Trade, Growth, and Convergence in a Dynamic Heckscher-Ohlin Model

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Trade and Growth

In 2007 Mexico has income per capita of 9600 U.S. dollars. In 1940 the United Stated had income per capita of about 9400 U.S. dollars (real 2007 U.S. dollars).

To study what will happened in Mexico over the next 70 years, should we study what happened to the United States since 1940?

…or should we take into account that the United States was the country with the highest income in the world in 1940, while Mexico has a very large trade relation with the United States — a country with a level of income per capita approximately 5 times larger in 2007?

We study this question using the Heckscher-Ohlin model of international trade: Countries differ in their initial endowments of capital per worker.

Contributions of This Paper

- A complete characterization of equilibria of the dynamic Heckscher-Ohlin model: a classic problem in economic theory studied by, for example, Oniki and Uzawa (1965) and Stiglitz (1970).

- A counterexample for the growth literature: introducing international trade into the standard growth model can completely reverse convergence results.

- Empirical relevance: to the extent that growth is driven by accumulation of some factor like physical or human capital, opening a less developed country to international trade can lower growth rates at the same time as it raises welfare.

• The General Dynamic Heckscher-Ohlin Model

n countries

countries differ in initial capital-labor ratios $\bar{k}_0^i$

and in size of population L^i.

two traded goods — a capital intensive good and a labor intensive good

$$y_j = \phi_j(k_j, \ell_j)$$

$$\frac{\phi_{1L}(k/\ell,1)}{\phi_{1K}(k/\ell,1)} < \frac{\phi_{2L}(k/\ell,1)}{\phi_{2K}(k/\ell,1)}$$

nontraded investment good

$$x = f(x_1, x_2)$$

Feasibility:

$$\sum_{i=1}^{n} L^i (c^i_{jt} + x^i_{jt}) = \sum_{i=1}^{n} L^i y^i_{jt} = \sum_{i=1}^{n} L^i \phi_j (k^i_{jt}, \ell^i_{jt}).$$

$$k^i_{1t} + k^i_{2t} = k^i_t$$

$$\ell^i_{1t} + \ell^i_{2t} = 1$$

$$k^i_{t+1} - (1-\delta) k^i_t = x^i_t = f(x^i_{1t}, x^i_{2t})$$

Infinitely-Lived Consumers

consumer in country i, $i = 1,...,n$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_{1t}^i, c_{2t}^i)$$

$$\text{s.t. } p_{1t}c_{1t}^i + p_{2t}c_{2t}^t + q_t^i x_t^i + b_{t+1}^i = w_t^i + (1 + r_t^{bi})b_t^i + r_t^i k_t^i$$

$$k_{t+1}^i - (1-\delta)k_t^i = x_t^i$$

$$c_{jt}^i \geq 0,\ x_t^i \geq 0,\ b_t^i \geq -B$$

$$k_0^i = \bar{k}_0^i, b_0^i = 0.$$

Notice that since p_{1t} and p_{2t} are equalized across countries by trade, we can set

$$q_t^i = q_t = 1.$$

The factor prices w_t^i and r_t^i are potentially different across countries.

International borrowing and lending:

$$\sum_{i=1}^{n} L^i b_t^i = 0,$$

No international borrowing and lending:

$$b_t^i = 0.$$

International borrowing and lending implies that $r_t^{bi} = r_t^b$, $t = 1, 2, \ldots$. No arbitrage implies that $r_t^i = r_t = r_t^b + \delta$.

Setting $q_t^i = q_t = 1$ and $b_t^i = 0$, we can write the problem of the consumer in country i, $i = 1,...,n$, as

$$\max \sum_{t=0}^{\infty} \beta^t u(c_{1t}^i, c_{2t}^i)$$

$$\text{s.t. } p_{1t} c_{1t}^i + p_{2t} c_{2t}^t + x_t^i = w_t^i + r_t^i k_t^i$$

$$k_{t+1}^i - (1-\delta) k_t^i = x_t^i$$

$$c_{jt}^i \geq 0,\ x_t^i \geq 0$$

$$k_0^i = \overline{k}_0^i .$$

Integrated Equilibrium Approach

Characterization and computation of equilibrium is relatively easy when we can solve for equilibrium of an artificial world economy in which we ignore restrictions on factor mobility and then disaggregate the consumption, production, and investment decisions.

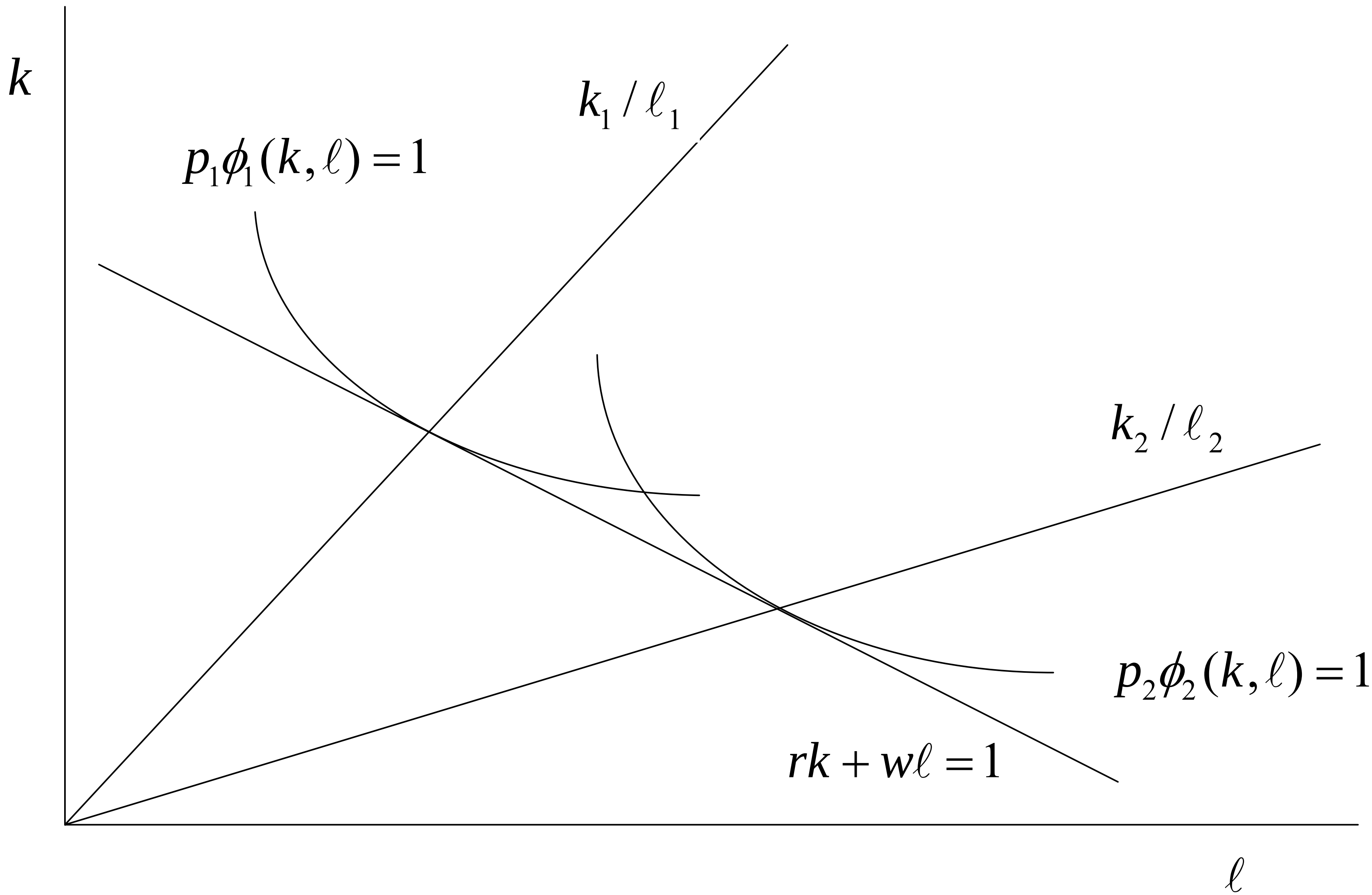
This is a guess-and-verify approach: We first solve for the integrated equilibrium of the world economy and then we see if we can disaggregate the consumption, production, and investment decisions.

Potential problem: We cannot assign each country nonnegative production plans for each of the two goods while maintaining factor prices equal to those in the world equilibrium.

