August 2024
(First version: August 2023)

Risk Loving and Fat Tails in the Wealth Distribution

Aloisio Pessoa de Araujo

IMPA and EPGE/FGV

Juan Pablo Gama

Universidade Federal de Minas Gerais

Timothy J. Kehoe

University of Minnesota,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

ABSTRACT

We study the dynamic properties of the wealth distribution in an overlapping generations model with warm-glow bequests and heterogeneous attitudes towards risk. Some dynasties of agents are risk averters, and others are risk lovers. Agents can invest in two types of Lucas trees. The two types of trees are symmetric in the sense that one type has a high return in states where the other has a return of zero. This symmetry allows risk averters to perfectly ensure their future income and eliminates aggregate uncertainty in the model. Furthermore, risk lovers take extreme portfolio positions, which make it easy for us to characterize the evolution of their wealth holdings over time. We show that the model has an equilibrium in which the aggregate wealth distribution converges to a unique invariant distribution. The invariant distribution of wealth of the risk lovers has fat tails for high bequest rates. The existence of fat tails is endogenously generated by the behavior of risk lovers rather than by the exogenous existence of fat tails in the endowments or in the returns of the assets. Therefore, the invariant distribution of wealth of risk averters does not have fat tails.

Keywords: Inequality; Overlapping generations; Invariant distribution; Fat tails; Risk loving.
JEL Codes: C62, D51, D53

*We thank participants at seminars at IMPA, FGV, and the University of Minnesota for helpful comments. We also thank participants at the SAET Conference 2023 and the SWET-SAET Conference 2023 in honor of David K. Levine. The authors gratefully acknowledge financial support from CAPES, FAPERJ, and CNPq. Gama thanks IMPA for generous financial support. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

We have written this paper to honor David K. Levine for his contributions to economics. Some of the other papers in this volume attest to David's contributions in areas of economics where the coauthors of this paper have little experience. There are areas of overlap, however, between David's contributions and our own. Like David, Aloisio Araujo and Tim Kehoe have been active members of the Society for the Advancement of Economic Theory and have served as its president. Like David, Aloisio and Tim have supervised graduate student research in economics over periods of many years. In fact, Tim was the co-supervisor of David's Ph.D. thesis at MIT. And the research project that has produced this paper started when Juan Pablo Gama was working on his Ph.D. at IMPA under Aloisio's supervision.

The connection between this paper and David is even more direct, however. In papers written with Tim, David has used simple dynamic general equilibrium models as building blocks in constructing more general theories. In that sense, the simple models in Kehoe and Levine (2001, 2008) served as building blocks for the general theory of dynamic general equilibrium with constraints on the enforcement of debt contracts developed in Kehoe and Levine (1993), even though the order of publication of these papers does not indicate the direction of the dependence. Similarly, the coauthors of this paper intend our simple model to be a building block in constructing a more general theory of dynamic general equilibrium with agents with heterogeneous attitudes towards risk. Furthermore, the formal modeling of general equilibrium with time and uncertainty, and even the notation used in this paper, follow that in Kehoe and Levine (1993, 2001, 2008).

Different attitudes towards risk can generate an unequal allocation among agents both in the short run and in the long run. In the short run, different degrees of risk aversion generate different portfolio allocations, causing unequal distribution of returns, as Buhlman (1980) has shown. In the long run, these inequalities can generate unequal distributions of wealth and income. A number of researchers — for example, Kihlstrom and Laffont — have studied the effects of entrepreneurs in economic models due to their tendency to take more risk than other type of agents. In general, researchers have modeled entrepreneurs as risk-neutral or risk-averse agents. Nonetheless, other researchers have pointed out that some behavior of entrepreneurs is not consistent with risk aversion or risk neutrality. See Shane, Locke and Collins (2003) and Araujo, Chateauneuf, Gama and Novinski (2018).

