077.4 & 711.3 & 366.1 & 5,803.2 & 0.0631 & 0.1226 \\
2000 & 2,085.8 & 1,241.3 & 844.5 & 9,872.9 & 0.0855 & 0.1257 \\
\end{array}
\]

**Calculation of parameters**

\[
\frac{(K_{t+1} - K_t)}{Y_t} = (\gamma \eta - 1)\tilde{k} / (\theta \tilde{k}^\alpha) = 0.0824
\]

\[ \theta \tilde{k}^\alpha = 36,854 \]

\[ \tilde{k} = 0.0979 \left( \theta \tilde{k}^\alpha / (\gamma \eta - 1) \right) = 0.0824(36,854 / 0.0345) = 86,335 \]

\[ K_t / Y_t = \tilde{k} / (\theta \tilde{k}^\alpha) = 86,335 / 36,854 = 2.3426 \]

\[ \delta K_t / Y_t = 0.1169 \]

\[ \delta = 0.0499 \]

\[ r_t K_t / Y_t = 0.3114 \]

\[ r = 0.1329 \]

\[ \theta = 36,854 / (86,335^{0.3114}) = 1,070.37 \]

\[ r - \delta = 0.1329 - 0.0499 = 0.0830 = \gamma \eta / \beta - 1 \]

\[ \beta = \gamma \eta / (1 + r - \delta) = 1.0345 / 1.0830 = 0.9552 \]

**Summary**

\[ \beta = 0.9552 \]

\[ \eta = 1.0192 \]

\[ \gamma = 1.0150 \]

\[ \delta = 0.0499 \]

\[ \theta = 1,070.37 \]

\[ \alpha = 0.3114 \]
Kaldor's stylized facts (again)

1. \( Y_t / N_t = (1.0150)^t \times 36,854 \) \( (t = 0 \text{ in 1960}) \)

2. \( K_t / N_t = (1.0150)^t \times 86,335 \)

3. \( r_t - \delta = 0.0830 \)

4. \( K_t / Y_t = 2.3426 \)

5. \( r_t K_t / Y_t = 0.3114, \ w_t N_t / Y_t = 0.6886 \)

6. \( \gamma = 1.0150 \)

A puzzle: Interest rates on bonds

\( i_t \) = Corporate bond rate (Moody's Aaa) (B-73)

\( \pi_t \) = Percentage change in GNP deflator (B-3)

<table>
<thead>
<tr>
<th></th>
<th>( i_t ) (%)</th>
<th>( \pi_t ) (%)</th>
<th>( i_t - \pi_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960-1969</td>
<td>5.01</td>
<td>2.36</td>
<td>2.65</td>
</tr>
<tr>
<td>1980-1989</td>
<td>11.34</td>
<td>4.80</td>
<td>6.54</td>
</tr>
<tr>
<td>1990-1999</td>
<td>7.72</td>
<td>2.30</td>
<td>5.42</td>
</tr>
<tr>
<td>2000</td>
<td>7.62</td>
<td>2.30</td>
<td>5.32</td>
</tr>
</tbody>
</table>

Arbitrage implies that

\[ r_t - \delta = 0.0830 \approx i_t - \pi_t = 0.0409 \]

There was a large equity premium up until the 1980s. See