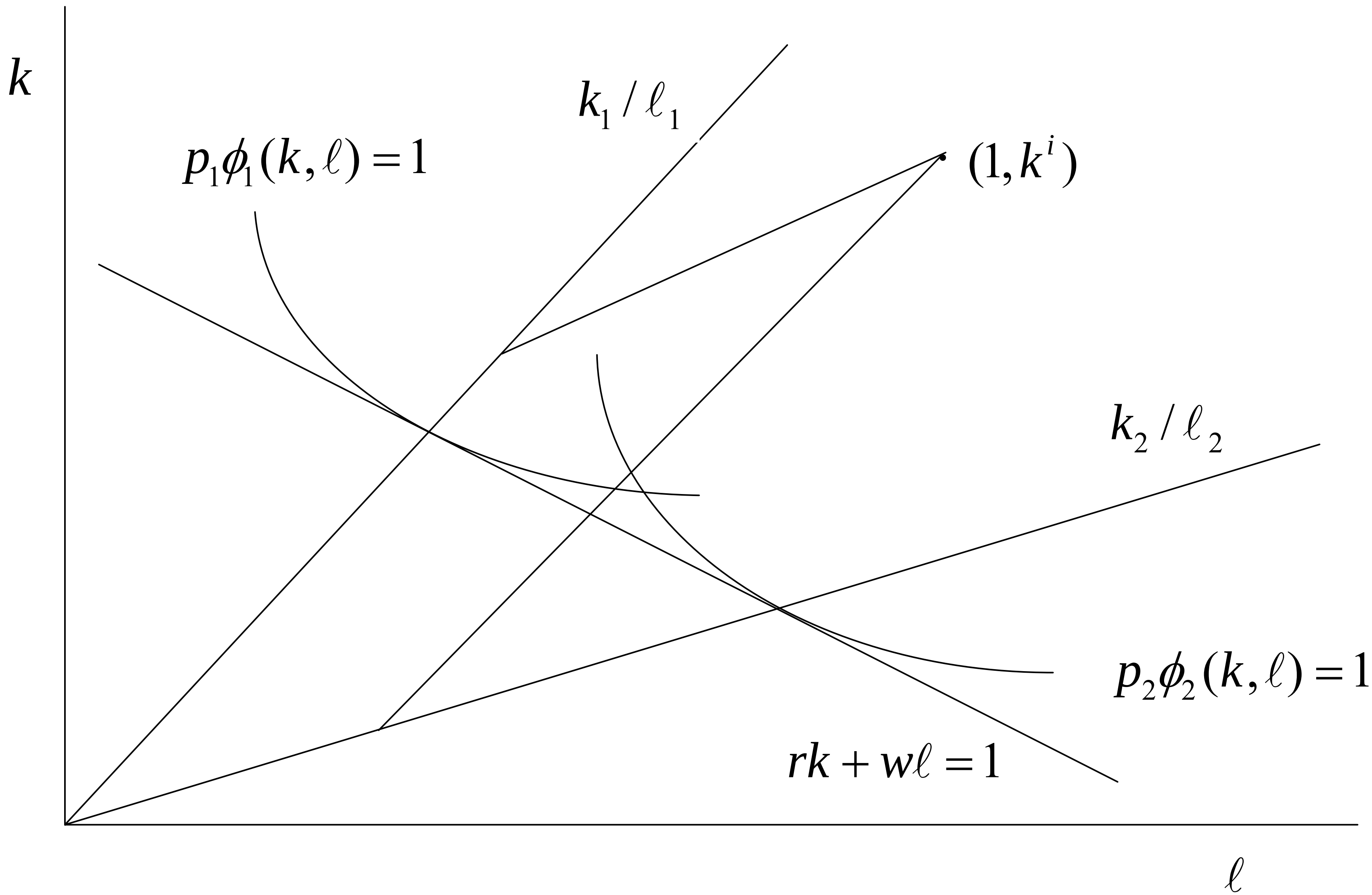
Another potential problem: We cannot assign each country nonnegative investment.

If the integrated equilibrium approach does not work, it could be very difficult to calculate an equilibrium.

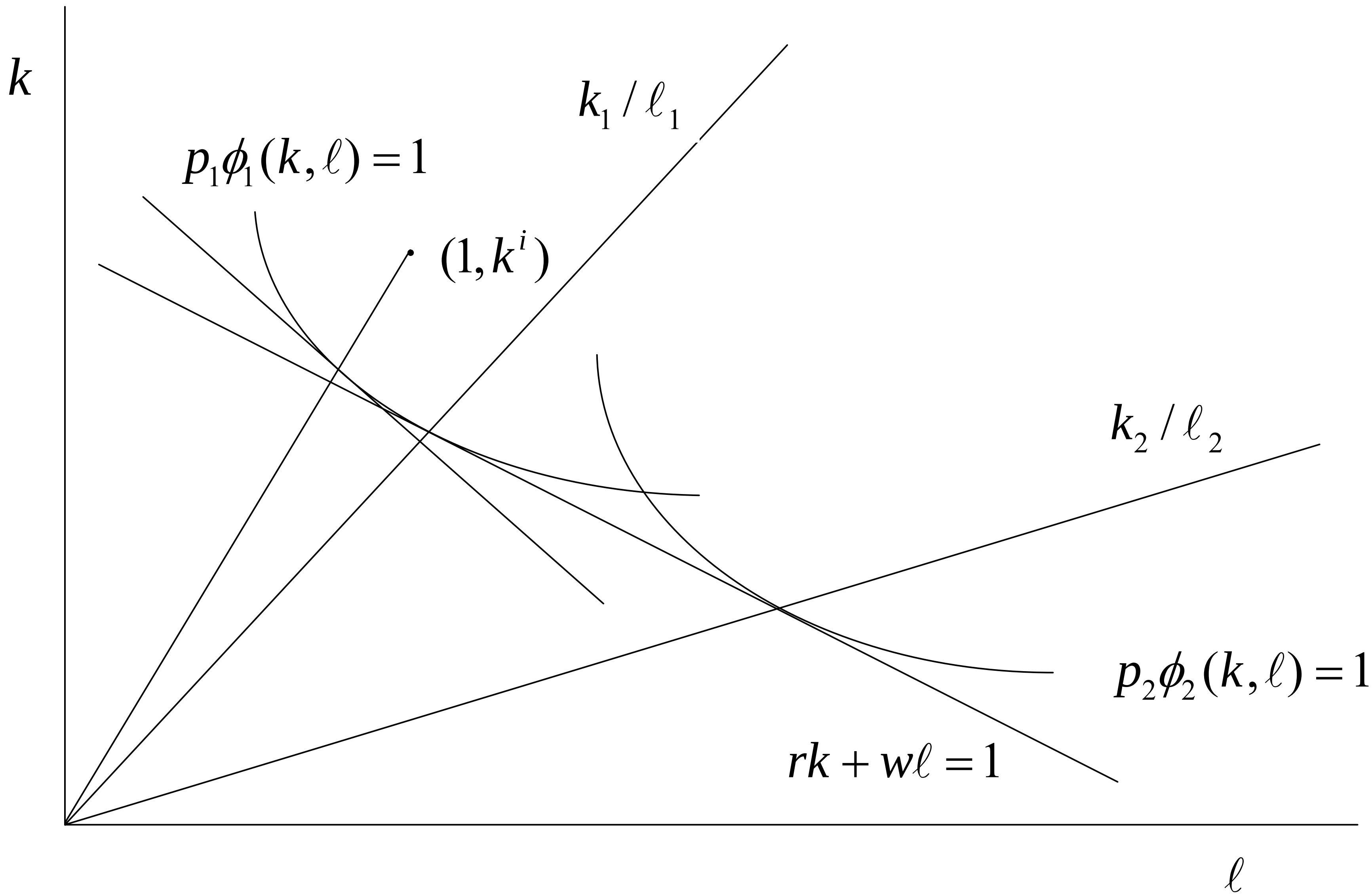
We would have to determine the pattern of specialization over an infinite time horizon.



k
k_1 / ℓ_1
$p_1\phi_1(k,\ell)=1$
k_2 / ℓ_2
$p_2\phi_2(k,\ell)=1$
$rk + w\ell = 1$
ℓ



k
k1 / ℓ1
p1φ1(k, ℓ) = 1
(1, k^i)
k2 / ℓ2
p2φ2(k, ℓ) = 1
rk + wℓ = 1
ℓ


k
k_1/ℓ_1
$p_1\phi_1(k,\ell)=1$
$(1,k^i)$
k_2/ℓ_2
$p_2\phi_2(k,\ell)=1$
$rk+w\ell=1$
ℓ

Results for General Model

International borrowing and lending implies factor price equalization in period $t = 1, 2, \ldots$ Production plans and international trade patterns are indeterminate.

Any steady state or sustained growth path has factor price equalization.

If there exists a steady state in which the total capital stock is positive or a sustained growth path, then there exists a continuum of such steady states or sustained growth paths, indexed by the distribution of world capital $\hat{k}^1 / \hat{k}, \ldots, \hat{k}^n / \hat{k}$.

International trade occurs in every steady state or sustained growth path of the model in which $\hat{k}^i / \hat{k} \neq 1$ for some i.