Araujo, Chateauneuf, Gama, and Novinski (2018) and Araujo, Gama, and Suarez (2022) have incorporated risk-loving agents into static general equilibrium models. In this paper, we analyze the properties of equilibria in dynamic general equilibrium models with both risk-averse and risk-loving agents. We analyze the existence of fat tails in the wealth distribution. We study an overlapping generation model with warm-glow bequests, as in Andreoni (1989), and two types of Lucas trees. We show that, in this model, there is a unique invariant distribution of wealth that, for some parameters, can have fat tails due to the existence of risk-loving agents. To analyze the impact of different assets on the distribution of wealth, we study a simple class of models where our assumptions of symmetry on assets and probabilities of events rule out aggregate uncertainty. We start by considering two types of dynasties — risk-averters and risk-lovers ones — that is, if an agent is a risk averse, all her predecessors and successors are risk averters, and, is and agent is a risk lover, all her predecessors and successors are risk lovers. To ensure the existence of equilibrium in this economy, we assume that there is a continuum of each type of dynasties. Araujo, Chateauneuf, Gama, and Novinski (2018) and Araujo, Gama, and Suarez (2022) show that aggregate uncertainty conditions are necessary to ensure the existence of equilibrium with a finite number of agents. Additionally, because of the behavior of risk lovers, wealth holdings tend to accumulate in the hands of a small proportion of these risk lovers, which generates fat tails in the distribution of wealth.

We start by briefly discussing the literature on wealth and income inequality. Piketty and Saez (2014) argue that the gap between the interest rate and the real GDP growth rate has a strong impact on inequality. Authors such as Lindert and Williamson (2016) study long-run data to analyze the inequality in the U.S. economy since colonial times. Most of this research generates distributions of wealth and income similar to Pareto distributions. Moreover, Benhabib, Bisin, and Zhu (2011, 2015, 2016) find conditions to generate fat tails for transformation processes induced by investment risk. On the other hand, Beare and Toda (2017) show that tails of wealth distribution decay exponentially in a heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk. Additionally, Gouin-Bonenfant and Toda (2023) develop an analytical framework to analyze models with heterogeneous risk-averse agents that endogenously generates fat tails wealth distributions. Our model has more similarities with Benhabib, Bisin, and Zhu (2011, 2015, 2016), however, than with Beare and Toda (2017) or with Gouin-Bonenfant and Toda (2023), since the fat tails in our model are a consequence of risk lovers' extreme specialization in their asset holdings and not of fat tails in the distribution of endowments.

In Section 2, we set up the model. In Subsection 2.1, we define the concept of equilibrium and, in Subsection 2.2, we characterize some basic properties of the equilibrium and the consumption plans of the agents. In Section 3, we analyze the dynamic properties of the equilibrium. In Subsection 3.1, we develop the analytical expressions for the invariant wealth distribution and present results con convergence to this distribution. In Subsection 3.2, we present results on fat tails of this distribution. In Subsection 3.3, we explain how we can obtain all of our results in an alternation model where agents face idiosyncratic uncertainty. In Section 4, we generalize the model to allow dynasties to change their attitude towards risk over time. Finally, in Section 5, we provide some concluding remarks.

2. Model

We analyze an overlapping generation model with warm-glow bequests and uncertainty. Our model has two events, 1 and 2, that can occur in each period. As in Kehoe and Levine (1993), a state $s = (\eta_1, \eta_2, \dots, \eta_t)$ at date $t \geq 1$ is a history of events $\eta_k \in \{1,2\}$ for $k \in \{1,2,\dots,t\}$. Each event η_k is equally probable. In each state s, there is only one consumption good. There are two types of agents with different attitudes towards risk, risk lovers and risk averters, each of which has measure 1. We denote risk lovers as $\{l_i\}_{i\in[0,1]}$ and risk averters agents as $\{a_i\}_{i\in[0,1]}$. There are two assets, two types of Lucas trees of measure 1, with real returns $(R_{1,1}, R_{1,2}) = (R, 0)$ for the asset 1 and $(R_{2,1}, R_{2,2}) = (0, R)$ for the asset 2 with $R > 0$ in each state with $t \geq 1$, where $R_{j,1}, R_{j,2}$ are the returns of asset j in event 1 and event 2, respectively. In $t = 1$, asset 1 has a return of $R_{1,1} = R$, and the asset 2 has no return, $R_{2,1} = 0$. In other words, we choose to start with $\eta_1 = 1$. In $t = 1$, the old generation, whom we refer to as generation 0 because we imagine them as having been born before the period before the model starts, has initial amount of assets: $(\theta_{1,0}^{a_i}, \theta_{2,0}^{a_i})$ for risk averters and $(\theta_{1,0}^{l_i}, \theta_{2,0}^{l_i})$ for risk lovers where $\theta_{j,0}^{a_i}$ is the asset holdings of tree j of risk averter a_i

with $i \in [0,1]$, and $\theta^{l_i}_{j,0}$ is the asset holdings of tree j of risk lover l_i with $i \in [0,1]$. The initial asset holdings satisfy

$$\int_0^1 \theta^{a_i}_{j,0}\, di + \int_0^1 \theta^{l_i}_{j,0}\, di = 1 \text{ for } j = 1,2.$$

