Calculating $\bar{b}$:

Utility of repaying even if bankers do not lend:

$$u((1-\theta)\bar{y},\theta\bar{y}-B)+\frac{\beta u((1-\theta)\bar{y},\theta\bar{y})}{1-\beta}$$

Utility of defaulting if bankers do not lend:

$$\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}.$$

$\bar{b}$ is determined by

$$u((1-\theta)\bar{y},\theta\bar{y}-\bar{b})+\frac{\beta u((1-\theta)\bar{y},\theta\bar{y})}{1-\beta}=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}$$

$$\log((1-\theta)\bar{y}) + \gamma\log(\theta\bar{y} - \bar{b}) + \frac{\beta\log((1-\theta)\bar{y} + \beta\gamma\log(\theta\bar{y})}{1-\beta}$$

$$= \frac{\log((1-\theta)Z\bar{y}) + \beta\gamma\log(\theta Z\bar{y})}{1-\beta}$$

This has a simple analytical solution for $\bar{b}$.

It is easiest to work with a grid of debt levels $[0, \tilde{B}]$, where $\tilde{B}$ is a number large enough so that the government would always want to default if it had debt equal to $\tilde{B}$. In the example, $\tilde{B} = 150$ is large enough.

For every grid point $B \in (\overline{b}, \tilde{B}]$, we can calculate the expected utility of the government if it reduces decides the debt to $\overline{b}$ in T periods, $T = 1, 2, \ldots, 6$. First-order conditions imply that $g_t = g^T(B)$ is constant as long as $B > \overline{b}$. We can solve for $g^T(B)$:

$$g^1(B) = \theta\overline{y} - B + \beta\overline{b}$$

$$g^T(B) = \theta\overline{y} - \frac{1-\beta(1-\pi)}{1-(\beta(1-\pi))^T}\left(B - (\beta(1-\pi))^{T-1}\beta\overline{b}\right).$$

Compute $V^T(B)$:

$$V^1(B) = u((1-\theta)\bar{y}, g^1(B)) + \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{b})}{1-\beta}$$

$$V^T(B) = \frac{1-(\beta(1-\pi))^T}{1+\beta(1-\pi)} u((1-\theta)\bar{y}, g^T(B)) + \frac{1-(\beta(1-\pi))^{T-1}}{1+\beta(1-\pi)} \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} + (\beta(1-\pi))^{T-2} \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{b})}{1-\beta}$$

For every $B \in (\bar{b}, \tilde{B}]$, we can calculate the optimal number of periods $T(B)$ to run down the debt finding what T maximizes $\left[V^1(B), V^2(B), \ldots, V^6(B)\right]$. That is,

$$T(B) = \arg\max_T V^T(B)$$

$$\tilde{V}(B) = \max_T V^T(B).$$

To find $\bar{B}$, we solve

$$\max\left[V^1(\bar{B}), V^2(\bar{B}), \ldots, V^6(\bar{B})\right]$$
$$= u((1-\theta)Z\bar{y}, \theta Z\bar{y} + \beta(1-\pi)\bar{B}) + \frac{\beta u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta}.$$