We focus on models with no international borrowing and lending.

For analysis of general model with infinitely lived consumers and comparison with model with overlapping generations, see

C. Bajona and T. J. Kehoe (2006), "Demographics in Dynamic Heckscher-Ohlin Models: Overlapping Generations versus Infinitely Lived Consumers."

Ventura Model

$$u(c_1, c_2) = v\left(f(c_1, c_2) \right) = \log\left(f(c_1, c_2) \right)$$

$$\phi_1(k_1, \ell_1) = k_1$$

$$\phi_2(k_2, \ell_2) = \ell_2$$

$$f(x_1, x_2) = \begin{cases} d\left(a_1 x_1^b + a_2 x_2^b\right)^{1/b} & \text{if } b \neq 0 \\ d x_1^{a_1} x_2^{a_2} & \text{if } b = 0 \end{cases}.$$

Ventura (1997) examines the continuous-time version of this model.

In the Ventura model, we can solve for the equilibrium of the world economy by solving a one-sector growth model in which $c_t = f(c_{1t}, c_{2t})$:

$$\begin{gathered} \max \sum_{t=0}^{\infty} \beta^t \log c_t \\ \text{s.t. } c_t + x_t = f(k_t, 1) \\ k_{t+1} - (1-\delta)k_t = x_t \\ c_t \geq 0,\ k_t \geq 0 \\ k_0 = \bar{k}_0. \end{gathered}$$

If $b < 0$ and $1/\beta - 1 + \delta > da_1^{1/b}$, the equilibrium converges to $\hat{k} = 0$.

If $b > 0$ and $1/\beta - 1 + \delta < da_1^{1/b}$, the economy grows without bound, and the equilibrium converges to a sustained growth path.

In every other case, the equilibrium converges to a steady state in which $f_K(\hat{k}, 1) = 1/\beta - 1 + \delta$.

The 2 sectors matter a lot for disaggregating the integrated equilibrium!

In particular, we cannot solve for the equilibrium values of the variables for one of the countries by solving an optimal growth problem for that country in isolation.

Instead, the equilibrium path for k_t^i and the steady state value of $\hat{k}^i$ depends on $\overline{k_0^i}$ as well as on the path for k_t and the steady state value of $\hat{k}$.

Proposition: Let $y_t^i = p_{1t} y_{1t}^i + p_{2t} y_{2t}^i = r_t k_t^i + w_t$. Suppose that $x_t^i > 0$ for all i and all t. Then

$$\frac{y_{t+1}^i - y_{t+1}}{y_{t+1}} = \frac{r_{t+1} c_t / y_{t+1}}{r_t c_{t-1} / y_t} \left(\frac{y_t^i - y_t}{y_t} \right)$$

If $\delta = 1$,

$$\frac{y_{t+1}^i - y_{t+1}}{y_{t+1}} = \frac{s_{t+1}}{s_t} \left(\frac{y_t^i - y_t}{y_t} \right)$$

where $s_t = c_t / y_t$.

Proof: The first-order conditions from the consumers' problems are

$$\frac{c_t^i}{c_{t-1}^i} = \frac{c_t}{c_{t-1}} = \beta(1 + r_t - \delta).$$

The demand functions are

$$c_t^i = (1-\beta)\left[\sum_{s=t}^{\infty}\left(\prod_{\tau=t+1}^{s}\frac{1}{1+r_\tau-\delta}\right)w_s + (1+r_t-\delta)k_t^i\right]$$

$$c_t^i - c_t = (1-\beta)(1+r_t-\delta)(k_t^i - k_t).$$

The budget constraint implies that

$$c_t^i - c_t + k_{t+1}^i - k_{t+1} = (1+r_t-\delta)(k_t^i - k_t).$$

Combining these conditions, we obtain

$$k_{t+1}^i - k_{t+1} = \frac{c_t}{c_{t-1}}(k_t^i - k_t).$$

The difference between a country's income per worker and the world's income per worker can be written as

$$y^i_{t+1} - y_{t+1} = r_{t+1}(k^i_{t+1} - k_{t+1}).$$

Using the expression for $k^i_{t+1} - k_{t+1}$ found above and operating, we obtain:

$$\frac{y^i_{t+1} - y_{t+1}}{y_{t+1}} = \frac{r_{t+1}c_t / y_{t+1}}{r_t c_{t-1} / y_t}\left(\frac{y^i_t - y_t}{y_t}\right).$$

In the case $\delta = 1$ this becomes (using $c_{t+1} / c_t = \beta r_{t+1}$),

$$\frac{y^i_{t+1} - y_{t+1}}{y_{t+1}} = \frac{s_{t+1}}{s_t}\left(\frac{y^i_t - y_t}{y_t}\right),$$

where $s_t = c_t / y_t$. ■

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_0}\left(\frac{y_0^i - y_0}{y_0}\right)$$

Proposition. Suppose that $\delta = 1$, that $0 < k_0 < \hat{k}$, and that $x_t^i > 0$ for all i and all t. Then

if $b > 0$, differences in relative income levels decrease over time;

if $b = 0$, differences in relative income levels stay constant over time; and

if $b < 0$, differences in relative income levels increase over time.

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_0}\left(\frac{y_0^i - y_0}{y_0}\right)$$

Proposition. Suppose that $\delta = 1$, that $0 < k_0 < \hat{k}$, and that $x_t^i > 0$ for all i and all t. Then

if $b > 0$, differences in relative income levels decrease over time;

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if $b < 0$, differences in relative income levels increase over time.

Notice contrast with convergence results for world of closed economies!

What about corner solutions in investment?

If $x_t^i > 0$ for all i and all t, then

$$\frac{k_{t+1}^i - k_{t+1}}{k_{t+1}} = \frac{c_t / k_{t+1}}{c_{t-1} / k_t}\left(\frac{k_t^i - k_t}{k_t}\right) = \frac{z_{t+1}}{z_t}\left(\frac{k_t^i - k_t}{k_t}\right)$$

where $z_t = c_{t-1} / k_t$ and $z_0 = c_0 / \left(\beta r_0 k_0\right)$.

The sequence z_t has the same monotonicity properties as the sequence $s_t = c_t / y_t$.

Proposition: Suppose that the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is constant or strictly decreasing. There exists an equilibrium where $x_t^i > 0$ for all i and all t.

Proposition: Suppose that the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is strictly increasing. Let

$$\hat{z} = \lim_{t\to\infty} \frac{c_{t-1}}{k_t},$$

and let $\overline{k}_0^{i_{min}} \leq \overline{k}_0^i$, $i = 1,\ldots,n$. If

$$\frac{\hat{z}}{z_0}\left(\frac{\overline{k}_0^{i_{min}} - \overline{k}_0}{\overline{k}_0}\right) \geq -1,$$

then there exists an equilibrium where $x_t^i > 0$ for all i and all t. Otherwise, there is no equilibrium where $x_t^i > 0$ for all i and all t. When there exists an equilibrium with no corner solutions in investment, it is the unique such equilibrium.

Numerical example 1: Two countries. $\beta = 0.95$, $\delta = 1$, and $L^1 = L^2 = 10$.