A consumer i from the old generation uses the Lucas trees and their returns to purchase consumption, $c^i_{(1)}$, and to leave a bequest to her successor, $b^i_{(1)}$. In particular, a risk-averse consumer $a_i \in [0,1]$ solves the problem

$$\begin{array}{lrcl} \max & & & U^{a_i}_0(c,b) = \big((1-\delta)\log c + \delta \log b\big) \\ \text{s.t.} & c+b & \leq & \big(R_{1,(1)} + q_{1,(1)}\big)\theta^{a_i}_{1,0} + \big(R_{2,(1)} + q_{2,(1)}\big)\theta^{a_i}_{2,0} \\ & 0 & \leq & c, b. \end{array}$$

where $\delta \in (0,1)$ is the bequest rate and $q_{j,s}$ is the price of the asset j in state s. Bequests are transfers of wealth from the old generation to the new one. Notice that we normalize the price of the consumption good to be equal to 1 in all states. The initial wealth of risk averter a_i from the old generation is $w^{a_i}_0 = \sum_{j=1,2}\big(R_{j,(1)} + q_{j,(1)}\big)\theta^{a_i}_{j,0}$. Similarly, a risk-loving consumer $l_i \in [0,1]$ solves the problem

$$\begin{array}{lrcl} \max & & & U^{l_i}_0(c,b) = \big(c^{1-\delta} b^{\delta}\big)^2 \\ \text{s.t.} & c+b & \leq & \big(R_{1,(1)} + q_{1,(1)}\big)\theta^{l_i}_{1,0} + \big(R_{2,(1)} + q_{2,(1)}\big)\theta^{l_i}_{2,0} \\ & 0 & \leq & c, b. \end{array}$$

In the first period of her life, an agent receives a bequest $b^i_s \geq 0$ from her predecessor and an endowment $\omega > 0$, and the agent decides on purchases of assets that pay off in states $(s,1)$ and $(s,2)$. In second period of her life, the agent decides on consumption and the bequest that she leaves to her successor depending on the returns of the Lucas trees $R_{j,\eta}$ for $j, \eta = 1,2$. A risk averter, $a_i \in [0,1]$, in state s at date $t \geq 1$ solves the problem

$$\begin{array}{lrcll} \max & & & U^{a_i}(c,b) = \frac{1}{2}\big((1-\delta)\log c_1 + \delta \log b_1\big) + \frac{1}{2}\big((1-\delta)\log c_2 + \delta \log b_2\big) & \\ & q_{1,s}\theta_1 + q_{2,s}\theta_2 & \leq & \omega + b^{a_i}_s, & \\ \text{s.t.} & c_\eta + b_\eta & \leq & \big(R_{1,(s,\eta)} + q_{1,(s,\eta)}\big)\theta_1 + \big(R_{2,(s,\eta)} + q_{2,(s,\eta)}\big)\theta_2 & \text{for } \eta = 1,2, \\ & 0 & \leq & c_1, c_2, b_1, b_2, & \end{array}$$

The solution to the problem is a consumption plan, a bequest plan, and a Lucas tree portfolio $\left(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i}\right)$ that maximizes the utility of the agent a_i subject to her budget constraints in state s, state $(s,1)$, and state $(s,2)$.

A risk lover, $l_i \in [0,1]$, in state s at date $t \geq 1$ solves the problem

$$
\begin{array}{llcll}
\max & & & U^{l_i}(c,b) = \frac{1}{2}\left(c_1^{1-\delta} b_1^{\delta}\right)^2 + \frac{1}{2}\left(c_2^{1-\delta} b_2^{\delta}\right)^2 & \\
& q_{1,s}\theta_1 + q_{2,s}\theta_2 & \leq & \omega + b_s^{l_i}, & \\
\text{s.t.} & c_\eta + b_\eta & \leq & \left(R_{1,(s,\eta)} + q_{1,(s,\eta)}\right)\theta_1 + \left(R_{2,(s,\eta)} + q_{2,(s,\eta)}\right)\theta_2 & \text{for } \eta = 1,2, \\
& 0 & \leq & c_1, c_2, b_1, b_2. &
\end{array}
$$

Notice that the risk averters allocate the same proportions, $1-\delta$ and δ, of wealth to consumption and bequests in each event. Risk lovers, however, have extreme behavior: They choose to allocate their consumption and bequests as much as possible to one event in their second period of life depending on the prices of the two assets available in the economy. If the prices are such that it is cheaper to invest in event 1 instead of event 2, all risk lovers specialize in event 1. If the prices of the two assets are the same, however, a positive measure of the agents can specialize in each of the events. Notice that, in the model, both types of agents allocate the same proportions, $1-\delta$ and δ, of their wealth to consumption and bequests.

