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Demographics in Dynamic Heckscher-Ohlin Models: Overlapping Generations versus Infinitely Lived Consumers*

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ABSTRACT

We contrast the properties of dynamic Heckscher-Ohlin models with overlapping generations with those of models with infinitely lived consumers under both closed and open international capital markets. In both environments, if capital is mobile, factor price equalization occurs after the initial period. If capital is not mobile, the properties of equilibria differ drastically across environments: With infinitely lived consumers, factor prices equalize in any steady state or cycle and, in general, there is positive trade in any steady state or cycle. With overlapping generations, we construct examples with steady states and cycles in which factor prices are not equalized, and any equilibrium that converges to a steady state or a cycle with factor price equalization has no trade after a finite number of periods.

Keywords: International trade; Factor price equalization; Equilibrium cycle; Chaos
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1. Introduction

Traditional Heckscher-Ohlin models are static and are designed to capture changes in the reallocation of resources across sectors that happen within a country when it opens to trade. Opening to international trade has other effects that are dynamic in nature. In particular, opening may alter the pattern of capital accumulation in a country and, thus, have effects on the country's growth and development path. As economists have become more interested in the dynamic aspects of international trade, the need for models that introduce trade in a dynamic framework has arisen. One of the early options chosen in the literature was to combine a Heckscher-Ohlin model of trade with two-sector growth models. Starting with the work of Oniki and Uzawa (1965), Bardhan (1965), and Stiglitz (1970), different versions of these models, with different (and sometimes contradictory) simplifying assumptions, have been used in the literature. For instance, Oniki and Uzawa (1965) assume that countries differ in either their savings rates or their rates of time preference, which implies that factor prices are not equalized in the long run. In contrast, Chen (1992) develops his analysis under the assumption that factor prices do equalize in the long run. Dynamic Heckscher-Ohlin models have also been constructed using an overlapping generations framework. In this area, the models and assumptions are even more variable: Bianconi (1995) assumes differences in tax rates across countries; Galor and Lin (1997) assume that countries differ in their rates of time preference; and Sayan (2005) assumes differences in population growth rates, just to cite a few. In view of this diversity of assumptions and frameworks, there is the need for a unified environment (departing as little as possible from the fundamental Heckscher-Ohlin assumptions) that allows us to determine the equilibrium properties of dynamic Heckscher-Ohlin models.

In this paper, we present such a unified framework. We study the properties of dynamic Heckscher-Ohlin models with two different demographic environments: an infinitely lived consumer environment and an overlapping generations environment. In our dynamic Heckscher-Ohlin model, a finite number of countries that differ only in population sizes and initial endowments of capital (as in the traditional Heckscher-Ohlin model) interact with each other by exchanging two traded goods, which are produced using capital and labor. The traded goods are used either in consumption or in the production of a nontraded investment good. Consumers supply labor inelastically and choose levels of consumption and capital accumulation to maximize their lifetime utility.

We find that the equilibrium properties of the model depend crucially on the assumptions made on international capital markets and on the choice of demographic environment. If international borrowing and lending are permitted, factor prices equalize after the first period, independently of the environment. Furthermore, the levels of capital and of international borrowing are not determined in equilibrium. At any given point in the equilibrium path, there is a continuum of possible continuation paths that have all the same prices and aggregate variables, but differ in the distribution of capital and international borrowing across countries and in the pattern of production and trade.

If international borrowing and lending are not permitted, then the equilibrium properties vary depending on the demographic environment. In the infinitely lived consumer environment, if a nontrivial steady state exists, then there is a continuum of nontrivial steady states, indexed by the distribution of capital across countries. To which steady state the world economy converges depends on initial endowments of capital. Factor prices equalize in all steady states, and trade is positive in all steady states except the one where initial capital-labor ratios are equal across countries. In the overlapping generations environment, any nontrivial steady state with factor price equalization is autarkic and has no trade. Furthermore, we show using examples that factor price equalization does not need to occur in steady state. Therefore, the factor price equalization assumption is innocuous in long-run analyses of the model under the infinitely lived consumer framework, but not under the overlapping generations framework. Both environments also differ in the behavior of equilibrium paths that converge to a steady state. In the overlapping generations environment, any equilibrium converging to a steady state where factor prices equalize becomes autarkic in a finite number of periods. No corresponding result exists for the infinitely lived consumer environment.

As in two-sector closed economy models, equilibrium paths may exhibit cycles and chaotic behavior. In the infinitely lived consumer environment, we show that factor prices equalize in any equilibrium cycle. In the overlapping generations environment, we show using examples that there may exist equilibrium cycles in which factor prices are not equalized. The paper develops a methodology for constructing two-country, two-sector, overlapping generations models from closed one- and two-sector economies in such a way that preserves their properties in terms of multiplicity of equilibria or cyclical behavior. Therefore, the overlapping generations models can present the same richness of equilibrium patterns as in closed economy models.

Our paper also contains an interesting methodological contribution. We develop a methodology to construct examples of trade economies for the overlapping generations' framework starting from one-sector closed economy models. This methodology is quite general and can be applied to a large number of settings.

The literature on dynamic Heckscher-Ohlin models was pioneered by Oniki and Uzawa (1965), Bardhan (1965), Stiglitz (1970), and Deardorff and Hanson (1978). In their models, countries produce two traded goods — a consumption good and an investment good — using production functions that differ across sectors but not across countries. In addition to differences in endowments, these papers also assume that countries have different savings rates or rates of time preference, so that no factor price equalization occurs in steady state. In steady state, factor prices are not equalized and, thus, the steady state is independent of initial conditions. Our model differs from theirs in two crucial aspects: we consider utility-maximizing consumers, and we do not impose any modeling assumptions restricting the equilibrium behavior of factor prices. Baxter (1992) studies the long-run behavior of a dynamic Heckscher-Ohlin model where countries differ in tax policy and shows how changes in tax policy may lead to reversals in comparative advantage. Cuñat and Maffezzoli (2004a) calibrate a specific dynamic trade model and study issues of convergence in income levels across countries under the assumption that factor prices do not equalize over time.

In contrast to the early literature, more recent papers impose, by assumption or by the choice of production functions, factor price equalization along the equilibrium path. Cuñat and Maffezzoli (2004b) introduce technology shocks and study the business cycle properties of a dynamic Heckscher-Ohlin model under the assumption of factor price equalization. Chen (1992) studies the equilibrium properties of a dynamic Heckscher-Ohlin model with elastic labor supply under the assumption that factor prices equalize along the equilibrium path. Ventura (1997) adds additional structure to the model that guarantees that factor prices equalize in equilibrium, independently of initial conditions, and that rules out the possibility of cyclical and chaotic behavior. He derives results regarding convergence of income distribution across countries over time. Bajona and Kehoe (2010) study the properties of a generalized version of Ventura's model without imposing factor price equalization. They show that the convergence results of Ventura (1997) depend crucially on the factor price equalization assumption.

A related literature considers two-sector growth models with infinitely lived consumers under the small open economy assumption. In the infinitely lived consumer environment, these papers include Findlay (1970), Mussa (1978), Smith (1984), Atkeson and Kehoe (2000), Chatterjee and Shukayev (2006), and Obiols-Homs (2005).

The literature on dynamic Heckscher-Ohlin models in an overlapping generations environment is less abundant. Galor (1992) characterizes the dynamics of a two-sector, two-period-lived overlapping generations model of growth in a closed economy. Papers using two-country versions of Galor (1992) assume some difference across countries besides factor endowments, usually without justifying the reason for their choice. For example, Bianconi (1995) assumes differences in tax rates across countries; Galor and Lin (1997) and Mountford (1998) assume that countries differ in their rates of time preference; Sayan (2005) assumes differences in population growth rates. All these papers study the long-run properties of the model under the factor price equalization assumption. Papers that study the two-sector overlapping generations environment under the small open economy assumption include Serra (1991), Gokcekus and Tower (1998), Kemp and Wong (1995), and Fisher (1992).

A recent literature constructs dynamic Heckscher-Ohlin models that exhibit endogenous growth. In the infinitely lived environment, Nishimura and Shimomura (2002) and Bond, Trask and Wang (2003) derive some results regarding indeterminacy of equilibria. Guilló (1999) and Mountford (1999) introduce production externalities in the overlapping generations environment.

2. The model

There are n countries in the model, which differ in their population sizes and their initial endowments of capital. Each country can produce three goods: two traded goods — a capital intensive good and a labor intensive good — and a nontraded investment good. The technologies available to produce these goods are the same across countries. Each traded good j, $j = 1,2$, is produced using capital and labor according to the production function $y_j = \phi_j(k_j, \ell_j)$.

A.1. The functions ϕ_j are increasing, concave, continuously differentiable, and homogeneous of degree one.

We assume that ϕ_j is continuously differentiable to simplify the exposition, and we let additional subscripts — $\phi_{jK}(k_j, \ell_j)$, $\phi_{jL}(k_j, \ell_j)$ — denote partial derivatives. It is an open question whether any substantive conclusion depends on this assumption. In particular, our analysis is easily extended to the fixed coefficient production functions, $y_j = \min[k_j / a_{jK}, \ell_j / a_{jL}]$.

Producers minimize costs taking prices as given and earn zero profits:

$$r \geq p_j \phi_{jK}(k_j, \ell_j), \text{ with equality if } k_j > 0 \tag{1}$$

$$w \geq p_j \phi_{jL}(k_j, \ell_j), \text{ with equality if } \ell_j > 0 \tag{2}$$

for $j = 1,2$. Here r is the rental rate, w is the wage, and p_1 and p_2 are the prices of the traded goods.

A.2. Good 1 is relatively capital intensive and there is no capital intensity reversal. That is,

$$\frac{\phi_{1L}(k/\ell, 1)}{\phi_{1K}(k/\ell, 1)} < \frac{\phi_{2L}(k/\ell, 1)}{\phi_{2K}(k/\ell, 1)} \text{ for all } k/\ell > 0. \tag{3}$$

This condition and the concavity of ϕ_1 and ϕ_2 imply that for any wage-rental ratio w/r, the profit maximizing capital-labor ratios satisfy $k_1/\ell_1 > k_2/\ell_2$. Notice that, if the production functions ϕ_j, $j = 1,2$, display constant elasticities of substitution, assumption A.2 implies that both production functions have the same elasticity of substitution.

The investment good is produced using the two traded goods with production function $x = f(x_1, x_2)$.

A.3. The function f is increasing, concave, continuously differentiable, and homogeneous of degree one. Capital depreciates at the rate δ, $1 \geq \delta > 0$.

The first-order conditions for profit maximization are

$$p_1 \geq q f_1(x_1, x_2), \text{ with equality if } x_1 > 0 \tag{4}$$

$$p_2 \geq q f_2(x_1, x_2), \text{ with equality if } x_2 > 0, \tag{5}$$

where q is the price of the investment good.

A.4. Labor and capital are not mobile across countries, but are mobile across sectors within a country.

2.1. Infinitely lived consumers

In the environment with infinitely lived consumers, each country i, $i = 1,...,n$, has a continuum of measure L^i of consumers, each of whom is endowed with $\bar{k}_0^i > 0$ units of capital in period 0 and one unit of labor at every period, which is supplied inelastically. Consumers have the same utility functions, within countries and across countries. In each period, the representative consumer in country i decides how much to consume of each of the two traded goods in the economy, c_{1t}^i, c_{2t}^i, how much capital to accumulate for the next period, k_{t+1}^i, and how much to lend, b_{t+1}^i. Consumers derive their income from wages, w_t^i, returns to capital, r_t^i, and returns to lending, r_t^{bi}. The representative consumer in country i solves the problem

$$\max \sum\nolimits_{t=0}^{\infty} \beta^t u(c_{1t}^i, c_{2t}^i)$$

$$\text{s.t.} \quad p_{1t}c_{1t}^i + p_{2t}c_{2t}^i + q_t^i x_t^i + b_{t+1}^i \le w_t^i + r_t^i k_t^i + (1 + r_t^{bi}) b_t^i \qquad (6)$$

$$k_{t+1}^i - (1-\delta) k_t^i \le x_t^i$$

$$c_{jt}^i \ge 0,\ x_t^i \ge 0,\ b_t^i \ge -B$$

$$k_0^i \le \bar{k}_0^i,\ b_0^i \le 0.$$

Here, B is a positive number large enough so that the constraint $b_t^i \ge -B$ rules out Ponzi schemes but does not otherwise bind in equilibrium.

A.5. The period utility function $u(c_1, c_2)$ is homothetic, strictly increasing, strictly concave, and continuously differentiable, with $\lim_{c_j \to 0} u_j(c_1, c_2) = \infty$, and $\lim_{c_j \to \infty} u_j(c_1, c_2) = 0$.

The first-order conditions of this consumer's problem (6) imply that

$$\frac{u_2(c_{1t}^i, c_{2t}^i)}{u_1(c_{1t}^i, c_{2t}^i)} = \frac{p_{2t}}{p_{1t}} \qquad (7)$$

$$\frac{u_1(c_{1t}^i, c_{2t}^i)}{\beta u_1(c_{1t+1}^i, c_{2t+1}^i)} \geq \frac{p_{1t}}{p_{1t+1} q_t^i}\left(q_{t+1}^i (1-\delta) + r_{t+1}^i\right), \text{ with equality if } x_t^i > 0. \quad (8)$$

Furthermore, the returns to capital and to international bonds, if borrowing and lending are permitted, have to be the same:

$$1 + r_{t+1}^{bi} \geq \frac{q_{t+1}^i (1-\delta) + r_{t+1}^i}{q_t^i}, \text{ with equality if } x_t^i > 0. \quad (9)$$

The feasibility condition for each traded good j, $j = 1, 2$, in period t, $t = 0, 1, \ldots$, is

$$\sum_{i=1}^n L^i (c_{jt}^i + x_{jt}^i) = \sum_{i=1}^n L^i y_{jt}^i. \quad (10)$$

Here, y_{jt}^i and x_{jt}^i denote, respectively, the output and input into investment of traded good j in country i, both expressed in per worker terms. Notice that, because each consumer is endowed with one unit of labor in every period, these quantities are also the same quantities per unit of labor. It is easy to modify the model, as does Ventura (1997), so that the endowment of labor per worker differs across countries, as long as these differences remain constant over time. The feasibility conditions for factors and for the investment good are

$$k_{1t}^i + k_{2t}^i \leq k_t^i \quad (11)$$

$$\ell_{1t}^i + \ell_{2t}^i \leq 1 \quad (12)$$

$$k_{t+1}^i - (1-\delta) k_t^i \leq x_t^i. \quad (13)$$

The market clearing condition for international bonds is

$$\sum_{i=1}^n L^i b_t^i = 0 \quad (14)$$

when international borrowing and lending are permitted. If not, this condition becomes $b_t^i = 0$.

2.2. Overlapping generations

In the environment with overlapping generations, a new generation of consumers is born in each period in each country. Consumers in generation t, $t = 0, 1, \ldots$, are born in period t and live for m periods. Each of these generations in country i has a continuum of measure L^i of consumers. In period of life h, $h=1,...,m$, each consumer is endowed with $\overline{\ell}^h$ units of labor, which are supplied inelastically. Consumers can save through accumulation of capital and bonds. We assume that they are born without any initial endowment of capital or bonds. The representative consumer born in country i in period t, $t=0,1,...$, solves

$$\max \sum_{h=1}^{m} \beta_h u_h(c_{1t+h-1}^{it}, c_{2t+h-1}^{it})$$

$$\text{s.t. } p_{1t+h-1}c_{1t+h-1}^{it} + p_{2t+h-1}c_{2t+h-1}^{it} + q_{t+h-1}^{i}x_{t+h-1}^{it} + b_{t+h}^{it} \leq w_{t+h-1}^{i}\overline{\ell}^h + r_{t+h-1}^{i}k_{t+h-1}^{it} + (1+r_{t+h-1}^{bi})b_{t+h-1}^{it} \quad (15)$$

$$k_{t+h}^{it} - (1-\delta)k_{t+h-1}^{it} \leq x_{t+h-1}^{it}$$

$$c_{jt+h-1}^{it} \geq 0,\ x_{t+h-1}^{it} \geq 0$$

$$k_t^{it} \leq 0,\ b_t^{it} \leq 0,\ x_{t+m-1}^{it} \geq -(1-\delta)k_{t+m-1}^{it},\ b_{t+m}^{it} \geq 0,$$

where u_h is the utility function in period of life h and satisfies the analogue of assumption A.5:

A.5'. For every h, $h=1,...,m$, the utility function $u_h(c_1,c_2)$ is homothetic, strictly increasing, strictly concave, and continuously differentiable, with $\lim_{c_j \to 0} u_{hj}(c_1,c_2) = \infty$ and $\lim_{c_j \to \infty} u_{hj}(c_1,c_2) = 0$.