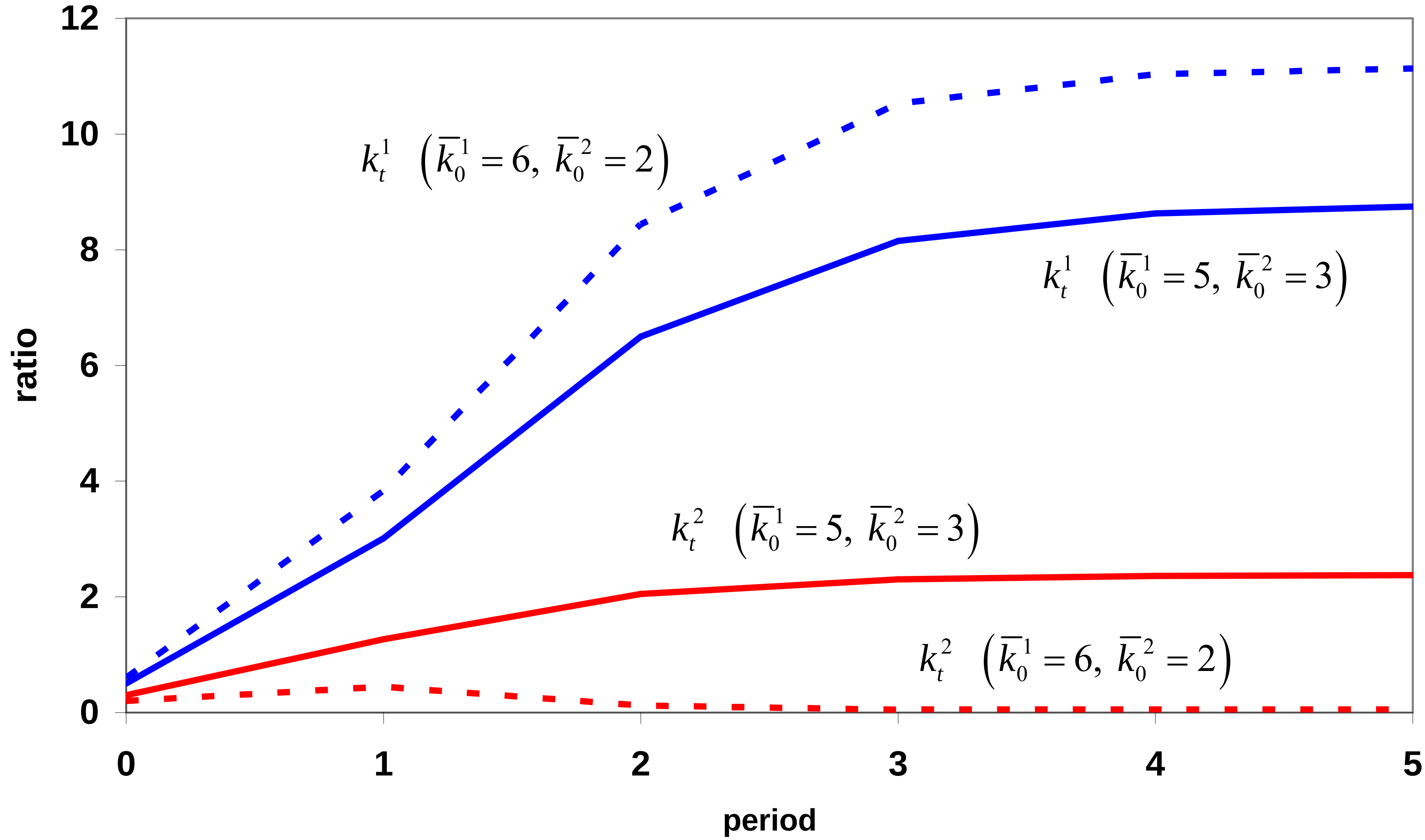
$$f(x_1, x_2) = 10\left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2}.$$

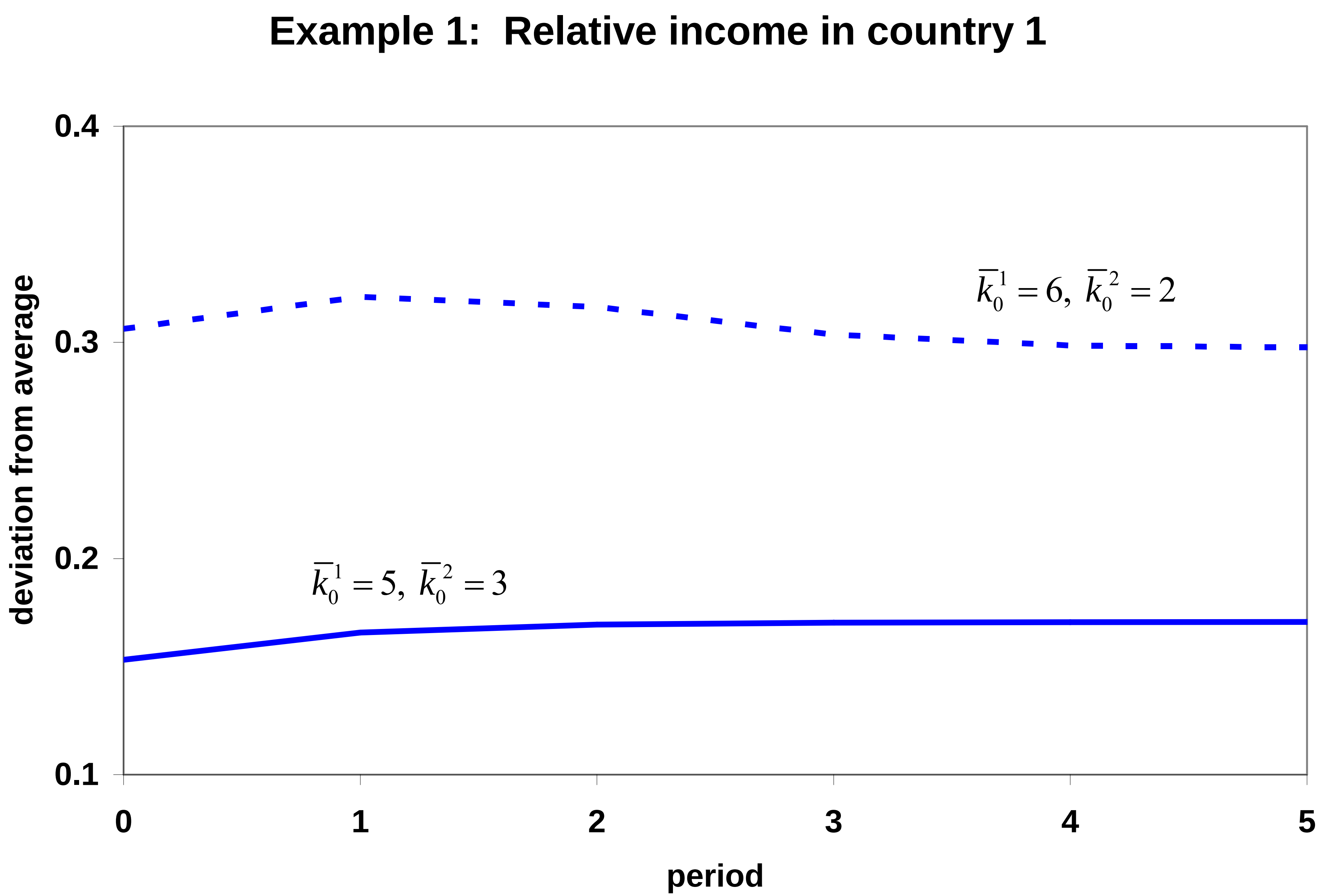
We contrast two different worlds:

In the first world, $\bar{k}_0^1 = 5$ and $\bar{k}_0^2 = 3$. Here there is an equilibrium with no corner solutions for investment.

In the second world, $\bar{k}_0^1 = 6$ and $\bar{k}_0^2 = 2$. Country 2 has $x_t^i = k_t^i = 0$ starting in period 3.

Example 1: Capital-labor ratios




Example 1: Relative income in country 1
0.4
0.3
0.2
0.1
deviation from average
$\overline{k}_0^1 = 6,\ \overline{k}_0^2 = 2$
$\overline{k}_0^1 = 5,\ \overline{k}_0^2 = 3$
0
1
2
3
4
5
period

Generalized Ventura Model

$u(c_1, c_2) = v\big(f(c_1, c_2)\big) = \log\big(f(c_1, c_2)\big)$, and f, ϕ_1, and ϕ_2 are general constant-elasticity-of-substitution functions

Define

$$F(k, \ell) = \max\ f(y_1, y_2)$$

$$\text{s.t. } y_1 = \phi_1(k_1, \ell_1)$$

$$y_2 = \phi_2(k_2, \ell_2)$$

$$k_1 + k_2 = k$$

$$\ell_1 + \ell_2 = \ell$$

$$k_j \geq 0,\ \ell_j \geq 0.$$

In Ventura model $F(k, \ell) = f(k, \ell)$.

C. E. S. Model

$$y_1 = \phi_1(k_1, \ell_1) = \theta_1\left(\alpha_1 k_1^b + (1-\alpha_1)\ell_1^b\right)^{1/b}$$

$$y_2 = \phi_2(k_2, \ell_2) = \theta_2\left(\alpha_2 k_2^b + (1-\alpha_2)\ell_2^b\right)^{1/b}$$

$$f(y_1, y_2) = d\left(a_1 y_1^b + a_2 y_1^b\right)^{1/b}$$

(All elasticities of substitution are equal.)