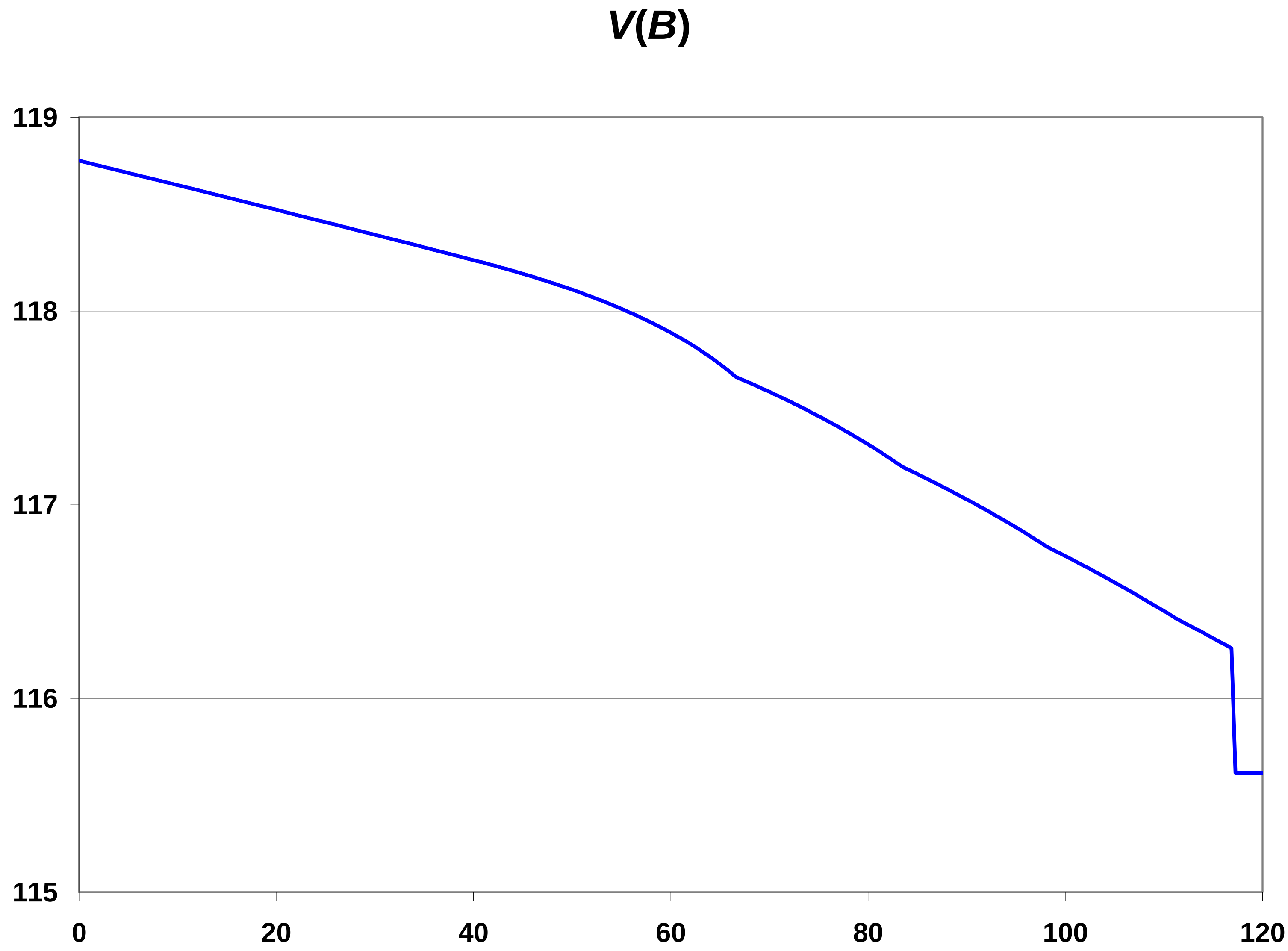
Now

$$V(B) = \begin{cases} \tilde{V}(B) & \text{if } B \le \overline{B} \\ u((1-\theta)Z\overline{y}, \theta Z\overline{y}) + \dfrac{\beta u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} & \text{if } B > \overline{B} \end{cases}.$$