In addition, there are $m-1$ generations of initial old consumers alive in period 0. Each generation s, $s=-m+1,...,-1$, in country i has a continuum of measure L^i of consumers, each of whom lives for $m+s$ periods and is endowed with $\overline{\ell}^{h-s}$ units of labor in period h, $h=1,...,m+s$. Each initial old consumer is also endowed with capital $\overline{k}_0^{is}$ and bonds $\overline{b}_0^{is}$. The representative consumer of generation t, $t=-m+1,...,-1$, in country i solves

$$\max \sum_{h=1-t}^{m} \beta_h u_h(c_{1t+h-1}^{it}, c_{2t+h-1}^{it})$$

$$\text{s.t. } p_{1t+h-1}c_{1t+h-1}^{it} + p_{2t+h-1}c_{2t+h-1}^{it} + q_{t+h-1}^{i}x_{t+h-1}^{it} + b_{t+h}^{it} \leq w_{t+h-1}^{i}\overline{\ell}^h + (1+r_{t+h-1}^{bi})b_{t+h-1}^{it} + r_{t+h-1}^{i}k_{t+h-1}^{it} \quad (16)$$

$$k_{t+h}^{it} - (1-\delta)k_{t+h-1}^{it} \leq x_{t+h-1}^{it}$$

$$c_{jt+h-1}^{it} \geq 0,\ x_{ht+h-1}^{it} \geq 0$$

$$k_0^{it} \leq \overline{k}_0^{it},\ b_0^{it} \leq \overline{b}_0^{it},\ x_{t+m-1}^{it} \geq -(1-\delta)k_{t+m-1}^{it},\ b_{t+m}^{it} \geq 0.$$

Notice that in each country i and each period t, the total population is mL^i and — normalizing the total supply of labor per period to be 1, $\sum_{h=1}^{m} \bar{\ell}^h = 1$ — the total amount of labor is L^i.

The feasibility condition for traded good j, $j = 1,2$, in period t, $t = 0,1,...,$ is

$$\sum_{i=1}^{n} L^i \left(\sum_{h=1}^{m} c_{jt}^{it-h+1} + x_{jt}^i \right) = \sum_{i=1}^{n} L^i y_{jt}^i . \tag{17}$$

Notice that, in this notation, y_{jt}^i and x_{jt}^i are expressed in terms of per unit of labor, not in terms of per capita. The output per capita of traded good j in country i in period t is y_{jt}^i / m, for example. The feasibility conditions for factor inputs and for the investment good are the same as in the infinitely lived consumer environment and are given by equations (11), (12), and (13).

Given intergenerational heterogeneity, we need to impose additional feasibility conditions in the overlapping generations environment:

$$x_t^i = \sum_{h=1}^{m-1} x_t^{it-h+1} \tag{18}$$

$$k_t^i = \sum_{h=2}^{m} k_t^{it-h+1} . \tag{19}$$

We also need to impose a market clearing condition on bonds. If international borrowing and lending are permitted, then $r_t^{bi} = r_t^b$ and this condition is

$$\sum_{i=1}^{n} L^i \sum_{h=2}^{m} b_t^{it-h+1} = \left(\prod_{s=0}^{t} (1 + r_s^b) \right) \sum_{i=1}^{n} L^i \sum_{h=1-m}^{-1} \bar{b}_0^{ih} . \tag{20}$$

If the initial nominal assets satisfy

$$\sum_{i=1}^{n} L^i \sum_{h=1-m}^{-1} \bar{b}_0^{ih} \neq 0 , \tag{21}$$

then the world has fiat money. If, however,

$$\sum_{i=1}^{n} L^i \sum_{h=1-m}^{-1} \bar{b}_0^{ih} = 0 , \tag{22}$$

then the world does not have fiat money.

If international borrowing and lending are not permitted, then the market clearing condition on bonds within each country becomes

$$\sum_{h=2}^{m} b_t^{it-h+1} = \left(\prod_{s=0}^{t} (1 + r_s^{bi}) \right) \sum_{h=1-m}^{-1} \bar{b}_0^{ih} , \; i = 1,...,n . \tag{23}$$

If

$$\sum_{h=1-m}^{-1} \bar{b}_0^{ih} \neq 0, \tag{24}$$

then country i has fiat money, and, if

$$\sum_{h=1-m}^{-1} \bar{b}_0^{ih} = 0, \tag{25}$$

then country i does not have fiat money.

3. Equilibrium

We give unified definitions of equilibrium and of steady states for Heckscher-Ohlin models with infinitely lived consumers and with overlapping generations.

Definition 1. There are n countries of different sizes, L^i, $i = 1,...,n$, and different initial endowments of capital and bonds: $\bar{k}_0^i$ and $\bar{b}_0^i$, $i = 1,...,n$, in the environment with infinitely lived consumers and $\bar{k}_0^{is}$ and $\bar{b}_0^{is}$, $s = -m+1,...,-1$, $i = 1,...,n$, in the environment with overlapping generations. An *equilibrium* is sequences of consumptions, investments, capital stocks, and bond holdings, $\{c_{1t}^i, c_{2t}^i, x_t^i, k_t^i, b_t^i\}$ in the environment with infinitely lived consumers and $\{c_{1t}^{is}, c_{2t}^{is}, x_t^{is}, k_t^{is}, b_t^{is}\}$, $s = t-m+1,...,t$, in the environment with overlapping generations, output and inputs for each traded industry, $\{y_{jt}^i, k_{jt}^i, l_{jt}^i\}$, $j = 1,2$, output and inputs for the investment sector $\{x_t^i, x_{1t}^i, x_{2t}^i\}$, and prices $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i, r_t^{bi}\}$, $i = 1,...,n$, $t = 0,1,2...$, such that

1. Given prices $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i, r_t^{bi}\}$, the consumption and accumulation plan $\{c_{1t}^i, c_{2t}^i, x_t^i, k_t^i, b_t^i\}$ solves the consumers' problem (6) in the environment with infinitely lived consumers, and the consumption and accumulation plan $\{c_{1t}^{is}, c_{2t}^{is}, x_t^{is}, k_t^{is}, b_t^{is}\}$ solves the consumers' problems (15) and (16) in the environment with overlapping generations.

2. Given prices $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i, r_t^{bi}\}$, the production plans $\{y_{jt}^i, k_{jt}^i, l_{jt}^i\}$ and $\{x_t^i, x_{1t}^i, x_{2t}^i\}$ satisfy the cost minimization and zero profit conditions (1), (2), (4), and (5).

3. The consumption, capital stock, $\{c_{1t}^i, c_{2t}^i, x_t^i, k_t^i, b_t^i\}$ or $\{c_{1t}^{is}, c_{2t}^{is}, x_t^{is}, k_t^{is}, b_t^{is}\}$, and production plans, $\{y_{jt}^i, k_{jt}^i, l_{jt}^i\}$ and $\{x_t^i, x_{1t}^i, x_{2t}^i\}$, satisfy (10)–(14) in the infinitely lived consumer environment and

(17)–(25) in the overlapping generations environment and the feasibility conditions in each environment.

Definition 2. A *steady state* is a set of consumption levels, an investment level, a capital stock, and bond holdings, $(\hat{c}_1^i, \hat{c}_2^i, \hat{x}^i, \hat{k}^i, \hat{b}^i)$ in the environment with infinitely lived consumers and $(\hat{c}_1^{is}, \hat{c}_2^{is}, \hat{x}^{is}, \hat{k}^{is}, \hat{b}^{is})$, $s = 1,...,m$, in the environment with overlapping generations, output and inputs for each traded industry, $(\hat{y}_j^i, \hat{k}_j^i, \hat{l}_j^i)$, $j = 1,2$, output and inputs for the investment sector, $(\hat{x}^i, \hat{x}_1^i, \hat{x}_2^i)$, and prices, $(\hat{p}_1, \hat{p}_2, \hat{q}, \hat{w}^i, \hat{r}^i, \hat{r}^{bi})$, $i = 1,...,n$, that satisfy the conditions of a competitive equilibrium for appropriate initial endowments of capital and bonds, $\bar{k}_0^i = \hat{k}^i$, $\bar{b}_0^i = \hat{b}^i$ in the environment with infinitely lived consumers and $\bar{k}_0^{is} = \hat{k}^{is}$ and $\bar{b}_0^{is} = \hat{b}^{is}$ in the environment with overlapping generations. Here we set $v_t = \hat{v}$ for all t, where v represents a generic variable.

Definition 3. An equilibrium displays *sustainable growth* if there exists a constant $1 < \gamma < \infty$ such that $\liminf_{t\to\infty} k_{t+1}^i / k_t^i = \liminf_{t\to\infty} c_{t+1}^i / c_t^i = \gamma$, $i = 1,...,n$.

We often assume in the paper that the initial conditions are such that all countries produce a positive amount of the investment good in every period, $x_t^i > 0$, and we normalize $q_t^i = q_t = 1$ for all t. We make two remarks regarding this assumption: First, since the investment good is produced using the two traded goods, and these prices are equalized across countries by trade, even if a country does not produce the investment good in a given period, the price of the investment good is the same as in countries that produce the investment good. Second, in the examples presented in this paper, we assume complete depreciation, $\delta = 1$, and the assumption of positive investment becomes an assumption of positive capital, $x_t^i = k_t^i > 0$. Positive capital in all countries in every period can be ensured by assuming Inada conditions on the production functions ϕ_j. Bajona and Kehoe (2010) construct an example with corner solutions in investment and in capital in a model where production functions do not satisfy Inada conditions. In their example, the convergence properties of the equilibrium differ from the properties of equilibrium with $x_t^i > 0$ along the whole equilibrium path. In more general cases where $\delta < 1$, the assumption $x_t^i > 0$ is still restrictive.

The characterization and computation of equilibrium of the models described above is difficult in general because it involves determining the pattern of specialization in production over an infinite horizon. In particular, for any prices of the traded goods, p_{1t}, p_{2t}, there exists threshold values $\kappa_1(p_{2t}/p_{1t})$ and $\kappa_2(p_{2t}/p_{1t})$ such that a country produces positive amounts of both traded goods if and only if its capital-labor ratio k_t^i satisfies $\kappa_1(p_{2t}/p_{1t}) > k_t^i > \kappa_2(p_{2t}/p_{1t})$. The set of capital-labor ratios that satisfy weak versions of these inequalities is called the *cone of diversification*. Figure 1, known as the Lerner diagram, depicts the cone of diversification graphically. Any two countries with endowments in the cone of diversification use capital and labor in the same proportions and face the same factor prices, r_t and w_t. If the assumption of no factor intensity reversal, A.2, is violated, there can be more than one cone of diversification and more than one pair of factor prices compatible with production of both goods in equilibrium.

If all countries have endowments in the cone of diversification at some prices of the traded goods, we say that *factor price equalization* occurs at those prices. This result is the factor price equalization theorem of static Heckscher-Ohlin theory. Notice that, given the endowments of capital and labor in each country in each period, the production of traded goods is identical to that in a static, two-sector Heckscher-Ohlin model. Consequently, the Rybszynski theorem and the Stolper-Samuelson theorem also hold in our framework.

To prove a version of the Heckscher-Ohlin theorem for this model — that, in any period, a country exports the good that is intensive in the factor in which it is abundant — we would need to make assumptions to ensure that trade is balanced for each country and that expenditures on the two traded goods are proportional across countries. In the next section, we assume that there is no international borrowing and lending, which ensures that trade is balanced for each country. Assumption A.3 ensures that x_{1t}^i / x_{2t}^i is equal across countries. In the infinitely lived consumer environment, assumption A.5 ensures that c_{1t}^i / c_{2t}^i is equal across countries, but our assumptions do not impose any restrictions on $(c_{1t}^i + x_{1t}^i)/(c_{2t}^i + x_{2t}^i)$. If we assume — as does Ventura (1997) — that $u(c_1, c_2) = v(f(c_1, c_2))$, where v is strictly concave and increasing, then we know that $c_{1t}^i / c_{2t}^i = x_{1t}^i / x_{2t}^i$, which implies that $(c_{1t}^i + x_{1t}^i)/(c_{2t}^i + x_{2t}^i)$ is equal across countries and that the Heckscher-Ohlin theorem holds. Similarly, in the overlapping generations environment, we would need to strengthen assumption A.5' by assuming that $u_h(c_1, c_2) = v_h(f(c_1, c_2))$ for all h to be able to prove the Heckscher-Ohlin theorem. If we do not make this sort of assumption on the relation between consumption and investment, there is no reason to expect the Heckscher-Ohlin theorem to hold. In this case, it is easy to construct examples in which a capital abundant country exports the labor intensive good.

If factor price equalization occurs in every period, the equilibrium prices and aggregate variables of the model can be determined by solving for the equilibrium of the *integrated economy*, a closed economy with factor endowments equal to the world endowments of the factors of production. (See Dixit and Norman 1980 for a description of the methodology.) The equilibrium of the n-country model is then computed by disaggregating the consumption, production, and investment allocations of the integrated economy across countries in a way that is consistent with initial conditions. The integrated economy approach greatly simplifies the characterization of equilibrium in static models. As we will see, it is even more useful in the dynamic models considered in this paper. The question arises of how general is a situation where factor prices are equalized along the equilibrium path. The existing literature abstracts away from this question either by assuming factor price equalization along the equilibrium path — as in Chen (1992), Ventura (1997), and Cuñat and Maffezzoli (2004a) — or by not allowing for factor prices to equalize — as in Baxter (1992) and Cuñat and Maffezzoli (2004b). In this paper, we abstract out from such assumptions and derive general results regarding factor price equalization in long-run equilibria and along equilibrium paths. In doing so, we shed light on the relationship between the assumptions made in the model and the results obtained in the literature.

4. General model with international borrowing and lending

When embedding a static trade model into a dynamic framework, the first question that arises is whether international borrowing and lending should be permitted. Most papers in the literature do not allow for such capital flows. In this section, we obtain two results for models with both infinitely lived consumers and overlapping generations when international borrowing and lending are permitted. The proofs are in appendix 2.

Proposition 1: In both a model with infinitely lived consumers that satisfies A.1–A.5 and a model with overlapping generations that satisfies assumptions A.1–A.4 and A.5', assume that international borrowing and lending are permitted. Also assume that $x_t^i > 0$ for all i and all t. Then factor price equalization occurs for all $t = 1, 2, \ldots$.

Proposition 2: In both a model with infinitely lived consumers that satisfies A.1–A.5 and a model with overlapping generations that satisfies assumptions A.1–A.4 and A.5', assume that international borrowing and lending are permitted. Also assume that $x_t^i > 0$ for all i and all t. Then countries' production plans and international trade patterns are not determinate for any period $t > 0$.

The intuition for these propositions is the classic result in static Heckscher-Ohlin theory that trade in goods is a substitute for factor mobility. (See, for example, Mundell 1957 and Markusen 1983.) The assumption that $x_t^i > 0$ for all i and all t is a far stronger assumption than we need to prove proposition 2, as we explain in the proof of the proposition in the appendix.

In the rest of the paper, we study versions of the general model where international borrowing and lending are not permitted. Therefore, we add the following assumptions:

A.6. In the infinitely lived consumer environment, assume that $b_t^i = 0$ for all $t = 0, 1, \ldots$, $i = 1, \ldots, n$.

A.6'. In the overlapping generations environment, assume that
$\sum_{h=2}^{m} b_t^{it-h+1} = \left(\prod_{s=0}^{t}(1 + r_s^{bi})\right)\sum_{h=1-m}^{-1} \bar{b}_0^{ih}$ for all $t = 0, 1, \ldots$, $i = 1, \ldots, n$.