In this case,

$$F(k,\ell) = D\left(A_1 k^b + A_2 \ell^b\right)^{1/b}$$

where

$$A_1 = \frac{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}}{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b} + \left[\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}}$$

$$A_2 = 1 - A_1$$

$$D = d\left\{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b} + \left[\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}\right\}^{\frac{1}{b}}.$$

The cone of diversification for the integrated economy has the form $\bar{\kappa}_1 k_t \geq k_t^i \geq \bar{\kappa}_2 k_t$.

$$\bar{\kappa}_i = \left(\frac{\alpha_i}{1-\alpha_i} \right)^{\frac{1}{1-b}} \frac{\left(a_1(1-\alpha_1)\theta_1^b \right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b \right)^{\frac{1}{1-b}}}{\left(a_1\alpha_1\theta_1^b \right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b \right)^{\frac{1}{1-b}}}.$$

The cone of diversification for the integrated economy has the form $\bar{\kappa}_1 k_t \geq k_t^i \geq \bar{\kappa}_2 k_t$.

$$\bar{\kappa}_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{\frac{1}{1-b}} \frac{\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}}{\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}}.$$

This is not the cone of diversification when factor prices are not equalized.

$$\kappa_1(p_2/p_1) = \left(\frac{\alpha_1}{1-\alpha_1}\right)^{\frac{1}{1-b}} \left[\frac{(1-\alpha_2)^{\frac{1}{1-b}}(\theta_2 p_2/p_1)^{\frac{b}{1-b}} - (1-\alpha_1)^{\frac{1}{1-b}}\theta_1^{\frac{b}{1-b}}}{\alpha_1^{\frac{1}{1-b}}\theta_1^{\frac{b}{1-b}} - \alpha_2^{\frac{1}{1-b}}(\theta_2 p_2/p_1)^{\frac{b}{1-b}}}\right]^{\frac{1}{b}}$$

$$\kappa_1(p_2/p_1) = \left[\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\alpha_1}{\alpha_1}\right)\right]^{\frac{1}{1-b}} \kappa_2(p_2/p_1).$$

Cobb-Douglas Model

$$y_1 = \phi_1(k_1, \ell_1) = \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1}$$

$$y_2 = \phi_2(k_2, l_2) = \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2}$$

$$f(y_1, y_2) = d y_1^{a_1} y_2^{a_2}$$

(This is the special case of the C.E.S. model where $b = 0$.)

In this case

$$F(k,\ell) = Dk^{A_1}\ell^{A_2}$$

where

$$A_1 = a_1\alpha_1 + a_2\alpha_2$$

$$A_2 = 1 - A_1$$

$$D = \frac{d\left[\theta_1 a_1 \alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\right]^{a_1}\left[\theta_2 a_2 \alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\right]^{a_2}}{A_1^{A_1} A_2^{A_2}}$$

$$\bar{\kappa}_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)\frac{A_2}{A_1}.$$

Proposition: In the Cobb-Douglas model with $\delta = 1$, suppose that factor price equalization occurs at period T. Then factor price equalization occurs at all $t \geq T$. Furthermore, the equilibrium capital stocks can be solved for as

$$k_t^i = \gamma^i k_t$$

where $\gamma^i = k_T^i / k_T$ and $k_{t+1} = \beta A_1 D k_t^{A_1}$ for $t \geq T$.

Proposition: In the C.E.S. model with $\delta = 1$, suppose that the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is weakly decreasing. Suppose that factor price equalization occurs in period T. Then there exists an equilibrium in which factor price equalization occurs at all $t \geq T$. Furthermore, this equilibrium is the only such equilibrium.

Proposition: In the C.E.S. model with $\delta = 1$, suppose that the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is strictly increasing. Again let $z_t = c_{t-1} / k_t$, $z_0 = c_0 / (\beta r_0 k_0)$, and $\hat{z} = \lim_{t\to\infty} c_{t-1} / k_t$. Let $\overline{k}_0^{i_{min}} \leq \overline{k}_0^{i} \leq \overline{k}_0^{i_{max}}$, $i = 1,\ldots,n$. If

$$\frac{\hat{z}}{z_0}\left(\frac{\overline{k}_0^{i_{min}} - \overline{k}_0}{\overline{k}_0}\right) \geq \kappa_2 - 1,\ \frac{\hat{z}}{z_0}\left(\frac{\overline{k}_0^{i_{max}} - \overline{k}_0}{\overline{k}_0}\right) \leq \kappa_1 - 1,$$

then there exists an equilibrium with factor price equalization in every period. If, however, either of these conditions is violated, there is no equilibrium with factor price equalization in every period. When there exists an equilibrium with factor price equalization in every period, it is the unique such equilibrium.

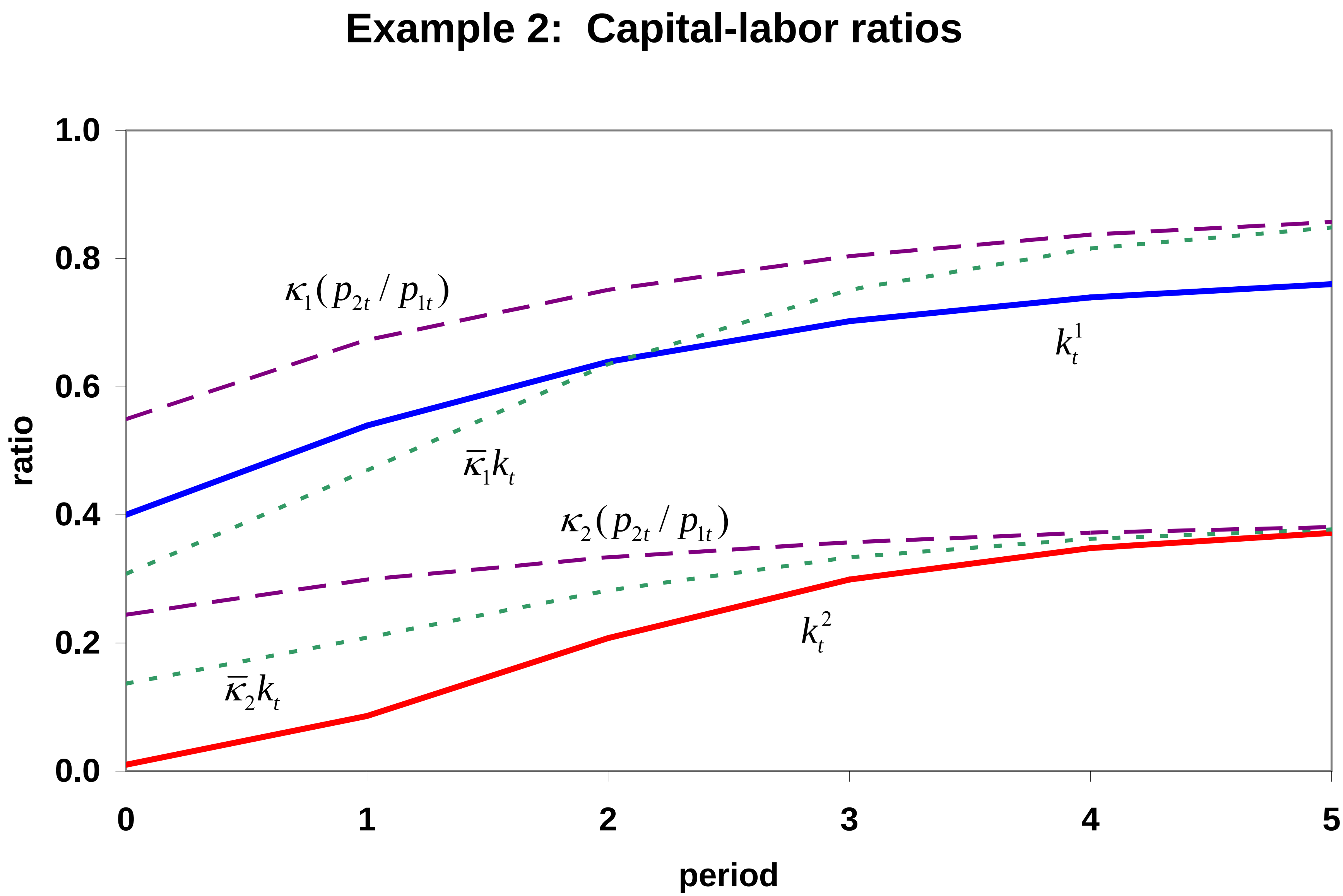
Numerical example 2: Two countries. $\beta = 0.95$, $\delta = 1$, and $L^1 = L^2 = 10$.