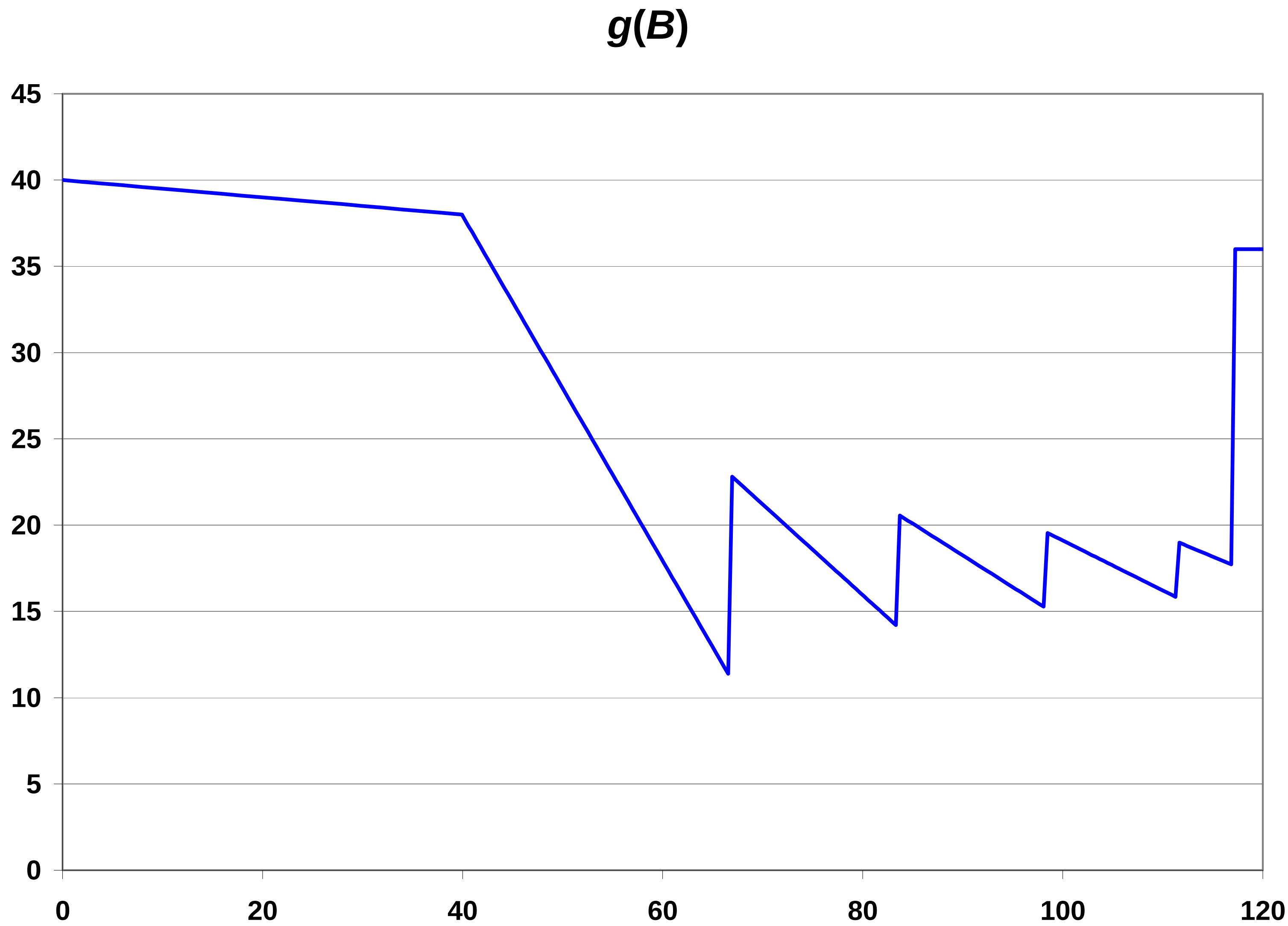
Notice that, if $B > \overline{B}$, the value function is not

$$u((1-\theta)Z\overline{y}, \theta Z\overline{y} + \beta(1-\pi)\overline{B}) + \frac{\beta u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta}$$