2.1. Equilibrium and equilibrium allocation

We now define an equilibrium for the model:

Definition 1. A sequence $\left(q_s, \left(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i}\right)_i, \left(c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i}\right)_i\right)_s$ is an equilibrium for the economy if, for each state s,

$$
\begin{array}{lcll}
\int_i \theta_{j,s}^{a_i} di + \int_i \theta_{j,s}^{l_i} di & = & 1 & \text{for } j = 1,2, \\
\int_i c_s^{a_i} di + \int_i c_s^{l_i} di & = & R + 2\omega &
\end{array}
$$

where $\left(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i}\right)$ is the optimal solution for risk averter a_i, and $\left(c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i}\right)$ is the optimal solution for risk lover l_i.

2.2 Characterization of equilibrium and equilibrium allocation

Lemma 1. In any equilibrium, the prices of two trees need to be identical at every state s:

$$q_{j,s} = \frac{2\omega + \delta R}{2(1-\delta)}, \qquad j = 1,2\,.$$

This result, whose proof can be found in the Appendix A, allows us to use the agents' contingent-claims market problem to calculate the equilibrium allocations. Based on the state-contingent price vectors, the Lucas tree asset prices satisfy

$$q_{1,s} = p_{(s,1)}\big(R + q_{1,(s,1)}\big) + p_{(s,2)}\big(q_{1,(s,2)}\big) \tag{2.1}$$

and

$$q_{2,s} = p_{(s,1)}\big(q_{2,(s,1)}\big) + p_{(s,2)}\big(R + q_{2,(s,2)}\big). \tag{2.2}$$

This allows us to transform the budget constraints in state s, state $(s,1)$, and state $(s,2)$ into a single budget constraint in state s using the state-contingent price vector $\big(p_{(s,1)}, p_{(s,2)}\big)$:

$$\begin{aligned} p_{(s,1)}(c_1 + b_1) + p_{(s,2)}(c_2 + b_2) &\leq \omega + b_s^i \\ c_1, c_2, b_1, b_2 &\geq 0, \end{aligned}$$

for a state of length $t \geq 1$.

Note that the Lucas trees are long-lived assets in positive net supply. Consequently, there is no asset pricing bubble for either asset in any state s, see Santos and Woodford (1997). Additionally, since we have not imposed any short-sale constraints on purchases of the Lucas trees, we can do this conversion from the sequential markets budget constraints to contingent-claims market budget constraints as long as there is no state s in which the two trees have the same gross return in the next period, whether they bear fruit or not. Then, markets must be complete, that is, the matrix

$$V(s) = \begin{pmatrix} R + q_{1,(s,1)} & q_{2,(s,1)} \\ q_{1,(s,2)} & R + q_{2,(s,2)} \end{pmatrix}$$

has full rank at prices q for every state s. See Hernandez and Santos (1996).

Consequently, an optimal allocation of a risk averter satisfies,

$$\frac{\delta}{(1-\delta)} c_{(s,1)}^{a_i} = b_{(s,1)}^{a_i} \text{ and } \frac{\delta}{(1-\delta)} c_{(s,2)}^{a_i} = b_{(s,2)}^{a_i} \tag{2.3}$$

for every state s. Therefore,

$$c_{(s,k)}^{a_i} = \frac{(1-\delta)}{2p_{(s,k)}}\left(\omega + b_s^{a_i}\right) \text{ for } k = 1,2. \tag{2.4}$$

In contrast, each risk lover specializes as much as possible in one event, but she also distributes her consumption and bequests as in Equation 2.3, since both agents distribute their wealth between consumption and bequests in the same proportion. Therefore, if $p_{(s,1)} < p_{(s,2)}$,

$$c_{(s,1)}^{l_i} = \frac{(1-\delta)}{p_{(s,1)}}\left(\omega + b_s^{l_i}\right) \text{ and } c_{(s,2)}^{l_i} = 0, \tag{2.5}$$

and analogously to the case in which $p_{(s,1)} > p_{(s,2)}$, for any state s. In these cases, all risk lovers invest in the same state generating problems with the symmetry with the aggregate endowments.