$$\phi_1(k,\ell) = 10k^{0.6}\ell^{0.4}$$

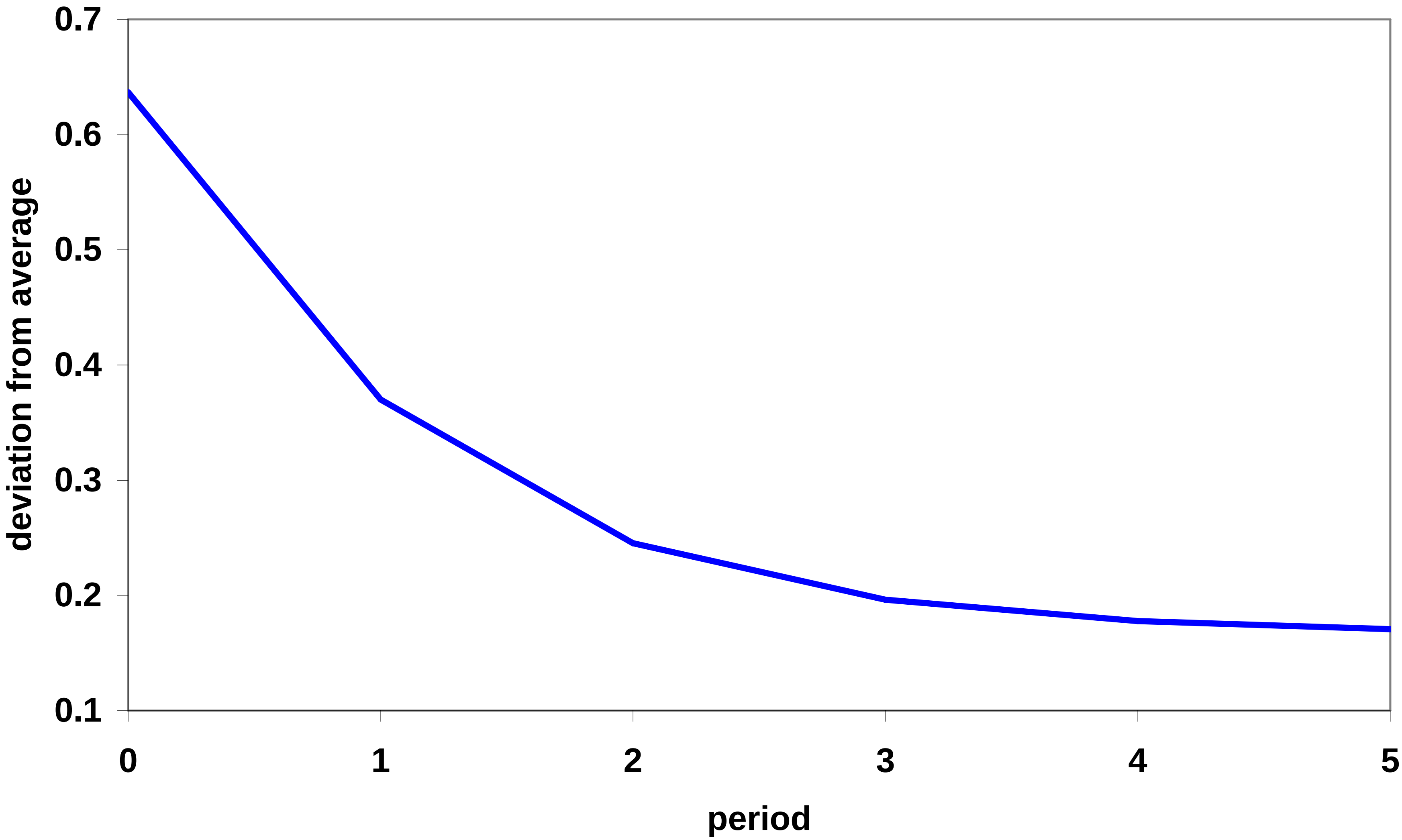
$$\phi_2(k,\ell) = 10k^{0.4}\ell^{0.6}$$

$$f(x_1, x_2) = x_1^{0.5} x_2^{0.5}$$

$\overline{k}_0^1 = 4$, $\overline{k}_0^2 = 0.1$.


Example 2: Capital-labor ratios
1.0
0.8
0.6
0.4
0.2
0.0
ratio
0
1
2
3
4
5
period
$\kappa_1(p_{2t} / p_{1t})$
k_t^1
$\bar{\kappa}_1 k_t$
$\kappa_2(p_{2t} / p_{1t})$
k_t^2
$\bar{\kappa}_2 k_t$

Example 2: Relative income in country 1



Numerical example 3: Two countries. $\beta = 0.95$, $\delta = 1$, and $L^1 = L^2 = 10$.

$$\phi_1(k,\ell) = 10\left(0.8k^{-0.5} + 0.2\ell^{-0.5}\right)^{-2}$$

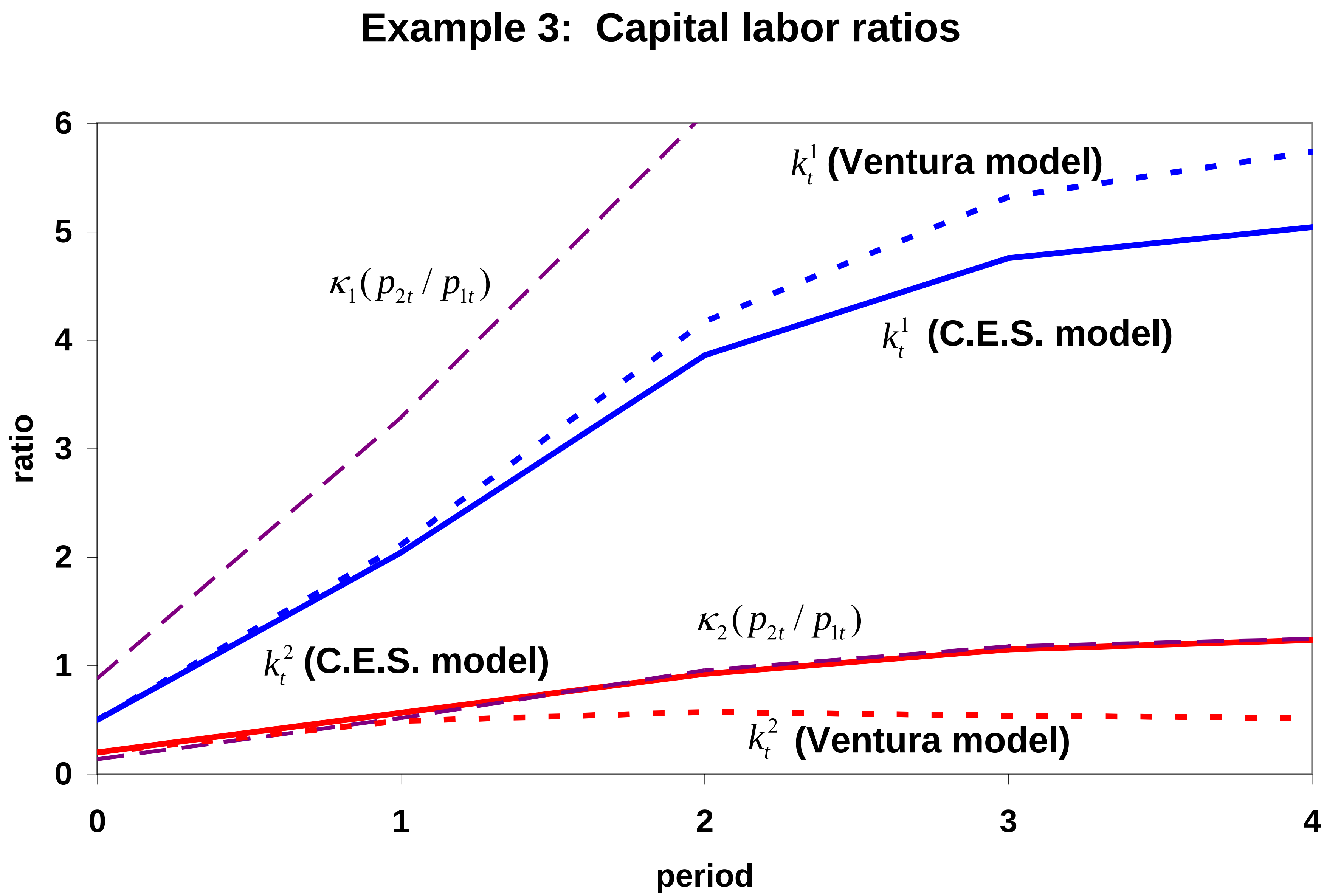
$$\phi_2(k,\ell) = 10\left(0.2k^{-0.5} + 0.8\ell^{-0.5}\right)^{-2}$$

