1. Consider a two sector growth model in which the representative consumer has the utility function

\[ \sum_{t=0}^{\infty} \beta^t \log(c_{1t}^{0} c_{2t}^{0}). \]

Here \( 0 < \beta < 1, \ a_1 \geq 0, \ a_2 \geq 0, \) and \( a_1 + a_2 = 1. \) Investment is produced according to

\[ k_{t+1} = (1 - \delta)k_t = dx_{1t}^{a_1}x_{2t}^{a_2}. \]

Feasible consumption/investment plans satisfy

\[ (c_1 + x_1) + (c_2 + x_2) = \phi(k_1, \ell_1) = k_1 \]

\[ (c_2 + x_2) + (c_2 + x_2) = \phi(k_2, \ell_2) = \ell_1, \]

where

\[ k_1 + k_2 = k, \]

\[ \ell_1 + \ell_2 = \ell. \]

The initial value of \( k \) is \( k_0. \) \( \ell \) is equal to 1. (In other words, all variables are expressed in per capita terms.)

a) Carefully define a competitive equilibrium for this economy.

b) Reduce the equilibrium conditions for this economy to two difference equations in \( k_t \) and \( c_t, \) and a transversality condition. Here \( c_t = dc_{1t}^{a_1} c_{2t}^{a_2} \) is aggregate consumption.

c) Suppose now that there is a world made up of \( m \) different countries all with the same technologies and preferences, but different endowments, \( \bar{L}/k_0^j \) and \( \bar{L}. \) (That is, there is a measure \( \bar{L} \) of consumers, each of whom is endowed with 1 unit of labor in every period and \( k_0 \) units of capital in period 0.) Suppose that there is no international borrowing or lending. Define an equilibrium for the world economy. Prove that in this equilibrium the variables \( c_t = \sum_{j=1}^{m} \bar{L}\bar{c}_{jt} / \sum_{j=1}^{m} \bar{L}, \ k_t = \sum_{j=1}^{m} \bar{L}\bar{k}_j / \sum_{j=1}^{m} \bar{L}, \ p_t, \ r_t, \) and \( w_t \) satisfy the equilibrium conditions for the equilibrium in part a where

\[ \bar{k}_0 = \sum_{j=1}^{m} \bar{L}\bar{k}_0 / \sum_{j=1}^{m} \bar{L}. \]
d) State and prove versions of the factor price equalization theorem, the Stolper-Samuelson theorem, the Rybczynski theorem, and the Heckscher-Ohlin theorem for this particular world economy.

e) Consider the case where $\delta = 1$. Let $z_0 = c_0 / (\beta r, k_0)$ and $z_t = c_{t-1} / k_t, \ t = 1, 2, \ldots$. Transform the two difference equations from part b into two difference equations in $k_t$ and $z_t$. Show that

$$\frac{k_i^t - k_i^{t-1}}{k_i} = \frac{z_t}{z_{t-1}} \left( \frac{k_i^t - k_i^{t-1}}{k_i^{t-1}} \right) = \frac{z_t}{z_0} \left( \frac{k_i^0 - k_i^0}{k_i^0} \right).$$

f) Suppose again that $\delta = 1$. Let $s_t = c_t / y_t$ where $y_t = p_n k_t + p_{2t} = dk_t^{\alpha}$. Transform the two difference equations from part b into two difference equations in $k_t$ and $s_t$. Show that

$$\frac{y_i^t - y_i^{t-1}}{y_i} = \frac{s_t}{s_{t-1}} \left( \frac{y_i^t - y_i^{t-1}}{y_i^{t-1}} \right) = \frac{s_t}{s_0} \left( \frac{y_i^0 - y_i^0}{y_i^0} \right)$$

where $y_i^t = p_t y_i^t + p_{2t} y_{2t} = r_t k_i^t + w_t$.

g) Suppose that $\delta = 1$, but that the utility function is

$$\sum_{t=0}^{\infty} \beta^t \log(a_i c_i^b + a_{2t} c_{2t}^b)^{1/b}$$

and that the production function for investment is

$$k_{i+1} = d(a_i x_i^b + x_{2i}^b)^{1/b}.$$ 

Explain the importance of the results in parts a-g in this world.

2. Suppose again that $\delta = 1$, that $c_t = dc_{2t}^a c_{2t}^b$ and that $k_{i+1} = dx_{i+1}^a x_{2t}^a$. Now suppose that

$$c_{i+1} + x_{i+1} = \phi_1(k_{i+1}, \ell_{i+1}) = \theta_1 \ell_{i+1}^{1-a} k_{i+1}^{a_1},$$

$$c_{2t+1} + x_{2t+1} = \phi_2(k_{2t+1}, \ell_{2t+1}) = \theta_2 \ell_{2t+1}^{1-a} k_{2t}^{a_2}.$$
a) Let $F(k, \ell)$ be the maximum value of

$$\max \ d_i y_{i_1}$$

s.t. $y_1 = \theta_1 \ell_1 k_1^\alpha$,

$y_2 = \theta_2 \ell_2 k_2^\alpha$,

$k_1 + k_2 = k$,

$\ell_1 + \ell_2 = \ell$,

$k_j, \ell_j \geq 0$.

Show that $F(k, \ell)$ has the form $Dk^{\delta} \ell^{1-\delta}$.

b) Suppose now that there is a world made up of $m$ different countries all with the same technologies and preferences, but different endowments, $L_0$ and $L_1$. Suppose that there is no international borrowing or lending and there are no international capital flows. Define an equilibrium for the world economy.

c) Using the answers to parts a and b, show that necessary and sufficient conditions for the integrated equilibrium approach to work for all $t = T, T+1, \ldots$, is that

$$\kappa_i k_i \geq k_i^i \geq \kappa_2 k_i$$

for all $i = 1, \ldots, m$ and all $t = T, T+1, \ldots$.

For some $\kappa_1, \kappa_2 > 0$.

d) Suppose that, in some period $T$,

$$\kappa_i k_T \geq k_T^i \geq \kappa_2 k_T$$

for all $i = 1, \ldots, m$.

Use the answers to parts a, b, and c and the answer to part f of question 3 to calculate analytical expressions for the equilibrium values of the variables in part b for all $t = T, T+1, \ldots$. [Hint: You can show that $\kappa_i k_i \geq k_i^i \geq \kappa_2 k_i$ for all $i = 1, \ldots, m$ and all $t = T, T+1, \ldots$.]

3. Find data to calculate the bilateral real exchange rate between two countries who have a bilateral trade relation that is important to at least one of the countries. Find data on the prices of traded goods in these two countries. Calculate a decomposition of the bilateral real exchange rate of the form

$$rer_i = rer_i^T + rer_i^N,$$

where $rer_i$ is the natural logarithm of the bilateral real exchange rate and $rer_i^T$ is the logarithm of the bilateral real exchange rate for traded goods. Calculate the correlation
between \( rer \) and \( rer^N \) in levels, in 1 year differences, and in 4 year differences. Calculate ratio of the standard deviations of \( rer \) and \( rer^N \) in levels, in 1 year differences, and in 4 year differences. Calculate a variance decomposition of \( rer \) in terms of \( rer^T \) and \( rer^N \) in levels, in 1 year differences, and in 4 year differences.