In addition, in what follows we sometimes assume that consumers aggregate the two traded goods to obtain utility in the same way that firms aggregate these goods to obtain the investment good.

Definition 4. A model with infinitely lived consumers that satisfies A.1–A.6 is *one-sector aggregatable* if $u(c_1, c_2) = v(f(c_1, c_2))$ for some v that is continuously differentiable, strictly concave, and strictly increasing. Similarly, a model with overlapping generations that satisfies A.1–A.4 and A.5'–A.6' is *one-sector aggregatable* if $u_h(c_1, c_2) = v_h(f(c_1, c_2))$ for some v_h, $h = 1, \ldots, m$, that satisfy these properties.

As we have seen, this assumption — which is very restrictive — guarantees that trade patterns obey the Heckscher-Ohlin theorem. As we shall see, it is also useful in restricting the possible dynamic behavior of equilibria.

5. Model economy with infinitely lived consumers

In this section, we study the behavior of equilibrium paths for the model with infinitely lived consumers. As mentioned in section 3, once the assumption of factor price equalization is relaxed, the integrated approach cannot be used. Nevertheless, properties of the solution can be derived if we take into account that equilibrium allocations in each country i solve a country planner's problem in a one-sector growth model with a time varying production function, following an approach similar to that used by Chen (1992). To construct this planner's problem, we begin by aggregating consumption of the two traded goods. The homotheticity assumption A.5 implies that $u(c_1,c_2) = v\left(g(c_1,c_2)\right)$, where g is strictly increasing, concave, and homogeneous of degree one, and v is strictly increasing and strictly concave. Define $c = g(c_1,c_2)$ to be an aggregate consumption good and $p(p_1,p_2)$ to be its unit cost function

$$\begin{aligned} p(p_1,p_2) = \min &\left\{ p_1c_1 + p_2c_2 \right\} \\ \text{s.t. } &\mathrm{g}(c_1,c_2) \geq 1 \\ &c_j \geq 0 . \end{aligned} \tag{26}$$

We next aggregate production of the two traded goods by defining the *revenue function*

$$\begin{aligned} \pi(p_1,p_2,k^i) = \max &\left\{ p_1\phi_1(k_1^i,\ell_1^i) + p_2\phi_2(k_2^i,\ell_2^i) \right\} \\ \text{s.t. } &k_1^i + k_2^i \leq k^i \\ &\ell_1^i + \ell_2^i \leq 1 \\ &k_j^i \geq 0 ,\ \ell_j^i \geq 0 . \end{aligned} \tag{27}$$

This revenue function indicates, for any given prices of the two traded goods, the maximum income that a country can obtain by allocating capital and labor over the production of the traded goods. (See, for example, Dixit and Norman 1980, who refer to this function as the revenue function; many other authors refer to it as the GDP function.) Figure 2 shows how the revenue function is constructed. As is seen in the figure, this function is strictly increasing, concave, but not strictly concave, and continuously differentiable but not twice continuously differentiable.

Using the first-order conditions for the revenue maximization problem (27), we obtain a characterization of the relationship between factor endowments, factor prices, and production patterns, which does not depend on the assumptions on the demographic framework. We state this relationship here, which is standard in Heckscher-Ohlin theory, as it is used several times in the paper.

Lemma 1. The optimal capital-labor ratios in interior solutions to the revenue maximization problem depend only on relative prices:

$$\frac{k_j^i}{\ell_j^i} = \kappa_j(p_2 / p_1), \; j = 1,2 . \tag{28}$$

If $\kappa_1(p_2 / p_1) \geq k^i \geq \kappa_2(p_2 / p_1)$, then factor prices only depend on goods prices,

$$r(p_1, p_2) = p_1\phi_{1K}(\kappa_1(p_2 / p_1),1) = p_2\phi_{2K}(\kappa_2(p_2 / p_1),1) \tag{29}$$

$$w(p_1, p_2) = p_1\phi_{1L}(\kappa_1(p_2 / p_1),1) = p_2\phi_{2L}(\kappa_2(p_2 / p_1),1) . \tag{30}$$

If $k^i > \kappa_1(p_2 / p_1)$, then country i produces only good 1, $r^i(p_1, p_2, k^i) = p_1\phi_{1K}(k^i,1) < r(p_1, p_2)$, and $w^i(p_1, p_2, k^i) = p_1\phi_{1L}(k^i,1) > w(p_1, p_2)$. If $k^i < \kappa_2(p_2 / p_1)$, then country i produces only good 2, $r^i(p_1, p_2, k^i) = p_2\phi_{2K}(k^i,1) > r(p_1, p_2)$, and $w^i(p_1, p_2, k^i) = p_2\phi_{2L}(k^i,1) < w(p_1, p_2)$.

Finally, given a sequence of prices $p^t = \left((p_{1t}, p_{2t}),(p_{1t+1}, p_{2t+1}),...\right)$ and an initial endowment of capital k_0^i, country *i*'s social planner takes the sequence of prices as given and solves

$$\begin{gathered} V(k_0^i; p^0) = \max \sum\nolimits_{t=0}^{\infty} \beta^t v(c_t^i) \\ \text{s.t. } p(p_{1t}, p_{2t})c_t^i + x_t^i \leq \pi(p_{1t}, p_{2t}, k_t^i) \\ k_{t+1}^i - (1-\delta)k_t^i \leq x_t^i \\ c_t^i \geq 0, \; x_t^i \geq 0 \\ k_0^i \text{ given.} \end{gathered} \tag{31}$$

Notice that this problem is like that of a planner in a one-sector model, except that the analogue of the production function, $\pi(p_{1t}, p_{2t}, k_t^i)$, changes every period as prices change, and consumption is weighted by the price index $p(p_{1t}, p_{2t})$. The next lemma relates the equilibrium of the model economy and the solution to the social planner's problem in (31).

Lemma 2: In a model with infinitely lived consumers that satisfies A.1–A.6, let the sequence $\{c_{1t}^i, c_{1t}^i, k_t^i, x_t^i, p_{1t}, p_{2t}\}$ be the equilibrium consumption, investment, and capital stock in country i and the equilibrium prices for the traded goods. Then, for any equilibrium prices $p^t = ((p_{1t}, p_{2t}), (p_{1t+1}, p_{2t+1}), \ldots)$, the country's value function $V(k; p^t)$ is continuous, strictly increasing, and strictly concave in k for all $k > 0$. Furthermore, for each $i = 1, \ldots, n$, the sequence $\{c_t^i, x_t^i, k_t^i\}$, where $c_t^i = g(c_{1t}^i, c_{2t}^i)$, solves the country planner's problem (31) in which the prices are the equilibrium prices and the initial capital stock is k_0^i.

Lemma 2 says that the country value function $V(k; p^0)$ completely summarizes the situation of a country i. We have not imposed conditions on ϕ_1, ϕ_2, f, and u to ensure that an equilibrium of the world economy exists. It may be that the economy is so productive that the representative consumer in some country can attain infinite utility, or it may be that the economy is so unproductive that consumption in some country converges so quickly to 0 that the consumer can attain no more than utility minus infinity. What lemma 2 says is that, if an equilibrium exists, $-\infty < V(k; p^0) < +\infty$, not just for the initial endowments $\bar{k}_0^i$, $i = 1, \ldots, n$, but for all $k > 0$, and that $V(k; p^t)$ has the characteristics of a dynamic programming value function.

The country social planner's problem and associated value function $V(k; p^t)$ are even closer to those of a planner in a one-sector growth model when the model is one-sector aggregatable. In this case, $p(p_{1t}, p_{2t}) = 1$ for any possible p_{1t} and p_{2t}. Notice that solving the set of *n* planner's problems and the fixed point problem is not an easy task. Theoretically, it is a convenient way to write the problem, though, since it highlights the similarities between the problems faced by each country. In particular, notice that the n countries' social planner's problems — one for each country — have the same sequence of prices and differ only in the initial endowments of capital, $\bar{k}_0^i$. Therefore, comparing equilibrium allocations of capital across countries is equivalent to doing comparative statics with respect to $\bar{k}_0^i$ on the planner's problem (31).

Proposition 3. In a model with infinitely lived consumers that satisfies assumptions A.1–A.6, let $\{k_t^i\}$, $\{k_t^{i'}\}$, $t = 0,1,...$, be the equilibrium capital stocks for two countries i and i'. Assume that $\bar{k}_0^i > \bar{k}_0^{i'}$. Then $k_t^i \geq k_t^{i'}$ for all t. Furthermore, if $x_t^i > 0$, then $k_t^i > k_t^{i'}$ implies that $k_{t+1}^i > k_{t+1}^{i'}$.

The proof of proposition 3 applies a monotonicity argument similar to those of Milgrom and Shannon (1994) to the country social planner's problem, and it is given in the appendix. Proposition 3 is powerful. It states that, under very general conditions, a country's ordering with respect to relative capital abundance does not change along the equilibrium path. Therefore, if the Heckscher-Ohlin theorem holds, the patterns of trade cannot reverse along the equilibrium path.

In the rest of the section, we restrict our analysis to specific types of equilibria.

5.1. Steady states

In this section, we derive properties of steady state equilibria for the model with infinitely lived consumers. These properties are standard in the literature on dynamic Heckscher-Ohlin models where countries only differ in their initial factor endowments. Chen (1992), Baxter (1992), and Bond, Trask, and Wang (2003) derive similar results in similar environments.

Definition 5. A *nontrivial steady state* is a steady state in which aggregate capital is positive, $\hat{k} = \sum_{i=1}^{n} L^i \hat{k}^i / \sum_{i=1}^{n} L^i > 0$, that is, $\hat{k}^i > 0$, for some $i = 1,...,n$.

Bajona and Kehoe (2010) construct an example in which $\phi_2(k,\ell) = \ell$ and in which one country has 0 capital in the steady state but the other country has positive capital.

Proposition 4: In a model with infinitely lived consumers that satisfies A.1–A.6, there is factor price equalization in any nontrivial steady state.

Proposition 5: In a model with infinitely lived consumers that satisfies A.1–A.6, if there exists a nontrivial steady state, there exists a continuum of them. These steady states have the same prices and the same aggregate capital-labor ratio, $\hat{k}$. The steady states are parameterized by the distribution of capital per worker across countries, $\hat{k}^1,...,\hat{k}^n$. Furthermore, international trade occurs in every steady state in which $\hat{k}^i \neq \hat{k}$ for some $i = 1,...,n$.

Consider an economy that has an equilibrium converging to a steady state. Proposition 5 implies that the long-run equilibrium cannot be determined independently of initial factor endowments. This is probably the reason why most existing papers introduce additional cross-country differences (like different rates of time preference or different tax rates across countries), which break down factor price equalization in steady state. We illustrate the dependence of the steady state distribution of capital on its initial distribution with an example.

Example 1. Consider a discrete-time version of the model studied by Ventura (1997). There are n countries. The production functions for the traded goods each use one factor of production; $\phi_1(k_1, \ell_1) = k_1$, and $\phi_2(k_2, \ell_2) = \ell_2$. Assume that the model is one-sector aggregatable with a Cobb-Douglas investment-consumption function, $f(x_1, x_2) = dx_1^a x_2^{1-a}$. Our technology assumptions ensure that factor price equalization always holds. Factor prices satisfy $r_t^i = r_t = p_{1t}$ and $w_t^i = w_t = p_{2t}$. Furthermore, as long as there are no corner solutions in investment, we can use the integrated approach to solve for equilibrium. To find the equilibrium of the integrated economy, we solve the social planner's problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t \log c_t \\ &\text{s.t. } c_t + k_{t+1} \leq dk_t^a \\ &c_t \geq 0,\ k_t \geq 0 \\ &k_0 \leq \bar{k}_0 . \end{aligned} \tag{32}$$

The textbook solution to this problem, first obtained by Brock and Mirman (1972), is

$$k_{t+1} = \beta a d k_t^a = (\beta a d)^{(1-a^{t+1})/(1-a)} \bar{k}_0^{a^{t+1}},\ c_t = (1-\beta a) dk_t^a = (1-\beta a) d (\beta a d)^{(1-a^t)/(1-a)} \bar{k}_0^{a^t} . \tag{33}$$

Using the first-order conditions, the feasibility conditions, and the solution (33), we obtain

$$c_{1t} = (1-\beta a) k_t,\ c_{2t} = 1-\beta a \tag{34}$$

$$x_{1t} = \beta a k_t,\ x_{2t} = \beta a \tag{35}$$

$$p_{1t} = a d k_t^{a-1},\ p_{2t} = (1-a) d k_t^{a}. \tag{36}$$

To disaggregate across countries, we start by comparing the first-order conditions for the country social planner's problem (31) with those for the integrated economy equilibrium. (It is here that the assumption of no corner solutions in investment is important.)

$$\frac{c_{t+1}^i}{c_t^i} = \beta p_{1t+1} = \beta r_{t+1} = \frac{c_{t+1}}{c_t}. \tag{37}$$

Using (37) and the problem's budget constraint, we can write the demand of the consumer in country i in period t as

$$c_t^i = (1-\beta)\left[\sum\nolimits_{s=t}^{\infty}\left(\prod\nolimits_{\tau=t+1}^{s}\frac{1}{p_{1\tau}}\right)p_{2s} + p_{1\tau}k_t^i\right]. \tag{38}$$

Subtracting the analogous condition for the integrated economy, we obtain

$$c_t^i - c_t = (1-\beta)p_{1t}(k_t^i - k_t). \tag{39}$$

The budget constraints for country i and for the integrated economy imply that

$$c_t^i - c_t + k_{t+1}^i - k_{t+1} = (1 + r_t - \delta)(k_t^i - k_t). \tag{40}$$

We can combine (37), (39), and (40) to obtain

$$k_{t+1}^i - k_{t+1} = \frac{c_t}{c_{t-1}}(k_t^i - k_t). \tag{41}$$

Setting $z_t = c_{t-1}/k_t = (1-\beta a)/(\beta a)$, $t = 1,2,\ldots$, and $z_0 = c_0/(\beta r_0 k_0) = (1-\beta a)/(\beta a)$, we obtain

$$\frac{k_{t+1}^i - k_{t+1}}{k_{t+1}} = \frac{z_{t+1}}{z_t}\left(\frac{k_t^i - k_t}{k_t}\right) = \frac{z_{t+1}}{z_0}\left(\frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0}\right) = \frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0}, \tag{42}$$

which implies that k_t^i / k_t is constant. Letting $\gamma^i = \bar{k}_0^i / \bar{k}_0$, we can solve for c_0^i / c_0 to produce

$$c_{1t}^i = \left(1 - a + \gamma^i a - \gamma^i \beta a\right)k_t,\ c_{2t}^i = 1 - a + \gamma^i a - \gamma^i \beta a \tag{43}$$

$$x_{1t}^i = \gamma^i \beta a k_t,\ x_{2t}^i = \gamma^i \beta a. \tag{44}$$

Comparing levels of income per capita, measured in current prices, across countries, we obtain

$$\frac{y_t^i}{y_t} = \frac{p_{1t}k_t^i + p_{2t}}{p_{1t}k_t + p_{2t}} = \frac{\gamma^i adk_t + (1-a)dk_t}{adk_t + (1-a)dk_t} = 1 - a + \gamma^i a \,. \tag{45}$$

We now see the strong consequences of proposition 3. In a world of closed countries, we expect every country to converge to the same steady state capital-labor ratio and level of income per capita,

$$\hat{k} = \left(\beta ad\right)^{1/(1-a)},\ \hat{y} = d\left(\beta ad\right)^{a/(1-a)}, \tag{46}$$

no matter what its initial endowment of capital. In a world of countries open to trade, however, differences in initial endowments of capital lead to persistently different capital stocks and income levels. In this example, in fact, differences stay proportionally fixed. As the world economy converges to its steady state, each country converges to a steady state that depends on its initial endowment of capital relative to the world average, $\gamma^i = \bar{k}_0^i / \bar{k}_0$,

$$\hat{k}^i = \gamma^i \left(\beta ad\right)^{1/(1-a)},\ \hat{y}^i = \left(1 - a + \gamma^i a\right) d\left(\beta ad\right)^{a/(1-a)}. \tag{47}$$

5.2. Sustained growth paths

Equilibria in both one- and two-sector closed economy growth models can exhibit sustained growth. (See, for example, Rebelo, 1991, for conditions under which this may occur.) Since our model generalizes these closed economy models to a world with trade, sustained growth is also possible here. The next proposition extends the results in the previous section to the limiting behavior of equilibria that exhibit sustained growth.