$$f(x_1, x_2) = \left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2}$$

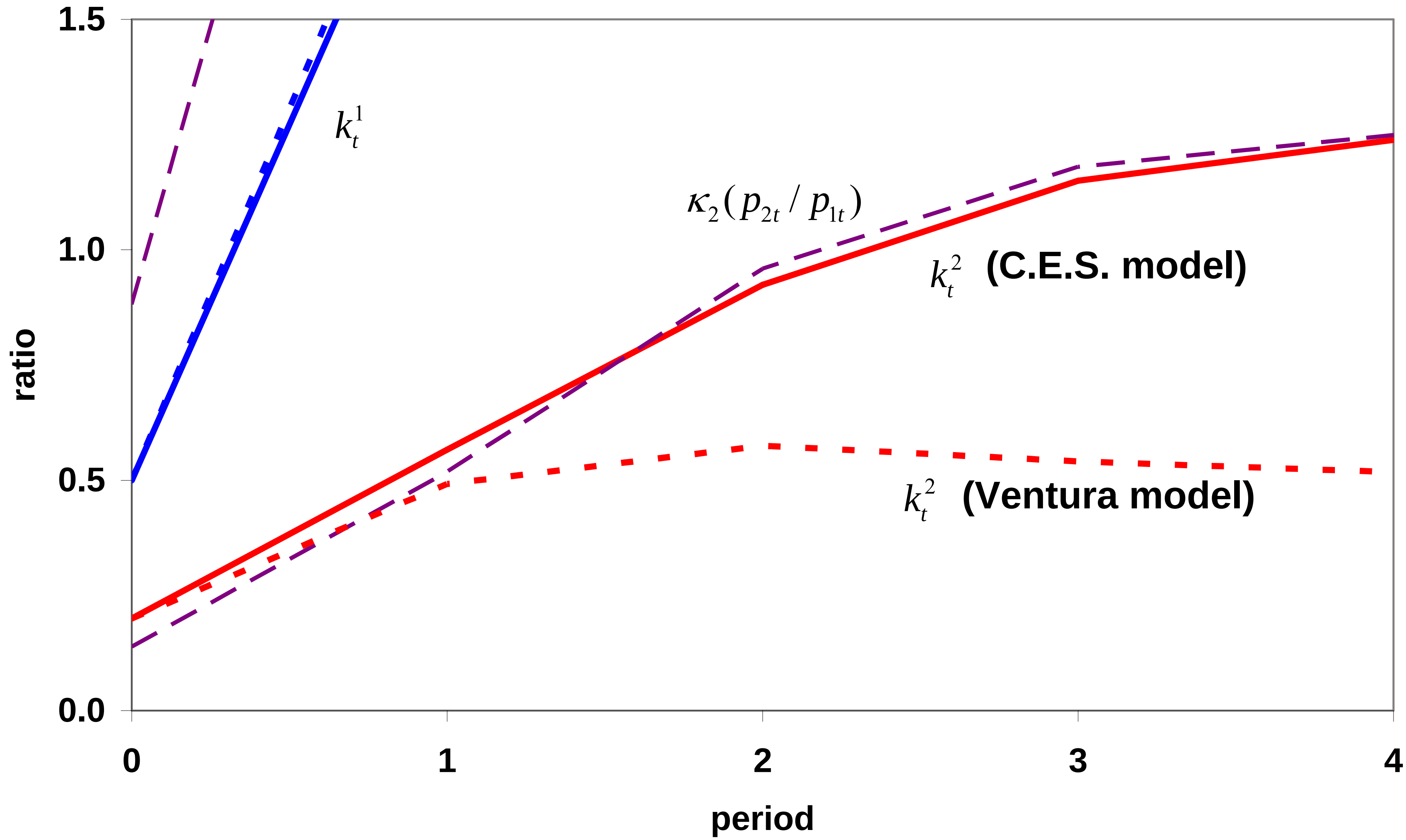
$$\bar{k}_0^1 = 5\ , \bar{k}_0^2 = 2.$$

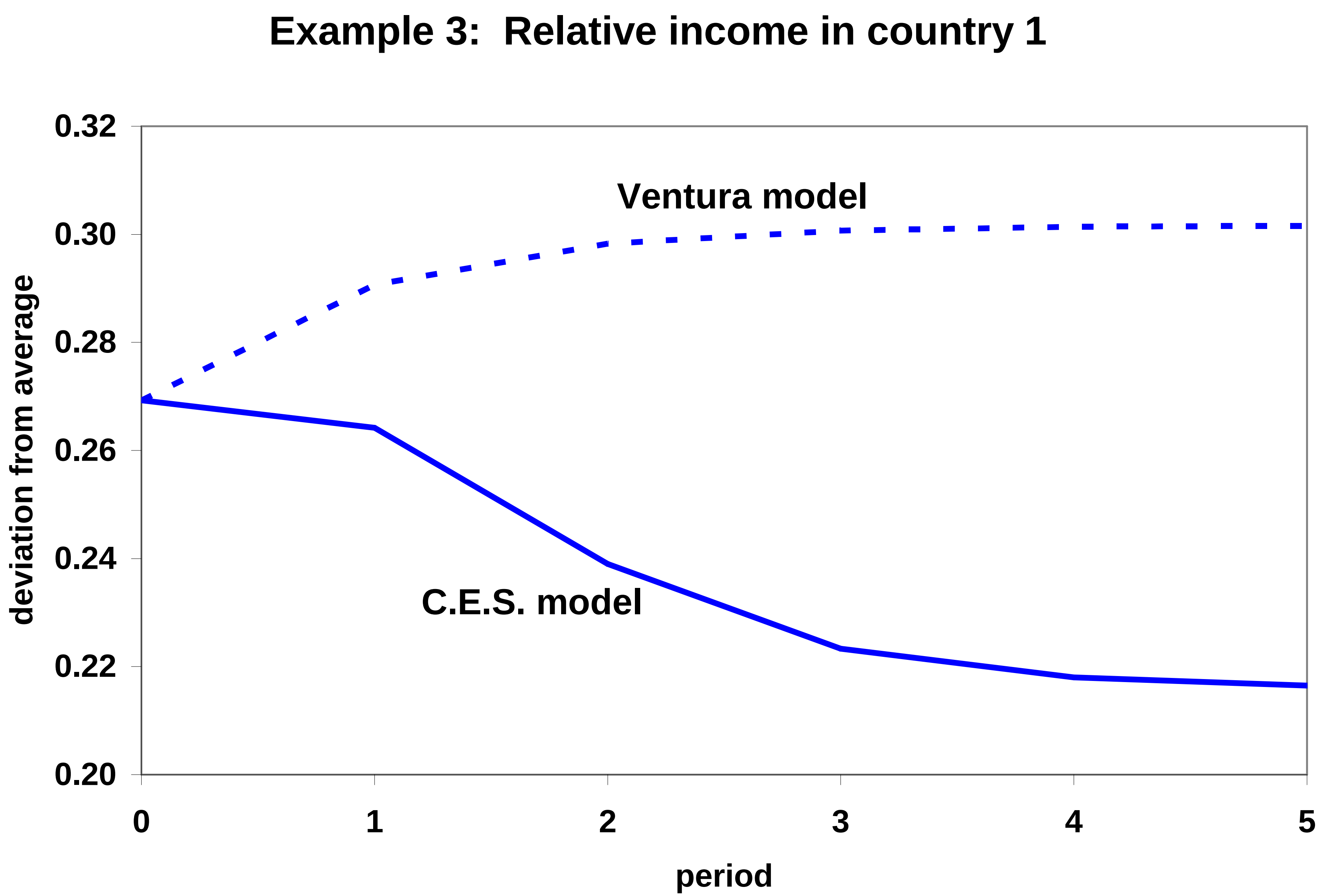
Contrast with the Ventura model with the same integrated equilibrium:

$$f(x_1, x_2) = 5.7328\left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2}.$$


Example 3: Capital labor ratios
6
5
4
3
2
1
0
ratio
0
1
2
3
4
period
$\kappa_1(p_{2t}/p_{1t})$
k_t^1 (Ventura model)
k_t^1 (C.E.S. model)
$\kappa_2(p_{2t}/p_{1t})$
k_t^2 (C.E.S. model)
k_t^2 (Ventura model)

Example 3: Capital labor ratios (detail)




Example 3: Relative income in country 1
Ventura model
C.E.S. model
deviation from average
0.32
0.30
0.28
0.26
0.24
0.22
0.20
0
1
2
3
4
5
period

Continuous-Time Ventura Model

$$\int_0^\infty e^{-\rho t} \log\left(f(c_1, c_2)\right) \, dt$$

$$y_1 = k_1$$

$$y_2 = \ell_2$$

$$\dot{k} + \delta k = x = f(x_1, x_2)$$

$$f(y_1, y_2) = \begin{cases} d\left(a_1 y_1^b + a_2 y_2^b\right)^{1/b} & \text{if } b \neq 0 \\ d x_1^{a_1} x_2^{a_2} & \text{if } b = 0 \end{cases}$$

We can find the integrated equilibrium by solving

$$\max \int_0^\infty e^{-\rho t} \log c \; dt$$

$$\text{s.t. } c + x = f(k,1) - \delta k = g(k)$$

$$\dot{k} + \delta k = x$$

$$c \geq 0, \; x \geq 0$$

$$k(0) = \overline{k}_0.$$

Ventura (1997) shows that

$$\frac{k^i(t)-k(t)}{k(t)} = \frac{c(t)/k(t)}{c(0)/k(0)}\left(\frac{k^i(0)-k(0)}{k(0)}\right) = \frac{z(t)}{z(0)}\left(\frac{k^i(0)-k(0)}{k(0)}\right)$$

and draws phase diagrams in (k, z) space to analyze convergence/divergence of k^i and k.