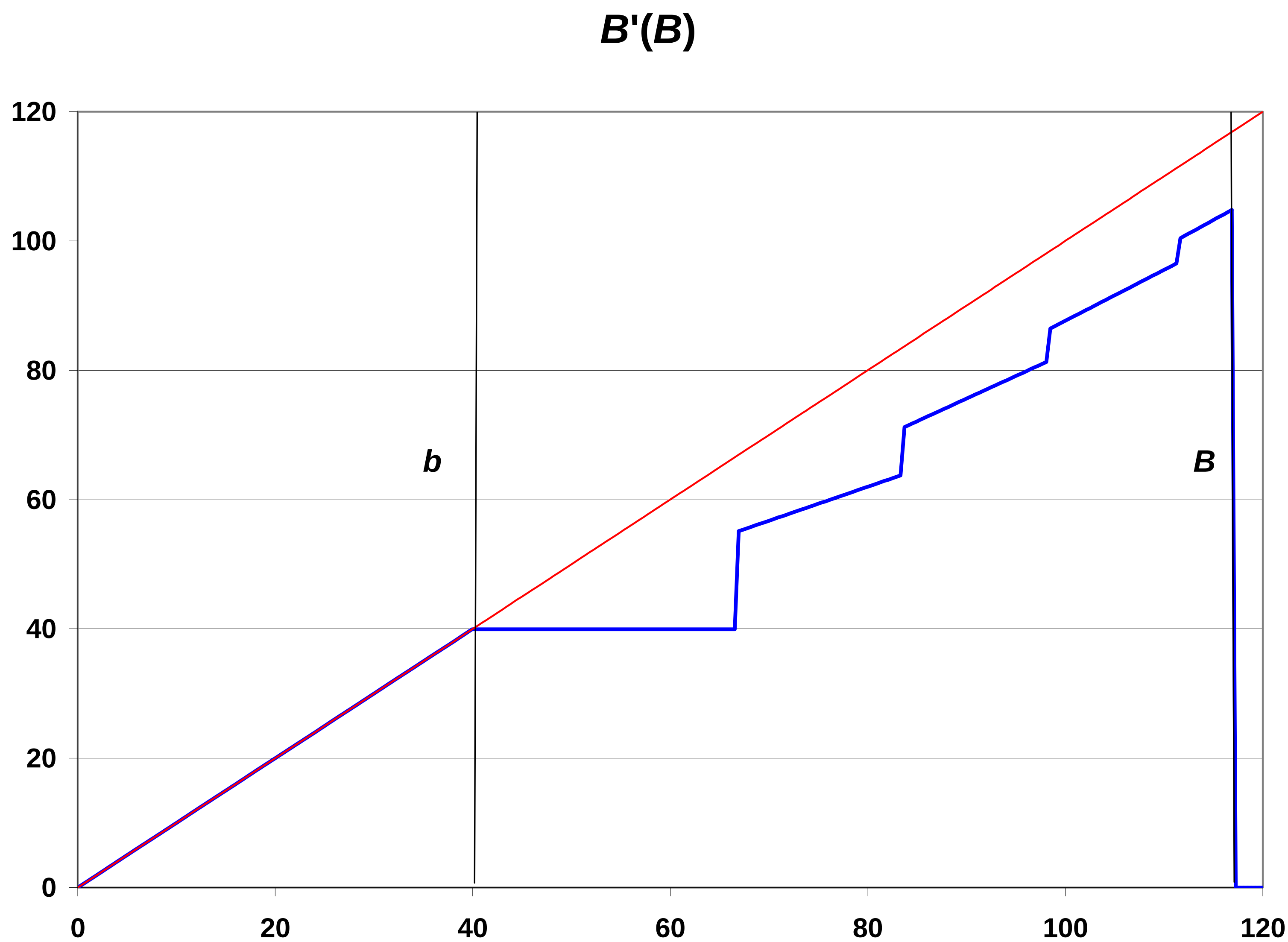
because the bankers realize that, if they lend to the government, the government would default. Therefore they do not lend. Consequently, $V(B)$ is discontinuous at $\overline{B}$.



V(B)
119
118
117
116
115
0
20
40
60
80
100
120



g(B)
45
40
35
30
25
20
15
10
5
0
0
20
40
60
80
100
120


B'(B)
120
100
80
60
40
20
0
b
B
0
20
40
60
80
100
120