Proposition 6: In a model with infinitely lived consumers that satisfies A.1–A.6, assume that there exists an equilibrium in which $\lim_{t\to\infty} k_{t+1}^i / k_t^i = \lim_{t\to\infty} c_{t+1}^i / c_t^i = \gamma^i$ for $0 < \gamma^i < \infty$ for all i and $\gamma^i > 1$ for some i. Then $\gamma^i = \gamma$ for all i. In this equilibrium with sustained growth, factor prices are equalized in the limit. Furthermore, if there exists a sustained growth path, there is a continuum of them, all of which have the same prices and aggregate capital-labor ratio, k_t, but differ in the initial allocation of capital per worker, $\bar{k}_0^i$, and the limiting distribution of capital across countries,

$\lim_{t\to\infty} \; k_t^i / k_t$. International trade occurs in the limit of any equilibrium with sustained growth in which $\lim_{t\to\infty} \; k_t^i / k_t \neq 1$ for some $i = 1,...,n$.

A limitation of proposition 6 is worth noting. Although sustained growth is defined in terms of the infimum limit of k_{t+1}^i / k_t^i and c_{t+1}^i / c_t^i, the proposition, which characterizes the limiting behavior of equilibria, requires that the limits of these variables exist. On the other hand, rather than assuming that these limits are equal across countries, the proposition proves that they are equal. Also worth noting is that the proposition does not rule out the possibility that $\lim_{t\to\infty} w_t^i = \infty$. If $\lim_{t\to\infty} w_t^i = \infty$ for some i, however, the proposition proves that it does so for all i . Furthermore, it is easy to show that, even if $\lim_{t\to\infty} w_t^i = \infty$,

$$\lim_{t\to\infty} \frac{w_t^i}{p_{1t} y_{1t}^i + p_{2t} y_{2t}^i} = 0 . \qquad (48)$$

Consequently, even if the wage grows without bound, it does so slowly enough that, in the limit, the economy behaves like an economy with no labor. (See Bajona and Kehoe 2010.)

The next example shows that the limiting distribution of capital in an equilibrium with sustained growth depends on the initial distribution, just as example 1 does in the case of an equilibrium that converges to a steady state.

Example 2. Consider a world economy identical to that in example 1 except that the production function for consumption and investment is of the general CES form

$$g(c_1, c_2) = f(c_1, c_2) = d\left(ac_1^b + (1-a)c_2^b\right)^{1/b} . \qquad (49)$$

Assume that $b > 0$ and $\beta a^{1/b} d > 1$. To find the equilibrium of the integrated economy, we solve the social planner's problem:

$$\begin{aligned} &\max \textstyle\sum_{t=0}^{\infty} \beta^t \log c_t \\ &\text{s.t. } c_t + k_{t+1} \leq d\left(ak_t^b + (1-a)\right)^{1/b} \qquad (50) \\ &c_t \geq 0 ,\; k_t \geq 0 \\ &k_0 \leq \bar{k}_0 . \end{aligned}$$

The equilibrium path for this problem exhibits sustained growth. Bajona and Kehoe (2010) show that along this path $z_t = c_{t-1} / k_t$ decreases and that

$$\hat{z} = \lim_{t \to \infty} \frac{c_t}{k_{t+1}} = \frac{1-\beta}{\beta}. \tag{51}$$

The analogue of equation (42) holds:

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_0} \left(\frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0} \right), \tag{52}$$

although z_t / z_0 is not equal to 1 as it is in example 1. The limiting distribution of capital is determined by the equation

$$\frac{\hat{k}^i - \hat{k}}{\hat{k}} = \frac{\hat{z}}{z_0} \left(\frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0} \right). \tag{53}$$

5.3. Cycles and chaos

Equilibria in two-sector closed economy growth models do not need to converge to a steady state or to a sustained growth path. Instead, the equilibrium may exhibit cycles or complex dynamics. General conditions for the existence of two-period cycles in two-sector growth models are presented by Benhabib and Nishimura (1985), and conditions for chaos are presented by Deneckere and Pelikan (1986), Boldrin and Montrucchio (1986), and Boldrin (1989), among others. In what follows, we present a specific example that has complex dynamics based on the two-sector closed economy model developed by Boldrin and Deneckere (1990).

Example 3. Consider a world with two countries, each of which has a measure one of consumers. Consumers have the period utility function

$$u(c_1, c_2) = c_1^{\alpha} c_2^{1-\alpha}, \tag{54}$$

where $\alpha = 0.03$. The production function of the investment good uses the traded goods in fixed proportions:

$$f(x_1, x_2) = \min \left[x_1, x_2 / \gamma \right], \tag{55}$$

where $\gamma = 0.09$. The production for the traded goods is such that each of the traded goods uses only one factor of production as in example 1. Furthermore, $\delta = 1$.

Boldrin and Deneckere (1990) show that such an economy exhibits stable two-period cycles for $\beta \in [0.093, 0.095]$ and chaos for $\beta \in [0.099, 0.112]$. If $\bar{k}_0^1 = \bar{k}_0^2$, then the equilibrium of the two-country economy coincides exactly with the equilibrium of the closed economy. If $\bar{k}_0^1 \neq \bar{k}_0^2$, if one country finds it optimal to increase its capital stock, so does the other country. Therefore, capital-labor ratios in both countries cycle in the same direction, mimicking the oscillations of the integrated equilibrium. Chen (1992) makes a similar argument for a different model.

Notice that in this example, u is not strictly concave and f is not continuously differentiable. Given that the property of having cycles or chaos is structurally stable, however, it would be easy to perturb the utility and investment functions to construct examples that satisfy assumptions A.3 and A.5 and that have equilibria with cycles or chaos.

The methodology used in this example is general and allows the construction of a two-country trade model starting from any closed economy model with a consumption sector and an investment sector, such as the model developed by Uzawa (1964). Let $g(k,\ell)$ be the production function for the consumption good, $f(k,\ell)$ be the production function for the investment good, and $v(c)$ be the utility function in the two-sector closed economy model. We set $\phi_1(k,\ell) = k$, $\phi_2(k,\ell) = \ell$, and $u(c_1, c_2) = v\left(g(c_1, c_2)\right)$.

In the cycle in example 3, factor prices are equalized in every period. In fact, this is the only sort of equilibrium cycle that is possible, at least if investment is positive.

Proposition 7: In a model with infinitely lived consumers that satisfies A.1–A.6, assume that there exists an equilibrium *s*-period cycle, $1 < s < \infty$ with $x_t^i > 0$ for all i and all t. Then factor price equalization occurs in every period of the cycle.

To prove this proposition in the appendix, we argue that, if an equilibrium with a finite cycle exists, countries have to change relative factor abundance at least once over the cycle. This implies that they have to change relative factor abundance an infinite number of times along the equilibrium path, which contradicts proposition 3.

As we have seen in lemma 2 and example 1, if consumers aggregate the two traded goods to obtain utility in the same way that firms aggregate these goods to obtain investment, then the equilibrium allocation of the integrated economy solves a one-sector social planner's problem.

Although the integrated economy approach to characterizing equilibria applies only when we can ensure factor price equalization, the assumption of one-sector aggregatability (made by Ventura 1997 and Bajona and Kehoe 2010, among others) coupled with proposition 7 puts strong restrictions on equilibrium dynamics.

Proposition 8: Assume that a model with infinitely lived consumers satisfies A.1–A.6 and is one-sector aggregatable. Also assume that $x_t^i > 0$ for all i and all t. Then there cannot exist an equilibrium with cycles.

6. A dynamic Heckscher-Ohlin model with overlapping generations

Departures from basic Heckscher-Ohlin assumptions are stronger in dynamic versions of the model made under an overlapping generations framework. For instance, it is common in the literature to depart from the assumption of identical consumers and to assume that consumers in different countries differ in their rates of time preference (Galor and Lin 1997, Mountford 1998). In this section, we derive some general results for equilibrium paths of our overlapping generations model, which do not include any of these additional assumptions. In view of the results, we also rationalize the reasons why these assumptions are made. Heckscher-Ohlin models with overlapping generations are significantly different from infinitely lived consumer models. They exhibit a richer variety of possible behavior, in the same way as they do in closed economy settings.[1]

6.1. Steady state analysis

A dynamic Heckscher-Ohlin model with overlapping generations may have steady states in which factor prices do not equalize. If prices do equalize in a given steady state, however, then all countries behave in exactly the same way and there is no trade. This result is in sharp contrast to propositions 4 and 5, which state that in the infinitely lived model, factor prices equalize in any steady state and that international trade occurs in (almost all) steady states. In the overlapping generations model, steady states with factor price equalization are not "interesting": any equilibrium converging to them has no trade in a finite number of periods. Therefore, in order to

[1] It is also worth noting that, in closed economy models with infinitely lived consumers, equilibria are generically determinate, wheras in closed economy models with overlapping generations, there are robust examples with indeterminate equilibria. (See Kehoe and Levine 1985 and the related literature.) Although these sorts of results can be expected to carry over to dynamic Heckscher-Ohlin models, we do not pursue these matters in this paper.

have factor price equalization and trade in the long run in an overlapping generations environment, the basic assumptions of the model (countries are identical except in their initial endowments) need to be modified. This is what is done in the existing literature.

Proposition 9: In a model with overlapping generations that satisfies assumptions A.1–A.4 and A.5'–A.6', assume that there is a nontrivial steady state in which factor prices equalize. Then $\hat{k}^i = \hat{k}$ is in the interior of the cone of diversification and there is no international trade in this steady state.

Proposition 10: In a model with overlapping generations that satisfies assumptions A.1–A.4 and A.5'–A.6', any equilibrium that converges to a steady state in which there is factor price equalization reaches factor price equalization and no trade within a finite number of periods. In particular, the equilibrium becomes autarkic once all generations alive have been born under factor price equalization.

As in the infinitely lived consumer environment, these results can be extended to economies with equilibria that converge to sustained growth paths.[2]

6.2. General structure of examples

As we have mentioned, overlapping generations economies can have steady states where factor prices do not equalize across countries. In what follows, we describe a general methodology for the construction of model economies with such properties starting from one-sector closed economy models. We then use our methodology to derive four different examples that have steady states without factor price equalization.

The general structure of our examples is that of a model with two countries, $i = 1, 2$ with CES production functions for the traded goods:

$$\phi_j(k,\ell) = \theta_j \left(\alpha_j k_j^{\rho} + (1-\alpha_j)\ell_j^{\rho}\right)^{1/\rho}, \tag{56}$$

for $\rho < 1$, $\rho \neq 0$, and

[2] Sustained growth is harder to obtain in the overlapping generations environment. In particular, sustained growth can occur only for economies that are not one-sector aggregatable and under strong conditions (see Jones and Manuelli, 1992, and Fisher, 1992, for details.)

$$\phi_j(k,\ell) = \theta_j k_j^{\alpha_j} \ell_j^{1-\alpha_j} \tag{57}$$

for $\rho = 0$. The production function for the investment good is Cobb-Douglas:

$$f(x_1, x_2) = d x_1^a x_2^{1-a}, \tag{58}$$

where $0 < a < 1$. Capital depreciates completely, $\delta = 1$.

On the consumers' side, we assume that there is a measure one of consumers of each generation in each country $\left(L^1 = L^2 = 1\right)$. The representative consumer in each generation t in country i lives for two periods, has labor endowments $(\bar{\ell}^1, \bar{\ell}^2)$, and has the utility function

$$\beta_1 \log\left(c_{1t}^t\right)^a \left(c_{2t}^t\right)^{1-a} + \beta_2 \log\left(c_{1t+1}^t\right)^a \left(c_{2t+1}^t\right)^{1-a}. \tag{59}$$

Notice that this model is one-sector aggregatable, since the parameter a in the utility function is the same as that in the production function for the investment good.

To find the cone of diversification for any given prices, we calculate the optimal capital-labor ratios in interior solutions to the revenue maximization problem (27). The solution is

$$\kappa_1(p_2/p_1) = \left(\frac{\alpha_1}{1-\alpha_1}\right)^{1/(1-\rho)} \left(\frac{(1-\alpha_2)^{1/(1-\rho)}(\theta_2 p_2/p_1)^{\rho/(1-\rho)} - (1-\alpha_1)^{1/(1-\rho)}\theta_1^{\rho/(1-\rho)}}{\alpha_1^{1/(1-\rho)}\theta_1^{\rho/(1-\rho)} - \alpha_2^{1/(1-\rho)}(\theta_2 p_2/p_1)^{\rho/(1-\rho)}}\right)^{1/\rho} \tag{60}$$

and

$$\kappa_2(p_2/p_1) = \left[\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\alpha_1}{\alpha_1}\right)\right]^{1/(1-\rho)} \kappa_1(p_2/p_1), \tag{61}$$

for $\rho \neq 0$. When $\rho = 0$, these functions become

$$\kappa_1(p_2/p_1) = \left[\frac{\theta_2 p_2/p_1}{\theta_1}\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_2}\right]^{1/(\alpha_1-\alpha_2)} \tag{62}$$

and

$$\kappa_2(p_2/p_1) = \left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\alpha_1}{\alpha_1}\right)\kappa_1(p_2/p_1). \tag{63}$$

Let us write $\alpha_2 = \alpha$, $\alpha_1 = \alpha + \varepsilon$, $\theta_2 = \theta$, and $\theta_1 = \theta + \lambda\varepsilon$ for $0 < \alpha < 1$, $\varepsilon > 0$, $\theta > 0$, and $\lambda \in R$. This relationship between the parameters allows us to express the capital-labor ratios that determine the cone of diversification as functions of α, ε, θ, and λ. Notice that $\varepsilon > 0$ guarantees that good 1 is the capital intensive good. Together with the endowments of capital and labor, the parameters α, θ, ε, λ, d, and a determine the pattern of specialization and trade. Notice that as ε approaches 0, the production functions for the two traded goods become more similar. The cone of diversification narrows, collapsing into a straight line when $\varepsilon = 0$. Setting $p_2 / p_1 = 1$ and using l'Hôpital's rule to take limits as ε tends to 0, we obtain:

$$\kappa_1(1) = \kappa_2(1) = \left(\frac{\theta - (1-\alpha)\rho\lambda}{\theta + \alpha\rho\lambda} \right)^{1/\rho} \tag{64}$$

for $\rho \neq 0$, and

$$\kappa_1(1) = \kappa_2(1) = e^{-\lambda/\theta} \tag{65}$$

for $\rho = 0$. By choosing λ appropriately, we can make this degenerate cone of diversification pass through any point. Furthermore, setting $p_1 = p_2 = 1$ determines the values of d and a as functions of α, θ, ε, and λ. Appendix 3 provides details on this derivation.

We derive each of our examples starting with a one-sector, closed-economy, overlapping generations model that has multiple steady states. Let $\phi_2(k, \ell)$ be the production function of this model. Let $\theta = \theta_2$ and $\alpha = \alpha_2$ be the parameters of this production function. We then construct another one-sector, closed economy model that preserves the steady state behavior of the original model by slightly perturbing this production function. Let $\phi_1(k, \ell)$ with parameters $\alpha_1 = \alpha + \varepsilon$ and $\theta_1 = \theta + \varepsilon\lambda$ be the production function of this perturbed model. The key is to find values of ε and λ so that there exists a steady state of the original model with capital-labor ratio and factor prices $(\hat{k}^2, \hat{r}^2, \hat{w}^2)$ and a steady state of the perturbed version of the model with capital-labor ratio and factor prices $(\hat{k}^1, \hat{r}^1, \hat{w}^1)$ that satisfy

$$\hat{k}^1 > \hat{k}^2, \ \hat{r}^1 < \hat{r}^2, \text{ and } \hat{w}^1 > \hat{w}^2 \tag{66}$$

and

$$\text{either } \hat{k}^1 > \kappa_1(1) \text{ or } \hat{k}^2 < \kappa_2(1) \text{ or both.} \tag{67}$$

Condition (67) implies that, in a two-sector, two-country economy with production functions $\phi_1(k,\ell)$ and $\phi_2(k,\ell)$, at least one country specializes in production if prices are $p_1 = p_2 = 1$ and countries' capital-labor ratios are $\hat{k}^i$, $i = 1,2$. Condition (66) implies that $\hat{r}^i$ and $\hat{w}^i$ are consistent with lemma 1, that is, the capital intensive country has a lower rental rate of capital. We can choose λ and ε so that condition (67) holds. Specifically, we choose λ so that the cone of diversification passes between the steady state $\hat{k}^2$ of the original one-sector model and the steady state $\hat{k}^1$ of the perturbed one-sector model when $\varepsilon = 0$ and then increase ε. Whether or not condition (66) holds in the constructed two-sector model depends on the properties of the original one-sector model, as we will see in the examples in the next section.