Notice that this is not the same as convergence/divergence of y^i and y, where

$$y^i = w + rk^i = f(y_1^i, y_2^i).$$

Instead, let us study the behavior of

$$\frac{y^i(t) - y(t)}{y(t)} = \frac{s(t)}{s(0)}\left(\frac{y^i(0) - y(0)}{y(0)}\right)$$

where

$$s(t) = \frac{r(t)c(t)}{y(t)} = \frac{(f_K(k,1) - \delta)c(t)}{f(k,1)} = \frac{g'(k)c(t)}{g(k)},$$

by analyzing phase diagrams in (k, s) space.

Here, of course,

$$g(k) = f(k,1) - \delta k$$

We use the first-order conditions

$$\frac{\dot{c}}{c} = g(k) - \rho$$

$$\frac{\dot{k}}{k} = \frac{g(k)}{k} - \frac{c}{k}$$

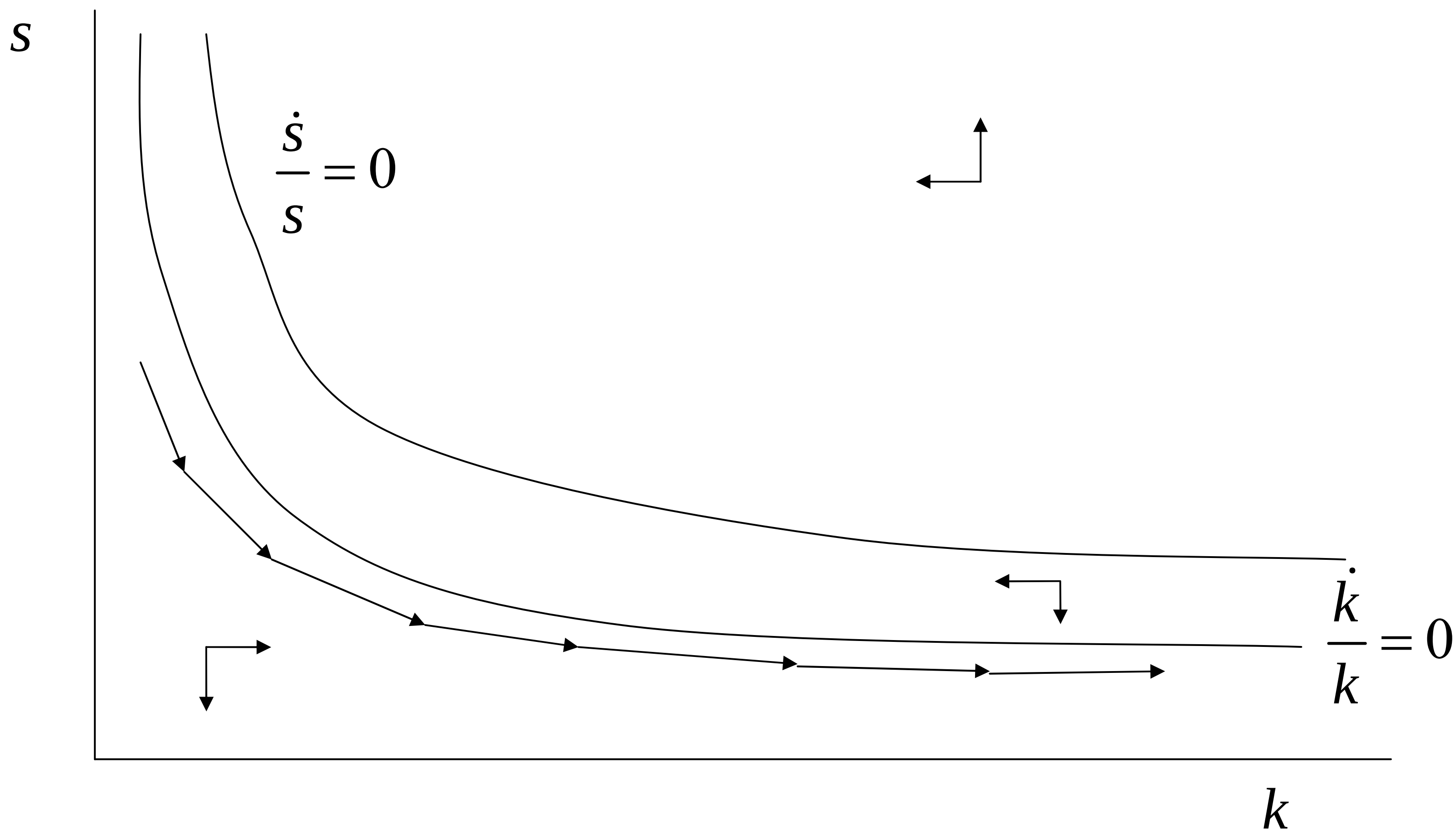
to obtain

$$\frac{\dot{s}}{s} = g'(k) - \rho - \left(\frac{g'(k)^2 - g(k)g''(k)}{g'(k)^2} \right) \left(g'(k) - s \right)$$

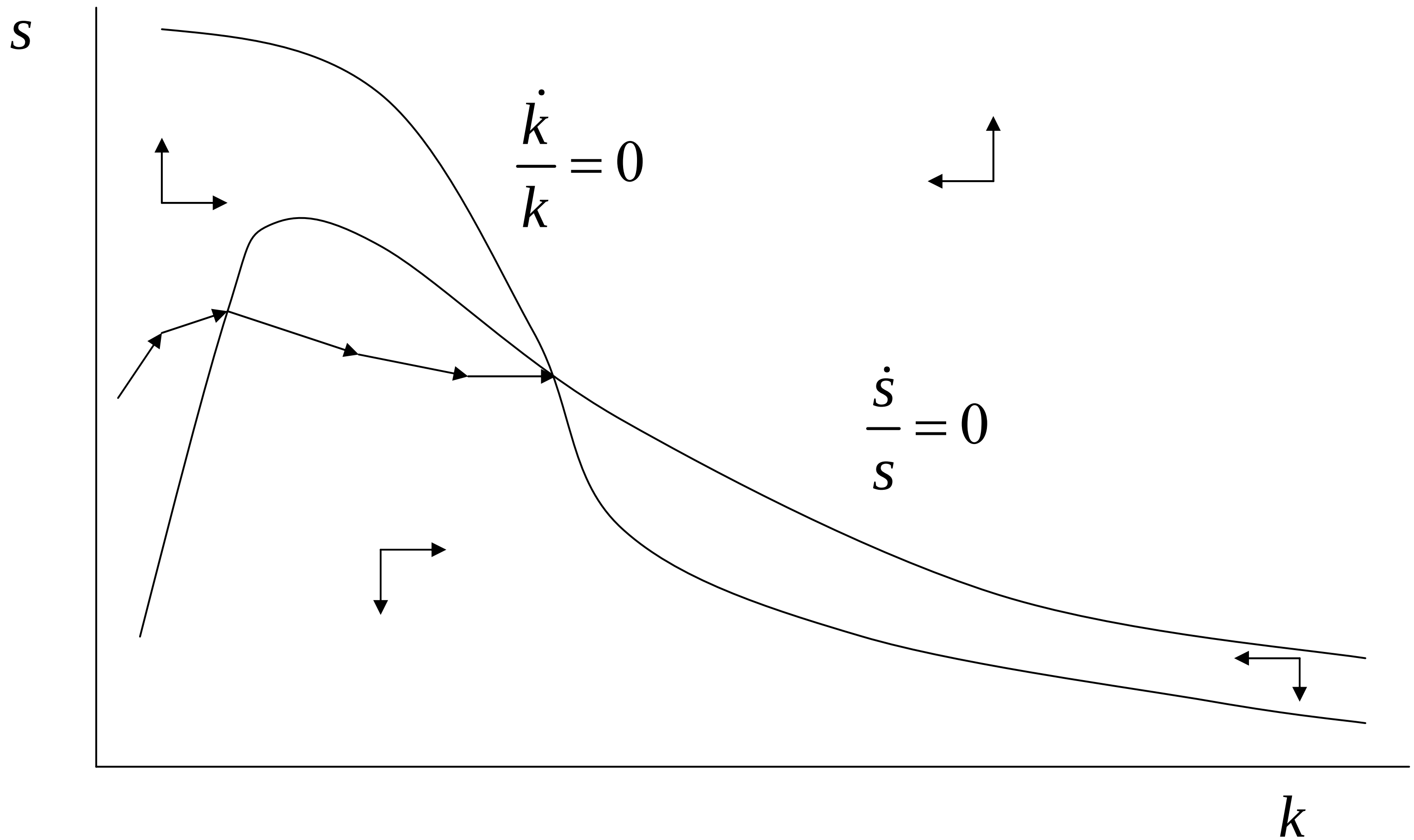
$$\frac{\dot{k}}{k} = \frac{g(k)}{g'(k)k} \left(g'(k) - s \right)$$

Notice that the same potential problems with corner solutions in investment or the capital stock arise in the continuous-time Ventura model as in the discrete-time model.

$b > 0$



$b < 0$



$b << 0$ and $\delta > 0$