6.3. Steady state examples

Using the methodology developed in the previous section, we construct four examples of two-country economies that have steady states where factor prices are not equal across countries. In the first two examples, there is positive fiat money in one of the countries but not the other. In the other two examples, there is no fiat money.

Example 4. Consider a one-sector, closed economy model where the representative consumer in generation t has the utility function

$$\log c_t^t + 2\log c_{t+1}^t \tag{68}$$

and the labor endowment is $(\bar{\ell}^1, \bar{\ell}^2) = (0.8, 0.2)$. The production function is Cobb-Douglas:

$$\phi_2(k,\ell) = 4k^{0.25}\ell^{0.75}. \tag{69}$$

This economy has two steady states: a steady state with no fiat money where $\hat{b} = 0$, $\hat{k} = 1.4675$, $\hat{r} = 0.75$, and $\hat{w} = 3.3019$ and a steady state with positive fiat money where $\hat{b} = 0.4$, $\hat{k} = 1$, $\hat{r} = 1$, and $\hat{w} = 3$. Notice that the steady state without fiat money has a higher capital-labor ratio, a lower rental rate, and a higher wage than does the steady state with fiat money, allowing us to construct an example that satisfies lemma 1.

To construct the perturbed economy, let us set $\varepsilon = 0.05$ and $\lambda = -0.8$. Then, ϕ_1 becomes

$$\phi_1(k,\ell) = 3.96k^{0.3}\ell^{0.7}. \tag{70}$$

The perturbed economy also has two steady states. The steady state without fiat money has $\hat{b}=0$, $\hat{k}=1.4219$, $\hat{r}=0.9286$, and $\hat{w}=3.0807$.

Consider now a two-country, two-sector economy where good 1 has production function ϕ_1 and good 2 has production function ϕ_2. The cone of diversification for the two-country economy for $\hat{p}_1=\hat{p}_2=1$ is determined by $\hat{\kappa}_2(1)=1.0757$ and $\hat{\kappa}_1(1)=1.3830$. Set the steady state capital-labor ratio in country 1 to $\hat{k}^1=1.4219$ (the non-monetary steady state of the perturbed economy) and in country 2 to $\hat{k}^2=1$ (the monetary steady state of the perturbed economy). We now have a steady state of the two-country, two-good economy where country 1 specializes in the production of good 1, producing $\hat{y}_1^1=4.4010$ and $\hat{y}_2^1=0$, and country 2 specializes in the production of good 2, producing $\hat{y}_1^2=0$ and $\hat{y}_2^1=4$. Figure 3 depicts the cone of diversification for this world economy. Finally, to have $p_1=p_2=1$ in steady state, we need $d=1.9977$ and $a=0.5239$. Factor prices do not equalize in this steady state: $\hat{r}^1=0.9286$ and $\hat{w}^1=3.0807$, but $\hat{r}^2=1$ and $\hat{w}^2=3$.

The calculation of the other variables is straightforward. Since $\hat{c}_1=1.0427$ and $\hat{c}_2=1.9364$ in the steady state of the perturbed one-sector economy, for example, and $(\hat{y}_1^1+\hat{y}_1^2)/(\hat{y}_1^1+\hat{y}_1^2+\hat{y}_2^1+\hat{y}_2^2)=4.4010/8.4010=0.5239$, we set $\hat{c}_{11}^1=(0.5239)1.0427=0.5462$, $\hat{c}_{21}^1=(1-0.5239)1.0427=0.4965$, $\hat{c}_{12}^1=1.0144$, and $\hat{c}_{22}^1=0.9220$. Similarly, since $\hat{x}^1=\hat{k}^1=1.4219$, we set $\hat{x}_1^1=0.7449$ and $\hat{x}_2^1=0.6770$. Notice that trade is balanced: country 1 exports $\hat{y}_1^1-\hat{c}_{11}^1-\hat{c}_{12}^1-\hat{x}_1^1=2.0955$ of good 1 and imports $\hat{c}_{21}^1+\hat{c}_{22}^1+\hat{x}_2^1-\hat{y}_1^1=2.0955$ of good 2.

Example 5. Consider a modification of example 4 where $\theta_1=3.92$ (achieved by setting $\lambda=-1.6$). The cone of diversification is now determined by $\hat{\kappa}_2(1)=1.3179$ and $\hat{\kappa}_1(1)=1.6944$, and $\hat{k}^1=1.4507$ is in its interior, as depicted in figure 4. In this steady state, country 1 diversifies in production, producing $\hat{y}_1^1=1.6201$ and $\hat{y}_2^1=2.7736$, and country 2 specializes in the production of good 2, producing $\hat{y}_1^2=0$ and $\hat{y}_2^1=4$. To ensure that $\hat{p}_1=\hat{p}_2=1$, we set $d=1.6332$ and $a=0.1930$. Factor prices do not equalize: $\hat{r}^1=0.8130$ and $\hat{w}^1=3.2143$, but $\hat{r}^2=1$ and $\hat{w}^2=3$.

Example 6. Consider now a model with the same utility function and labor endowments as in examples 4 and 5, but where the production function is

$$\phi_2(k,\ell)=4\left(0.25k^{-3}+0.75\ell^{-3}\right)^{-1/3}. \tag{71}$$

The one-sector closed economy model has three steady states. In one steady state, $\hat{b}=0.4$ and $\hat{k}=1$. In the other two, $\hat{b}=0$ and either $\hat{k}=0.5675$ or $\hat{k}=1.3355$. When $\hat{k}=0.5675$, $\hat{r}=0.75$ and $\hat{w}=3.3019$, and, when $\hat{k}=1.3355$, $\hat{r}=0.75$ and $\hat{w}=3.3019$, allowing us to construct an example that satisfies lemma 1.

Let us set $\varepsilon=0.05$ and $\lambda=0$, so that the production function for the perturbed economy is

$$\phi_1(k,\ell)=4\left(0.3k^{-3}+0.7\ell^{-3}\right)^{-1/3}. \tag{72}$$

The perturbed economy also has three steady states: one steady state with positive fiat money and two steady states with no fiat money.

The cone of diversification is now determined by $\hat{\kappa}_2(1)=0.9687$ and $\hat{\kappa}_1(1)=1.0315$. Set the capital-labor ratio in country 1 to $\hat{k}^1=1.3908$, which is the perturbation of the steady state of the one-sector model where $\hat{k}=1.3355$, and set the capital-labor ratio in country 2 to $\hat{k}^2=0.5675$. Country 1 specializes in good 1, producing $\hat{y}_1^1=4.2884$ and $\hat{y}_2^1=0$, and country 2 specializes in the production of good 2, producing $\hat{y}_1^2=0$ and $\hat{y}_2^1=3.1147$. To ensure that $\hat{p}_1=\hat{p}_2=1$, we set $d=1.9749$ and $a=0.5793$. Factor prices do not equalize: $\hat{r}^1=0.4237$ and $\hat{w}^1=3.6991$, but $\hat{r}^2=3.5452$ and $\hat{w}^2=1.1029$.

The essential step in the construction of example 6 is to start with a one-sector closed economy model that has multiple steady states without fiat money. It is impossible to do this with a model with logarithmic utility and Cobb-Douglas production. Consider such a model with production function $\phi_2(k,\ell)=\theta k^{\alpha}\ell^{1-\alpha}$, utility function $\beta_1\log c_t^t+\beta_2\log c_{t+1}^t$, and labor endowments $(\bar{\ell}^1,\bar{\ell}^2)$. The unique steady state without fiat money is determined by

$$\hat{k}=\left(\frac{\beta_2(1-\alpha)\alpha\theta\bar{\ell}^1}{\beta_1(1-\alpha)\bar{\ell}^2+\left(\beta_1+\beta_2\right)\alpha}\right)^{\frac{1}{1-\alpha}}, \tag{73}$$

and the rental rate of capital is given by

$$\hat{r}=\alpha\theta\hat{k}^{\alpha-1}=\frac{\beta_1\ell_2}{\beta_2\ell_1}+\frac{\left(\beta_1+\beta_2\right)\alpha}{\beta_2(1-\alpha)\ell_1}. \tag{74}$$

6.4. Cycles and chaos

To complete our analysis, we use the methodology developed in section 6.1 to construct two examples with a two-period cycle as a steady state: one in which both countries completely specialize in each period and one in which one of the countries diversifies in each period.

Example 7. We start by constructing a one-sector, pure exchange economy with an equilibrium two-period cycle. In this economy the representative consumer in generation t solves the problem

$$\max\ 100c_t^t - \frac{(c_{t+1}^t)^{-4}}{4}$$
$$\text{s.t. } c_t^t + b_{t+1}^t = 0.8 \qquad (75)$$
$$c_{t+1}^t = 0.2 + (1 + r_{t+1}^b) b_{t+1}$$
$$c_t^t \geq 0,\ c_{t+1}^t \geq 0 .$$

We can think of this economy as a production economy with the production function $y = \ell$ and the labor endowments $(\bar{\ell}^1, \bar{\ell}^2) = (0.8, 0.2)$. This economy has an equilibrium cycle of periodicity 2, with $(\hat{c}_t^t, \hat{c}_t^{t-1}, \hat{r}_t, \hat{b}_t)$ cycling between

$$(\hat{c}_1^1, \hat{c}_1^2, \hat{r}_1^b, \hat{b}_1) = (0.7670, 0.2330, -0.9314, 0.4803) \qquad (76)$$

and

$$(\hat{c}_2^2, \hat{c}_2^1, \hat{r}_2^b, \hat{b}_2) = (0.3197, 0.6803, 13.5766, 3.295 \times 10^{-2}) . \qquad (77)$$

Notice that $(1+\hat{r}_1^b)(1+\hat{r}_2^b) = 1$. Figure 5 depicts this cycle in an offer-curve diagram. The figure graphs the set of excess demands $(y^t, z^t) = (c_t^t - 0.8, c_{t+1}^t - 0.2)$ that are utility maximizing for some r_{t+1}^b and the set of y^t and z^t that satisfy the feasibility condition $z^{t-1} + y^t = 0$. (See, for example, Kehoe 1989.)

The next step is to perturb this economy to allow production in which capital plays a role:

$$y = \phi_2(k, \ell) = \left(0.1k^{3/5} + 0.9\ell^{3/5}\right)^{5/3} . \qquad (78)$$

This one-sector economy with production has a cycle of periodicity 2, with $(\hat{k}_t, \hat{r}_t, \hat{w}_t, \hat{m}_t)$ cycling between

$$(\hat{k}_1, \hat{r}_1, \hat{w}_1, \hat{b}_1) = (0.2214, 0.1754, 0.8639, 0.3958) \tag{79}$$

and

$$(\hat{k}_2, \hat{r}_2, \hat{w}_2, \hat{b}_2) = (3.422 \times 10^{-5}, 5.6996, 0.8391, 6.945 \times 10^{-2}) . \tag{80}$$

Notice that $\hat{r}_1 \hat{r}_2 = 1$.

We add another sector by perturbing production function in much the same way that we did in the steady state examples in the previous section:

$$y = \phi_1(k, \ell) = 1.09\left(0.15k^{3/5} + 0.85\ell^{3/5}\right)^{5/3} . \tag{81}$$

That is, we set $\varepsilon = 0.05$ and $\lambda = 1.8$. The two-sector economy has a two-period cycle where $p_1 = p_2 = 1$ over the cycle and countries alternate their positions between

$$(\hat{k}_1, \hat{r}_1, \hat{w}_1, \hat{b}_1) = (9.726 \times 10^{-2}, 0.3834, 0.8554, 0.3137) \tag{82}$$

and

$$(\hat{k}_2, \hat{r}_2, \hat{w}_2, \hat{b}_2) = (2.418 \times 10^{-4}, 2.6083, 0.8394, 0.1203) . \tag{83}$$

The cone of diversification for this economy is determined by $\hat{\kappa}_2(1) = 8.452 \times 10^{-3}$ and $\hat{\kappa}_1(1) = 2.687 \times 10^{-2}$. When a country has capital $\hat{k}_1 = 9.726 \times 10^{-2}$, it specializes in the production of good 1, producing $\hat{y}_{11} = 0.8926$ and $\hat{y}_{21} = 0$, and, when it has capital $\hat{k}_2 = 2.418 \times 10^{-4}$, it specializes in the production of good 2, producing $\hat{y}_{12} = 0$ and $\hat{y}_{22} = 0.8400$. In the equilibrium cycle, countries change their specialization pattern every period. Because the countries are symmetric and alternate between the same two positions, the relative production of the two goods stays constant, which allows us to ensure that $p_1 = p_2 = 1$ by setting $d = 1.9991$ and $a = 0.5151$.

Example 8. This example slightly modifies the production function of good 1 in example 7 so that one of the countries diversifies in production. In particular, we set $\theta_1 = 1.07$ by setting $\lambda = 1.4$.

The cone of diversification is now determined by $\hat{\kappa}_2(1) = 5.637 \times 10^{-2}$ and $\hat{\kappa}_1(1) = 0.1792$. This economy has a cycle where countries alternate their positions between

$$(\hat{k}_1, \hat{r}_1, \hat{w}_1, \hat{b}_1) = (0.1391, 0.2984, 0.8500, 0.3388) \quad (84)$$

and

$$(\hat{k}_2, \hat{r}_2, \hat{w}_2, \hat{b}_2) = (1.291 \times 10^{-4}, 3.3515, 0.8392, 0.1011). \quad (85)$$

When a country has capital $\hat{k}_1 = 0.1391$, it now diversifies, producing $\hat{y}_{11} = 0.6084$ and $\hat{y}_{21} = 0.2830$, and, when it has capital $\hat{k}_2 = 1.291 \times 10^{-4}$, it specializes in the production of good 2, producing $\hat{y}_{12} = 0$ and $\hat{y}_{22} = 0.8397$. Countries enter and exit the cone of diversification an infinite number of times. To ensure that $p_1 = p_2 = 1$, we set $d = 1.9124$ and $a = 0.3515$.

Examples 7 and 8 illustrate the richness of possible equilibrium behavior of dynamic Heckscher-Ohlin models with overlapping generations. In particular, the examples have long-run equilibria without factor price equalization in models that are one-sector aggregatable (in one of them, the countries leave the cone an infinite number of times). Propositions 7 and 8 show that this type of example cannot be constructed in models with infinitely lived consumers.

As a final note, we observe that, as in the infinitely lived consumer environment, any two-sector, closed economy model with overlapping generations can be made into a trade model. Suppose that $g(k, \ell)$ is the production function for the consumption good in the two-sector closed economy model, that $f(k, \ell)$ is the production function for the investment good, and that $v_h(c)$ is the utility function in period of life h. Set $\phi_1(k, \ell) = k$, $\phi_2(k, \ell) = \ell$, and $u_h(c_1, c_2) = v_h(g(c_1, c_2))$. Since the cone of diversification is the entire nonnegative quadrant, factor prices equalize in any equilibrium. The equilibrium is autarkic after $m+1$ periods, where m is the length of a consumer's life. Consequently, all of the possibilities for equilibrium dynamics that are present in two-sector growth models with overlapping generations are also present in our model. In particular, two-sector, closed-economy models with cyclical and chaotic behavior, like those constructed by Reichlin (1992), can be made into examples of Heckscher-Ohlin economies.

Appendix 1: Equations that characterize equilibrium

Common to both environments:

$i = 1,...,n$, $t = 0,1,...$

$$x_t^i = f(x_{1t}^i, x_{2t}^i) \tag{86}$$

$$k_{t+1}^i - (1-\delta) k_t^i = x_t^i \tag{87}$$

$$\ell_{1t}^i + \ell_{2t}^i = 1 \tag{88}$$

$$k_{1t}^i + k_{2t}^i = k_t^i . \tag{89}$$

$j = 1,2$

$$y_{jt}^i = \phi_j(k_{jt}^i, \ell_{jt}^i) \tag{90}$$

$$r_t^i \geq p_{jt} \phi_{jK}(k_{jt}^i, \ell_{jt}^i) \text{ , } = \text{ if } k_{jt}^i > 0 \tag{91}$$

$$w_t^i \geq p_{jt} \phi_{jL}(k_{jt}^i, \ell_{jt}^i) \text{ , } = \text{ if } \ell_{jt}^i > 0 \tag{92}$$

$$p_{jt} = f_j(x_{1t}^i, x_{2t}^i) . \tag{93}$$

Specific to infinitely lived consumer environment:

$i = 1,...,n$

$$k_0^i = \bar{k}_0^i ,\ \bar{b}_0^i = 0 \tag{94}$$

$t = 0,1,...$

$$\frac{u_2(c_{1t}^i, c_{2t}^i)}{u_1(c_{1t}^i, c_{2t}^i)} = \frac{p_{2t}}{p_{1t}} \tag{95}$$

$$\frac{u_1(c_{1t}^i, c_{2t}^i)}{\beta u_1(c_{1t+1}^i, c_{2t+1}^i)} = (1 + r_{t+1}^{bi}) \frac{p_{1t}}{p_{1t+1}} \geq (1 + r_{t+1}^i - \delta) \frac{p_{1t}}{p_{1t+1}} \text{ , } = \text{ if } x_t^i > 0 \tag{96}$$

$$p_{1t} c_{1t}^i + p_{2t} c_{2t}^i + x_t^i + b_{t+1}^i = w_{t+1}^i + (1 + r_t^b) b_t^i + (1 + r_t^i - \delta) k_t^i \tag{97}$$

$j = 1,2$

$$\sum\nolimits_{i=1}^n L^i (c_{jt}^i + x_{jt}^i) = \sum\nolimits_{i=1}^n L^i y_{jt}^i . \tag{98}$$

If there is international borrowing and lending,

$$\sum\nolimits_{i=1}^n L^i b_t^i = 0 \tag{99}$$

$$r_t^{bi} = r_t^b . \tag{100}$$

If there is no international borrowing and lending,

$$b_t^i = 0\,. \tag{101}$$

Specific to overlapping generations environment:

$i = 1,\ldots,n$

$$k_0^{ih} = \bar{k}_0^{ih},\ b_0^{ih} = \bar{b}_0^{ih},\ h = 1-m,\ldots,-1 \tag{102}$$

$t = 0,1,\ldots$

$$x_t^i = \sum_{h=1}^{m-1} x_t^{it-h+1} \tag{103}$$

$$k_t^i = \sum_{h=2}^{m} k_t^{it-h+1} \tag{104}$$

$j = 1,2$

$$\sum_{i=1}^{n} L^i \left(\sum_{h=1}^{m} c_{jt}^{it-h+1} + x_{jt}^i \right) = \sum_{i=1}^{n} L^i y_{jt}^i \tag{105}$$

$h = 1,\ldots,m$

$$\frac{u_2(c_{1t}^{it-h+1}, c_{2t}^{it-h+1})}{u_1(c_{1t}^{it-h+1}, c_{2t}^{it-h+1})} = \frac{p_{2t}}{p_{1t}} \tag{106}$$

$$p_{1t}c_{1t}^{it-h+1} + p_{2t}c_{2t}^{it-h+1} + x_{t+1}^{it-h+1} + b_{t+1}^{it-h+1} = w_t^i \ell^h + (1+r_t^{bi})b_t^{it-h+1} + (1+r_t^i - \delta)k_t^{it-h+1} \tag{107}$$

$h = 1,\ldots,m-1$

$$k_{t+1}^{it-h+1} - (1-\delta)\,k_t^{it-h+1} = x_t^{it-h+1} \tag{108}$$

$$\frac{\beta_h u_1(c_{1t}^{it-h+1}, c_{2t}^{it-h+1})}{\beta_{h+1} u_1(c_{1t+1}^{it-h+1}, c_{2t+1}^{it-h+1})} = (1+r_{t+1}^{bi})\frac{p_{1t}}{p_{1t+1}} \geq (1+r_{t+1}^i - \delta)\frac{p_{1t}}{p_{1t+1}},\ =\ \text{if } x_t^{it-h+1} > 0\,. \tag{109}$$

If there is international borrowing and lending,

$$\sum_{i=1}^{n} L^i \sum_{h=2}^{m} b_t^{it-h+1} = \left(\prod_{s=0}^{t} (1+r_s^b) \right) \sum_{i=1}^{n} L^i \sum_{h=1-m}^{-1} \bar{b}_0^{ih} \tag{110}$$

$$r_t^{bi} = r_t^b\,, \tag{111}$$

and the initial amount of fiat money is $\sum_{i=1}^{n} L^i \sum_{h=1-m}^{-1} \bar{b}_0^{ih}$.

If there is no international borrowing and lending,

$$\sum_{h=2}^{m} b_t^{it-h+1} = \left(\prod_{s=0}^{t} (1+r_s^{bi}) \right) \sum_{h=1-m}^{-1} \bar{b}_0^{ih}\,, \tag{112}$$

and the initial amount of fiat money per capita in country i is $\sum_{h=1-m}^{-1} \bar{b}_0^{ih}$.

Appendix 2: Proofs

Proof of proposition 1: The first-order conditions for the consumers' problems, (6) or (15) and (16), imply the no-arbitrage condition $r_t^i = r_t^b + \delta$ for all $t > 0$. Consequently, rental rates of capital are equalized for all $t > 0$. Since both goods are produced and each country produces at least one good, pick two countries, say 1 and 2, such that country 1 produces good 1 and country 2 produces good 2. Let $p_j(r_t, w_t)$ be the unit cost function in sector j. Since country 2 produces good 2, $p_{2t} = p_2(r_t, w_t^2) \leq p_2(r_t, w_t^1)$. Since ϕ_2 is capital intensive, ϕ_2 is strictly increasing in labor, which implies that p_2 is strictly increasing in the wage. This implies that $w_t^1 \geq w_t^2$.

There are two cases to examine. First, suppose that ϕ_1 is also strictly increasing in labor. Then $p_{1t} = p_1(r_t, w_t^1) \leq p_1(r_t, w_t^2)$ where p_1 is strictly increasing in the wage. This implies that $w_t^1 \leq w_t^2$. Consequently, $w_t^1 = w_t^2$. Second, suppose instead that ϕ_1 is not strictly increasing in labor. Then it cannot employ labor at a positive wage. Since $w_t^1 \geq w_t^2$, if $w_t^1 \neq w_t^2$, then $w_t^1 > w_t^2$. Consequently, country 1 must also produce good 2, which implies that $p_{2t} = p_2(r_t, w_t^2) = p_2(r_t, w_t^1)$. Since ϕ_2 is strictly increasing in labor, this implies that $w_t^1 = w_t^2$. ■

Proof of proposition 2: We prove the proposition for the model with infinitely lived consumers. The proposition for the model with overlapping generations follows the same argument with changes in notation. Assume that an equilibrium $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i\}$, $\{c_{1t}^i, c_{2t}^i, x_t^i, k_t^i, b_t^i\}$, $\{y_{jt}^i, k_{jt}^i, \ell_{jt}^i\}$, $\{x_t^i, x_{1t}^i, x_{2t}^i\}$ exists in which $x_t^i > 0$ for all i and all t. We provide a method for constructing an infinite number of equilibria that have the same values of all variables in all periods except for s and $s+1$. In periods s and $s+1$, we keep the values of prices p_{1t}, p_{2t}, $q_t^i = 1$, w_t^i, r_t^i, consumption c_{1t}^i, c_{2t}^i in individual countries, and of aggregate production and investment fixed. We change k_{s+1}^i to $\tilde{k}_{s+1}^i$ so as to change the production plans y_{js+1}^i, k_{js+1}^i, ℓ_{js+1}^i to $\tilde{y}_{js+1}^i$, $\tilde{k}_{js+1}^i$, $\tilde{\ell}_{js+1}^i$, $j = 1, 2$. To accumulate the capital stock $\tilde{k}_{s+1}^i$, we set $\tilde{x}_s^i = \tilde{k}_{s+1}^i - (1-\delta)k_s^i$, $\tilde{x}_{js}^i = (\tilde{x}_s^i / x_s^i)x_{js}^i$, $j = 1, 2$. To satisfy the budget constraint, we set $\tilde{b}_{s+1}^i = x_{s+1}^i + b_{s+1}^i - \tilde{x}_{s+1}^i$. We then set $\tilde{x}_{s+1}^i = k_{s+2}^i - (1-\delta)\tilde{k}_{s+1}^i$, $\tilde{x}_{js+1}^i = (\tilde{x}_{s+1}^i / x_{s+1}^i)x_{js+1}^i$, $j = 1, 2$. It is easy to check that the budget constraint in country i period $s+1$ is satisfied because the constraint in s is satisfied.

To find nonnegative production plans $\tilde{y}_{js+1}^i$, $\tilde{k}_{js+1}^i$, $\tilde{\ell}_{js+1}^i$ consistent with profit maximization at the equilibrium prices p_{1s+1}, p_{2s+1}, w_{s+1}, r_{s+1}, the capital-labor ratio $\tilde{k}_{s+1}^i$ needs to be in the cone of diversification,

$$\kappa_1(p_{2s+1} / p_{1s+1}) \geq \tilde{k}_{s+1}^i \geq \kappa_2(p_{2s+1} / p_{1s+1}). \tag{113}$$

We also want to keep investment nonnegative in periods s and $s+1$,

$$\tilde{k}_{s+1}^i \geq (1-\delta)k_s^i \tag{114}$$

$$\tilde{k}_{s+1}^i \geq k_{s+2}^i / (1-\delta). \tag{115}$$

Together, these inequalities become

$$\min\left[\kappa_1(p_{2s+1}/p_{1s+1}), k^i_{s+1}/(1-\delta)\right] \geq \tilde{k}^i_{s+1} \geq \max\left[\kappa_2(p_{2s+1}/p_{1s+1}), (1-\delta)k^i_s\right]. \tag{116}$$

Assumption A.2 ensures that the cone of diversification has an interior,

$$\kappa_1(p_{2s+1}/p_{1s+1}) > \kappa_2(p_{2s+1}/p_{1s+1}). \tag{117}$$

Since we have assumed that $x^i_t > 0$ for $t = s, s+1$,

$$k^i_{s+2}/(1-\delta) > k^i_{s+1} > (1-\delta)k^i_s. \tag{118}$$

Proposition 1 ensures that k^i_{s+1} is in the cone of diversification, although it may be on the boundary. Therefore, k^i_{s+1} satisfies inequality (116) and

$$\min\left[\kappa_1(p_{2s+1}/p_{1s+1}), k^i_{s+1}/(1-\delta)\right] > \max\left[\kappa_2(p_{2s+1}/p_{1s+1}), (1-\delta)k^i_s\right]. \tag{119}$$

Consequently, we can set $\tilde{k}^i_{s+1} = k^i_{s+1} + \varepsilon^i$ for some nonzero ε^i small enough and still satisfy inequality (116).

If $k^i_{s+2}/(1-\delta) > \kappa_1(p_{2s+1}/p_{1s+1}) = k^i_{s+1}$, then we have to set $\varepsilon^i < 0$, and, if $(1-\delta)k^i_s < \kappa_2(p_{2s+1}/p_{1s+1}) = k^i_{s+1}$, then we have to set $\varepsilon^i > 0$. Otherwise, we have freedom to set $n-1$ values of ε^i independently. We want to keep the aggregate capital stock the same,

$$\frac{\sum_{i=1}^n L^i \tilde{k}^i_{s+1}}{\sum_{i=1}^n L^i} = \frac{\sum_{i=1}^n L^i(k^i_{s+1} + \varepsilon^i)}{\sum_{i=1}^n L^i} = k_{s+1}, \tag{120}$$

which implies that

$$\varepsilon^n = -\sum_{i=1}^{n-1}(L^i/L^n)\varepsilon^i. \tag{121}$$

There are two possible cases. First, k^n_{s+1} is in the interior of the cone of diversification, in which case, by choosing ε^i, $i = 1,...,n-1$, small enough, we can ensure that $\tilde{k}^n_{s+1} = k^n_{s+1} - \sum_{i=1}^{n-1}(L^i/L^n)\varepsilon^i$ stays in the cone. Second, k^n_{s+1} is on the boundary of the cone, in which case, if we can choose some of the deviations ε^i, $i = 1,...,n-1$, to be positive and others negative, we can set them so that ε^n has any desired sign. If all of the deviations ε^i, $i = 1,...,n-1$, have to be of the same sign, then we know that all of the capital-labor ratios k^i_{s+1}, $i = 1,...,n-1$, are on the same boundary of the cone. Since $k_{s+1} = \sum_{i=1}^n L^i k^i_{s+1} / \sum_{i=1}^n L^i$ is in the interior of the cone, however, this implies that k^n_{s+1}, if it is on a boundary of the cone, is on the other boundary, which implies that ε^n has the desired sign.

By reshuffling savings between capital and bonds, we have shown that there is an $n-1$ dimensional indeterminacy in any period $s+1$ for which $x^i_t > 0$ for all i in periods s and $s+1$. If $x^i_t = 0$ in either of these periods for some country i, then it could be that

$$\min\left[\kappa_1(p_{2s+1}/p_{1s+1}), k_{s+1}^i/(1-\delta)\right] = \max\left[\kappa_2(p_{2s+1}/p_{1s+1}), (1-\delta)k_s^i\right], \tag{122}$$

which would imply that we have to set $\varepsilon^i = 0$ and would reduce the dimension of indeterminacy by 1 in period $s+1$. Notice that, if δ=1, then equation (122) can never hold, and we always have an $n-1$ dimensional indeterminacy in every period $t > 0$. ■

Proof of lemma 1: The result follows directly from the first-order conditions for the revenue maximization problem (27) and the strict concavity of ϕ_j. See, for example, Dixit and Norman (1980). ■

Proof of lemma 2: Suppose that the country social planner's problem (31) is well posed in the sense that there is a feasible $\{c_t^i, x_t^i, k_t^i\}$ such that $\liminf_{T\to\infty} \sum_{t=0}^{\infty} \beta^t v(c_t^i) > -\infty$ and there is no feasible $\{c_t^i, x_t^i, k_t^i\}$ such that $\limsup_{T\to\infty} \sum_{t=0}^{\infty} \beta^t v(c_t^i) = +\infty$. We can show that the equilibrium $\{c_t^i, x_t^i, k_t^i\}$ solves the country planner's problem (31) in which the prices p^0 are the equilibrium prices and the initial capital stock is $\bar{k}_0^i$ by showing that the equilibrium conditions imply that the first-order conditions and transversality condition of (31) are satisfied. Furthermore, if V is well defined in the sense that $-\infty < V(k; p^0) < +\infty$ for all $k > 0$, then it is straightforward to use the sorts of arguments in Stokey, Lucas, and Prescott (1989) to show that the properties of v and π imply that V is continuous, increasing, and concave as a function of k. Consequently, to prove the theorem we need to argue that $-\infty < V(k; p^0) < +\infty$ for all $k > 0$. Notice that $V(\bar{k}_0^i; p^0)$ is finite when the prices p^0 are the equilibrium prices and the initial capital stock in country i is $\bar{k}_0^i$. Similarly, $V(k_t^i; p^t)$ is finite for any k_t^i on the equilibrium path. Otherwise, the consumer's problem (6) would not be well posed and the prices p^0 would not be the equilibrium prices.

We first argue that $V(k; p^0) < +\infty$. Suppose, to the contrary, that $V(\hat{k}_0; p^0) = +\infty$ for some $\hat{k}_0 > 0$, and let $\{\hat{c}_t, \hat{x}_t, \hat{k}_t\}$ be the corresponding plan for consumption, investment, and the capital stock. If $\hat{k}_0 \le \bar{k}_0^{i_{max}}$, where $\bar{k}_0^{i_{max}} \ge \bar{k}_0^i$, $i = 1,...,n$, we could reduce $k_0^{i_{max}}$ to $\hat{k}_0$, follow the plan $\{\hat{c}_t, \hat{x}_t, \hat{k}_t\}$, and achieve infinite utility in equilibrium, which implies that the prices p^0 cannot be equilibrium prices. If $\hat{k}_0 > \bar{k}_0^{i_{max}}$, then assumption A.5 implies that we can reduce $c_0^{i_{max}}$ to $\tilde{c}_0$, $c_0^{i_{max}} > \tilde{c}_0 > 0$, and increase $x_0^{i_{max}}$ to $\tilde{x}_0 > x_0^{i_{max}}$, thereby increasing $k_1^{i_{max}}$ to $\tilde{k}_1$. Consider the continuation plan

$$(\tilde{c}_t, \tilde{x}_t, \tilde{k}_t) = (\theta\hat{c}_t + (1-\theta)c_t^{i_{max}}, \theta\hat{x}_t + (1-\theta)x_t^{i_{max}}, \theta\hat{k}_t + (1-\theta)k_t^{i_{max}}),\ t = 1,2,..., \tag{123}$$

where $\theta = (\hat{k}_1 - \tilde{k}_1)/(\hat{k}_1 - k_1^{i_{max}})$. Since $\{c_t^{i_{max}}, x_t^{i_{max}}, k_t^{i_{max}}\}$ satisfies the feasibility conditions in problem (31) for the initial capital $\bar{k}_0^{i_{max}}$ and $\{\hat{c}_t, \hat{x}_t, \hat{k}_t\}$ satisfies the feasibility for the initial capital $\hat{k}_0$, $\{\tilde{c}_t, \tilde{x}_t, \tilde{k}_t\}$ satisfies the feasibility conditions for initial capital $\bar{k}_0^{i_{max}}$. Notice that, although $v(\tilde{c}_0) < v(c_0^{i_{max}})$, the strict concavity of v implies that

$$\sum_{t=1}^{\infty} \beta^t v(\tilde{c}_t) = \sum_{t=1}^{\infty} \beta^t v(\theta\hat{c}_t + (1-\theta)c_t^{i_{max}}) > \theta\sum_{t=1}^{\infty} \beta^t v(\hat{c}_t) + (1-\theta)\sum_{t=1}^{\infty} \beta^t v(c_t^{i_{max}}) = +\infty, \tag{124}$$

which again implies that the prices p^0 cannot be equilibrium prices.

We now argue that $V(k;p^0) > -\infty$. Suppose, to the contrary, that $V(\hat{k}_0;p^0) = -\infty$ for some $\hat{k}_0 > 0$. If $\hat{k}_0 \geq \bar{k}_0^{i_{min}}$, where $\bar{k}_0^{i_{min}} \leq \bar{k}_0^i$, $i = 1,...,n$, we could reduce $\hat{k}_0$ to $k_0^{i_{min}}$, follow the equilibrium plan $\{c_t^{i_{min}}, x_t^{i_{min}}, k_t^{i_{min}}\}$ and achieve a finite value of the objective function in problem (31), which implies that $V(\hat{k}_0;p^0) > -\infty$. If $\hat{k}_0 < \bar{k}_0^{i_{min}}$, consider the plan $\left\{\theta c_t^{i_{min}}, \theta x_t^{i_{min}}, \theta k_t^{i_{min}}\right\}$, where $\theta = \hat{k}_0 / \bar{k}_0^{i_{min}}$. Since $\{c_t^{i_{min}}, x_t^{i_{min}}, k_t^{i_{min}}\}$ is feasible for initial capital $k_0^{i_{min}}$, $\{\theta c_t^{i_{min}}, \theta x_t^{i_{min}}, \theta k_t^{i_{min}}\}$ is feasible for initial capital $\hat{k}_0$ and

$$V(\hat{k}_0;p^0) \geq \sum_{t=0}^{\infty} \beta^t v(\theta c_t^{i_{min}}). \tag{125}$$

The strict concavity of v implies that

$$\sum_{t=0}^{\infty} \beta^t v(\theta c_t^{i_{min}}) > \sum_{t=0}^{\infty} \beta^t v(c_t^{i_{min}}) - (1-\theta)\sum_{t=0}^{\infty} \beta^t v'(c_t^{i_{min}}) c_t^{i_{min}} . \tag{126}$$

Since $\left\{c_t^{i_{min}}, x_t^{i_{min}}, k_t^{i_{min}}\right\}$ is an equilibrium plan, we know that $\sum_{t=0}^{\infty} \beta^t v(c_t^{i_{min}}) > -\infty$ and that

$$0 < \sum_{t=0}^{\infty} \beta^t v'(c_t^{i_{min}}) c_t^{i_{min}} = \frac{v'(c_0^{i_{min}})}{p(p_{10},p_{20})} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \frac{1}{1 + r_s^{i_{min}} - \delta} \right) p(p_{1t},p_{2t}) c_t^{i_{min}} < +\infty . \tag{127}$$

(In fact, the latter term is the present discounted value in period 0 of consumption in i_{min}.) Consequently, $V(\hat{k}_0;p^0) > -\infty$. ■

Proof of proposition 3. The proof follows the approach of Milgrom and Shannon (1994). We prove that $k_0^i \geq k_0^{i'}$ implies $k_1^i \geq k_1^{i'}$. By induction, the argument extends to any other period. Let us write the country social planner's problem as

$$V(k_0^i;p^0) = \max v(c_0(k_0^i,k_1^i;p^0)) + \beta V(k_1^i;p^1) \tag{128}$$
$$\text{s.t. } 0 \leq k_1^i \leq \bar{k}(k_0^i;p^0),$$

where $c_0(k_0^i,k_1^i;p^0) = (\pi(p_{10},p_{20},k_0^i) - k_1^i + (1-\delta)k_0^i)/p(p_{10},p_{20})$, and $\bar{k}(k_0^i;p^0) = \pi(p_{10},p_{20}) + (1-\delta)k_0^i$. Define $F(k_0^i,k_1^i;p^0) = v(c_0(k_0^i,k_1^i;p^0)) + \beta V(k_1^i;p^1)$. We prove our result by contradiction. If $k_0^i = k_0^{i'}$, then $k_1^i = k_1^{i'}$. Therefore, assume, without loss of generality, that $k_0^1 > k_0^2$ and $k_1^1 < k_1^2$. Notice that, since $k_1^1 < k_1^2 \leq \bar{k}(k_0^2;p^0) < \bar{k}(k_0^1;p^0)$, both k_1^1 and k_1^2 are feasible. Since k_1^i maximizes $F(k_0^i,k_1^i;p^0)$, $i = 1,2$, we know that

$$F(k_0^1,k_1^1;p^0) - F(k_0^1,k_1^2;p^0) \geq 0 \geq F(k_0^2,k_1^1;p^0) - F(k_0^2,k_1^2;p^0). \tag{129}$$

Define the function $g(k_0) = F(k_0,k_1^1;p^0) - F(k_0,k_1^2;p^0)$.

$$g'(k_0) = \left[v'\left(c_0(k_0,k_1^1;p^0)\right) - v'\left(c_0(k_0,k_1^2;p^0)\right)\right]\left(\frac{\pi_k(p_{10},p_{20},k_0) + 1 - \delta}{p\left(p_{10},p_{20}\right)}\right) < 0, \tag{130}$$

where $\pi_k(p_{10}, p_{20}, k_0)$ is the partial derivative of $\pi(p_{10}, p_{20}, k_0)$ with respect to k_0. The inequality follows from the fact that v is strictly concave and c_0 is strictly increasing in k_1. Therefore, $k_0^1 > k_0^2$ implies $g(k_0^1) < g(k_0^2)$, which contradicts (129).

Suppose now that $k_0^1 > k_0^2$. We have demonstrated that $k_1^1 \geq k_1^2$. We now demonstrate that, if $x_0^1 > 0$, $k_1^1 > k_1^2$. Suppose, to the contrary, that $k_0^1 > k_0^2$ and $x_0^1 > 0$, but that $k_1^1 = k_1^2$. Since

$$(1-\delta)k_0^1 + x_0^1 = k_1^1 = k_1^2 = (1-\delta)k_0^2 + x_0^2, \tag{131}$$

$k_0^1 > k_0^2$ and $x_0^1 > 0$ imply that $x_0^2 > x_0^1 > 0$. We can therefore write the first-order condition for x_0^i in (31) as an equality,

$$\frac{v(c_0^i)}{p(p_{10}, p_{20})} = \frac{\beta v(c_1^i)}{p(p_{11}, p_{21})}\left(\pi_k(p_{11}, p_{21}, k_1^i) + 1 - \delta\right),\ i = 1, 2. \tag{132}$$

Since $k_1^1 = k_1^2$, $c_1^1 = c_1^2$. Equation (132) therefore implies that

$$v(c_0^1) = v(c_0^2). \tag{133}$$

Since $c_0^1 > c_0^2$ and v is strictly concave, however,

$$v(c_0^1) < v(c_0^2), \tag{134}$$

a contradiction. ■

Proof of proposition 4: Without loss of generality, assume that $\hat{k}^1 > 0$. The first-order condition for the consumer's problem (6) implies that

$$\hat{r}^i \leq \frac{1}{\beta} - 1 + \delta \text{ and } = \text{ if } \hat{x}^i > 0. \tag{135}$$

Therefore, for all $i = 1, ..., n$ such that $\hat{x}^i > 0$, $\hat{r}^i = \hat{r}^1$. Assume that there is a country i for which $\hat{x}^i = 0$ and $\hat{r}^i < \hat{r}^1$. Since the rental rate of capital is lower in country i, lemma 1 implies that $\hat{k}^i > \hat{k}^1 > 0$. Since in a steady state $\hat{x}^i = \delta \hat{k}^i$, $\hat{x}^i = 0$ implies that $\hat{k}^i = 0$, which is a contradiction. Consequently, $\hat{r}^i = \hat{r}^1$ for all $i = 1, ..., n$. Following the same argument as in the proof of proposition 1, equal rental rates of capital across countries imply equal wages across countries. ■

Proof of proposition 5: Suppose that the economy has a nontrivial steady state with distribution of capital $(\hat{k}^1, ..., \hat{k}^n)$. We argue that there exists a nontrivial steady state for all $(k^1, ..., k^n)$ that satisfy

$$\sum\nolimits_{i=1}^n L^i k^i = \sum\nolimits_{i=1}^n L^i \hat{k}^i \tag{136}$$

$$\hat{\kappa}_2 \leq k^i \leq \hat{\kappa}_1, \tag{137}$$

where $\hat{\kappa}_j$ is the capital-labor ratio in industry j, $j = 1, 2$. Each of these steady states has the same prices and, by construction, the same aggregate capital and production. We need to define the other steady state variables and show that these steady state variables satisfy the equilibrium conditions listed in appendix 1. Let the distribution of capital and labor across industries in each country be

$$\ell_1^i = \frac{k^i - \hat{\kappa}_2}{\hat{\kappa}_1 - \hat{\kappa}_2},\ \ell_2^i = 1 - \ell_1^i \tag{138}$$

$$k_j^i = \hat{\kappa}_j \ell_j^i . \tag{139}$$

Since $k^i > 0$, we can use the first-order conditions for the investment good (93) to set

$$x_2^i = \frac{\delta k^i}{f(\hat{x}_1 / \hat{x}_2, 1)} \tag{140}$$

$$x_1^i = (\hat{x}_1 / \hat{x}_2) x_2^i . \tag{141}$$

We also set $\hat{b}^i = 0$. It is easy to check that all of the production feasibility conditions (86)–(94) are satisfied in the new steady state. We are left with defining consumption patterns that are consistent with the consumer's first-order conditions and the feasibility conditions in each goods market. Equation (95) and assumption A.5 imply that $\hat{c}_1^i / \hat{c}_2^i = \hat{c}_1 / \hat{c}_2$. We use the budget constraint (97) to define

$$c_2^i = \frac{\hat{w} + (\hat{r} - \delta)k^i}{\hat{p}_1 \hat{c}_1 / \hat{c}_2 + \hat{p}_2} \tag{142}$$

$$c_1^i = (\hat{c}_1 / \hat{c}_2) c_2^i . \tag{143}$$

With these definitions the rest of the conditions that characterize equilibrium, conditions (96)–(98), are satisfied. Condition (101) is satisfied by construction. ■

Proof of proposition 6: There are three steps in the argument. First, we show that, if $\gamma^i > 1$ for some i then $\gamma^i > 1$ for all i. Second, we show that, if $\gamma^i > 1$ for all i, then $\gamma^i = \gamma$ for all i. Third and finally, we use $\gamma^i = \gamma$ for all i to prove that factor prices are equal in the limit.

To prove that $\gamma^i > 1$ for some i implies that $\gamma^i > 1$ for all i, assume, to the contrary and without loss of generality, that $0 < \gamma^2 < 1 < \gamma^1$. Notice that $\gamma^1 > 1$ implies that there exists T^1 such that $c_t^1 < c_{t+1}^1$ for all $t = T^1, T^1 + 1, \ldots$. The strict concavity of utility implies that $v'(c_t^1) > v'(c_{t+1}^1)$ for all $t = T^1, T^1 + 1, \ldots$. Similarly, there exists T^2 such that $c_t^2 > c_{t+1}^2$ and $v'(c_t^2) < v'(c_{t+1}^2)$ for all $t = T^2, T^2 + 1, \ldots$. Consequently, for all $t = T, T+1, \ldots$, where $T > \max[T^1, T^2]$,

$$\frac{v'(c_t^2)}{v'(c_{t+1}^2)} < 1 < \frac{v'(c_t^1)}{v'(c_{t+1}^1)} . \tag{144}$$

Notice too that there exists T^3 such that $k_{t+1}^1 > k_t^1 > k_t^2$ for all $t = T^3, T^3 + 1, \ldots$. Consequently, we can use the consumers' first-order conditions (96) to obtain

$$(1+r_{t+1}^1-\delta)\frac{p(p_{1t},p_{2t})}{p(p_{1t+1},p_{2t+1})}=\frac{v'(c_t^1)}{\beta v'(c_{t+1}^1)}>\frac{v'(c_t^2)}{\beta v'(c_{t+1}^2)}\geq(1+r_{t+1}^2-\delta)\frac{p(p_{1t},p_{2t})}{p(p_{1t+1},p_{2t+1})}, \quad (145)$$

which implies that

$$r_{t+1}^1>r_{t+1}^2. \quad (146)$$

Lemma 1 says that $k_{t+1}^1>k_{t+1}^2$ and $r_{t+1}^1>r_{t+1}^2$ is not possible in equilibrium, however, which is the desired contradiction.

To prove that $\gamma^i>1$ for all i implies that $\gamma^i=\gamma$ for all i, assume, to the contrary and without loss of generality, that $\gamma^1>\gamma^2>1$. Consider the function

$$h(\gamma^i)=\lim_{c\to\infty}\frac{v'(c)}{v'(\gamma^i c)},\ i=1,...,n. \quad (147)$$

The strict concavity of v implies that $h'(\gamma^i)>0$, which implies that $h(\gamma^1)>h(\gamma^2)$. Since $\lim_{t\to\infty}c_{t+1}^i/c_t^i=\gamma^i>1$ and v' is continuous, there exists T such that

$$\frac{v'(c_t^1)}{v'(c_{t+1}^1)}>\frac{v'(c_t^2)}{v'(c_{t+1}^2)} \quad (148)$$

for all $t=T,T+1,...$. Consequently, following the same logic as in the argument that $\gamma^i>1$ for some i implies that $\gamma^i>1$ for all i, we can show that $k_t^1>k_t^2$ and $r_t^1>r_t^2$, which lemma 1 says is not possible in equilibrium and which is the desired contradiction.

Since $\gamma^i=\gamma$ for all i, we know that

$$\lim_{t\to\infty}(1+r_t^i-\delta)\frac{p(p_{1t},p_{2t})}{p(p_{1t+1},p_{2t+1})}=h(\gamma), \quad (149)$$

which implies that

$$\lim_{t\to\infty}r_t^i=r \quad (150)$$

for all i. The rest of the proof is a modification of the proofs of propositions 4 and 5. ■

Proof of proposition 7: Assume that the model has a cycle of periodicity s. Since $\hat{x}_t^i>0$, the first-order conditions for the consumer's problem (6) are satisfied with equality. Multiplying these conditions for $t=1,...,s+1$ and using the fact that $c_{j1}^i=c_{js+1}^i$, $j=1,2$, and $p_{11}=p_{1s+1}$, we obtain

$$\beta^s\prod_{t=1}^{s+1}(1-\delta+r_{t+1}^i)=1. \quad (151)$$

Assume that there is a period in which rental rates are not equal in two countries, that is, without loss of generality, assume that $r_t^1<r_t^2$ for some t, $t=1,...,s$. Equation (151) implies that there exists $t'=1,...,s+1$ such $r_{t'}^1>r_{t'}^2$. Lemma 1 implies that $k_t^1>k_t^2$ and $k_{t'}^1<k_{t'}^2$, which contradicts proposition 3. ■

Proof of proposition 8: Assume, to the contrary, that an equilibrium cycle exists. Proposition 7 says that factor prices are equalized along the equilibrium path. Therefore, since $x_t^i > 0$ for all i and t, there exists an equilibrium cycle for the integrated economy. Since the model is one-sector aggregatable, the equilibrium of the integrated economy solves the one-sector social planner's problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t v(c_t) \\ &\text{s.t. } c_t + x_t \le F(k_t, 1) \\ &k_{t+1} - (1-\delta)k_t \le x_t \\ &c_t \ge 0,\ x_t \ge 0 \\ &k_0 \le \bar{k}_0 . \end{aligned} \tag{152}$$

Here, F is the production function

$$\begin{aligned} &F(k,\ell) = \max f(y_1, y_2) \\ &\text{s.t. } y_j = \phi_j(k_j, \ell_j),\ j = 1,2 \\ &k_1 + k_2 \le k \\ &\ell_1 + \ell_2 \le \ell \\ &k_j \ge 0,\ \ell_j \ge 0 . \end{aligned} \tag{153}$$

See Bajona and Kehoe (2010) for details. Standard results from, for example, Stokey, Lucas, and Prescott (1989) imply that the aggregate k_t is either monotonically increasing or monotonically decreasing (or constant). Suppose, for example, that $\bar{k}_0 < \hat{k}$. Then proposition 3 implies that, since $k_t = \sum_{i=1}^{m} L^i k_t^i$ is strictly increasing, then so are all k_t^i, $i = 1,...,n$, which implies that there are no cycles. ■

Proof of proposition 9: Assumption A.5' implies that, for given prices, the solution to the consumer's maximization problem (15) is unique. In a steady state with factor price equalization, each generation in every country faces the same prices and, therefore, chooses the same consumption and saving plans. To see that the steady state is interior to the cone of diversification, observe that, since $\delta > 0$, there is positive investment in every state and, consequently, assumption A.3 implies that a positive amount of both goods is produced in every country. ■

Proof of proposition 10: Proposition 9 implies that, in a steady state, all countries produce positive amounts of each good, $j = 1,2$. Assumption A.1 implies that the functions κ_j, $j = 1,2$, that determine the cone of diversification are continuous functions of prices. Therefore, there exists a period T such that for all $t = T, T+1,...$ the equilibrium is inside the cone of diversification and factor prices are equalized. Furthermore, all generations born after period $t = T + m$ face the same prices. The same argument as the one used in proposition 9 ensures that these generations choose the same consumption and saving patterns, making the equilibrium autarkic. ■

Proof of proposition 11: See the proof of theorem 1 in Fisher (1992). ■

Appendix 3. Derivations

The first-order conditions for utility maximization and for profit maximization in the investment sector imply that

$$\frac{ac_2^i}{(1-a)c_2^i} = \frac{ax_2^i}{(1-a)x_2^i} = \frac{p_1}{p_2}. \tag{154}$$

Feasibility and $L^1 = L^2 = 1$ imply that

$$\frac{c_2^i}{c_1^i} = \frac{x_2^i}{x_1^i} = \frac{c_2^i + x_2^i}{c_1^i + x_1^i} = \frac{y_2^1 + y_2^2}{y_1^1 + y_1^2}, \tag{155}$$

which implies that

$$a = \frac{y_1^1 + y_1^2}{y_1^1 + y_1^2 + y_2^1 + y_2^2}. \tag{156}$$

If $p_1 = p_2 = q = 1$, we can use the first-order condition for profit maximization into the investment sector with respect to x_1^i to obtain

$$1 = ad\left(\frac{y_1^1 + y_1^2}{y_2^1 + y_2^2}\right)^{1-a}, \tag{157}$$

which implies that

$$d = \frac{1}{a}\left(\frac{y_1^1 + y_1^2}{y_2^1 + y_2^2}\right)^{1-a} = \frac{y_1^1 + y_1^2 + y_1^1 + y_1^2}{y_1^1 + y_1^2}\left(\frac{y_1^1 + y_1^2}{y_2^1 + y_2^2}\right)^{\frac{y_1^1 + y_1^2}{y_1^1 + y_1^2 + y_2^1 + y_2^2}}. \tag{158}$$

By choosing d and a so that $p_1 = p_2 = 1$, we ensure that trade is balanced. To see that this is so, we add up budget constraints in the steady state of the one-sector model to obtain

$$\hat{c}^i + \hat{x}^i = \sum_{h=1}^{2}(\hat{c}^{ih} + \hat{x}^{ih}) = \sum_{h=1}^{2}\hat{w}^i\bar{\ell}^h + \hat{r}^i\hat{k}^i = \hat{w}^i + \hat{r}^i\hat{k}^i. \tag{159}$$

We construct the two sectors so that

$$\hat{w}^i + \hat{r}^i\hat{k}^i = \hat{p}_1\hat{y}_1^i + \hat{p}_2\hat{y}_2^i. \tag{160}$$

Since $p_1 = p_2 = 1$, we know that $\hat{c}^i = d(\hat{c}_1^i)^a(\hat{c}_2^i)^{1-a} = \hat{c}_1^i + \hat{c}_2^i$ and analogously for $\hat{x}^i$. Consequently,

$$\hat{c}_1^i + \hat{c}_2^i + \hat{x}_1^i + \hat{x}_2^i = \hat{c}^i + \hat{x}^i = \hat{y}_1^i + \hat{y}_2^i . \tag{161}$$

$$(\hat{y}_1^i - \hat{c}_1^i - \hat{x}_1^i) + (\hat{y}_2^i - \hat{c}_2^i - \hat{x}_2^i) = 0 . \tag{162}$$

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Figure 1. Lerner diagram

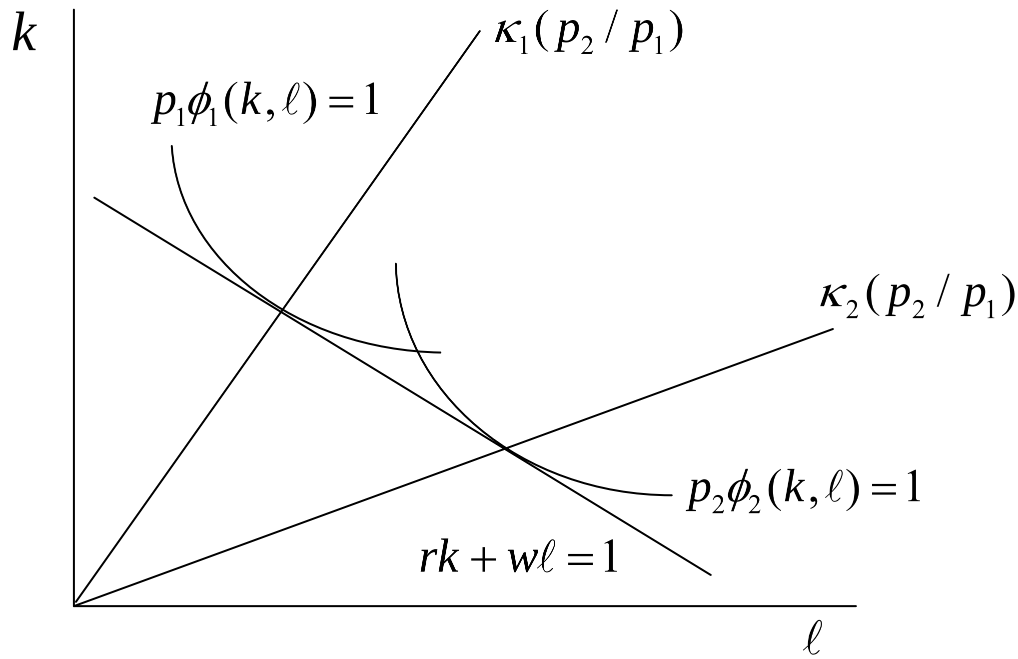

Figure 2. Revenue function

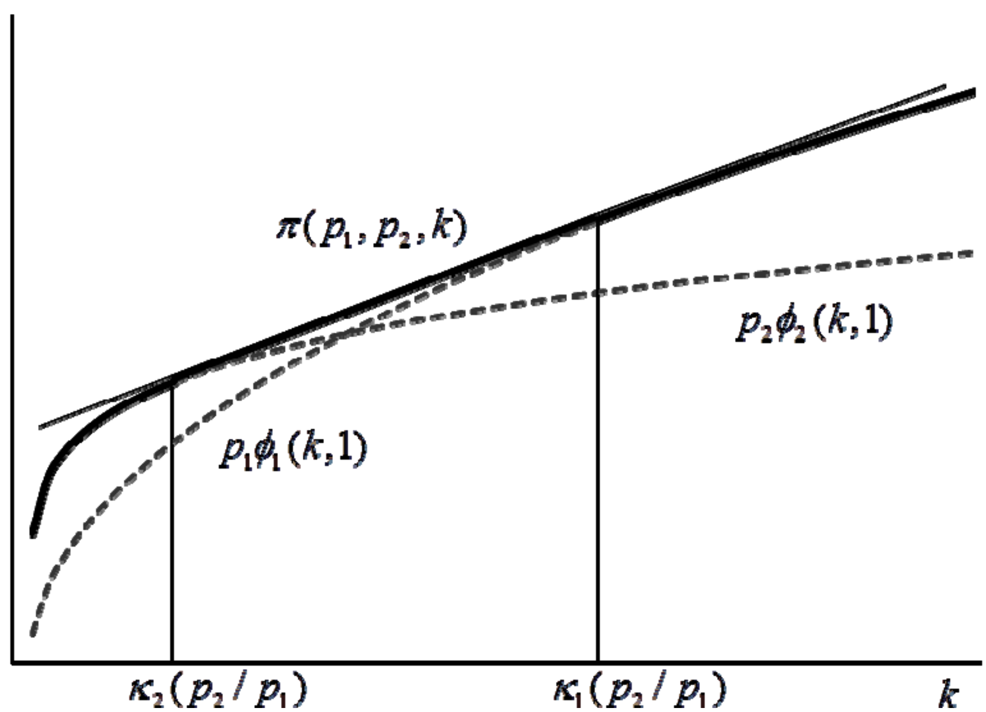

Figure 3. Steady state and cone of diversification in example 4

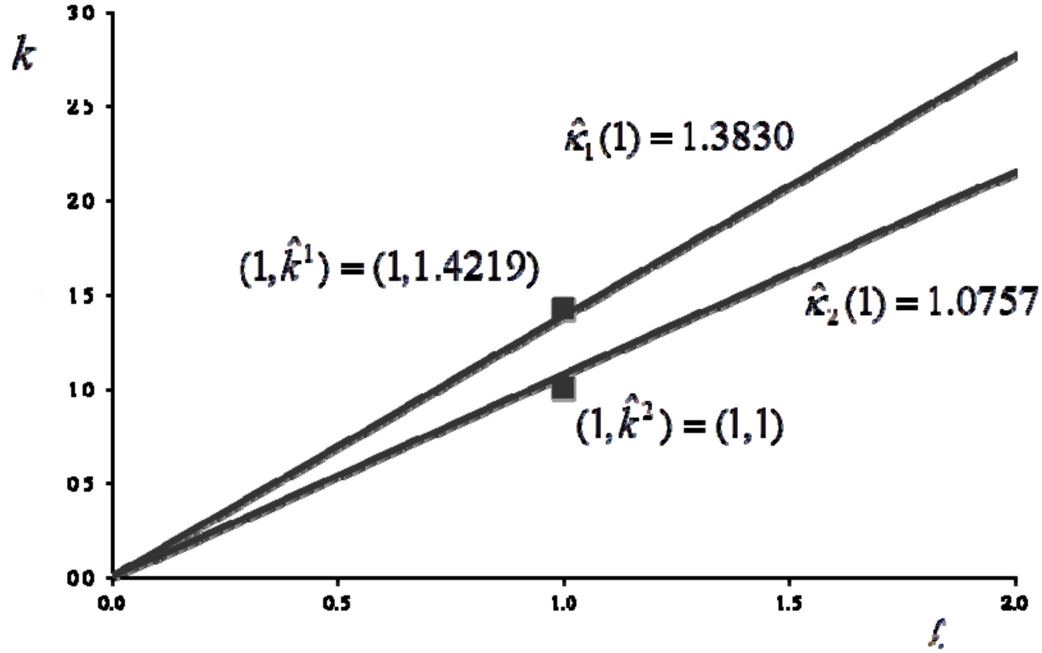

Figure 4. Steady state and cone of diversification in example 5

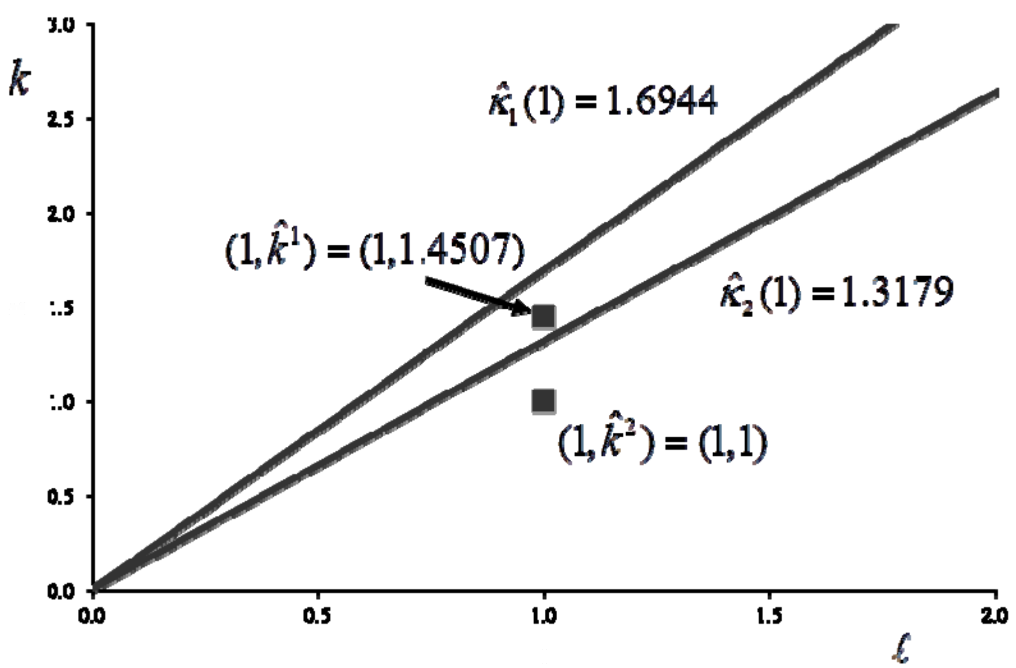

Figure 5. Cycle in the pure exchange economy in example 7