Lemma 2. There is no equilibrium in which the Arrow-Debreu state contingent prices satisfy $p_{(s,1)} < p_{(s,2)}$ or $p_{(s,1)} > p_{(s,2)}$ for any state s. Consequently, there is a unique sequence of Arrow-Debreu state contingent prices, and are given by

$$p_{(s,j)} = \frac{2\omega + \delta R}{2(2\omega + R)} \; \forall s, j.$$

This lemma implies that, because of the symmetry mentioned above, there is only a symmetric equilibrium in which $p_{(s,1)} = p_{(s,2)}$ and half of the aggregate wealth of each type of type of agent is held in each type of asset. The risk averters individually choose to divide their asset holdings evenly between the two types of Lucas trees. The risk lovers individually choose to hold only one type of asset, but their aggregate asset holdings are split evenly between the two types of trees.

The optimality conditions of risk averters and risk lovers imply that both aggregate consumption and aggregate bequests are constant fractions of aggregate wealth. Since in every state, half of aggregate wealth is transferred into each of the two future states, in equilibrium there is no aggregate uncertainty, and aggregate bequests are constant across the states.

We can now characterize the equilibrium allocation in the sequential markets version of the model. Lemma 1 says that, for $j = 1,2,$ and all s,

$$q_{j,s} = \frac{2\omega + \delta R}{2(1-\delta)} \ .$$

Using the optimality conditions, the Arrow-Debreu state-contingent budget constraint, and the Lucas tree asset prices, we can solve for the portfolio allocation of a risk averter a_i in the first period of her life:

$$\theta_{1,s}^{a_i} = \theta_{2,s}^{a_i} = \frac{(1-\delta)\left(\omega + b_s^{a_i}\right)}{2\omega + \delta R}.$$

In contrast, because of the symmetry of the returns of the Lucas trees and the asset prices, any risk lover l_i is indifferent between investing all her wealth in event 1 or in event 2. Then, the portfolio allocation in the first period of her life is either

$$\theta_{1,s}^{l_i} = \frac{(R + q_{1,s})(2(1-\delta))\left(\omega + b_s^{l_i}\right)}{R(2\omega + \delta R)}, \theta_{2,s}^{a_i} = -\frac{2q_{2,s}(1-\delta)\left(\omega + b_s^{l_i}\right)}{R(2\omega + \delta R)}$$

if agent l_i decides to invest everything in event 1 (in this case, $c_{(s,2)}^{l_i} = b_{(s,2)}^{l_i} = 0$), or

$$\theta_{1,s}^{l_i} = -\frac{2q_{1,s}(1-\delta)\left(\omega + b_s^{l_i}\right)}{R(2\omega + \delta R)}, \theta_{2,s}^{l_i} = \frac{(R + q_{2,s})(2(1-\delta))\left(\omega + b_s^{l_i}\right)}{R(2\omega + \delta R)}$$

if agent l_i decides to invest everything in event 2 (in this case, $c_{(s,1)}^{l_i} = b_{(s,1)}^{l_i} = 0$) in the second period of her life.

3. Analysis of equilibria and the invariant distribution of wealth

Before we analyze the invariant distribution of wealth, let us point out a particularity of the model: Although the equilibrium prices are unique, the equilibrium allocation is not unique. At each state, there is a continuum of distributions in which the risk lovers could specialize if $p_{(s,1)} = p_{(s,2)}$. We have argued that in equilibrium half of the aggregate wealth of risk lovers must be invested in each type of tree, but this does not enable us to identify which risk lovers hold tree 1 and which hold tree 2.

Lemma 3. There is a continuum of equilibrium allocations for the economy.

3.1. Invariant distribution with constant distribution of endowment among time

For risk averters, the wealth distribution is given by $w_1^{a_i} = \left(q_{1,(1)} + R\right)\theta_{1,0}^{a_i} + q_{2,(1)}\theta_{2,0}^{a_i}$ in $t = 1$. And therefore, $w_{(1,1)}^{a_i} = w_{(1,2)}^{a_i} = \omega + \delta w_1^{a_i}/\left(2p_{(1,1)}\right)$ where

$$p_{(1,1)} = p_{(1,2)} = p_{(s,1)} = p_{(s,2)} = \pi = \frac{2\omega + \delta R}{2(2\omega + R)},$$

and $w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = \omega + \delta w_s^{a_i}/\left(2p_{(s,1)}\right)$ for all state s of length $t \geq 1$. Then,

$$w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = \sum\nolimits_{k=0}^{t} \left(\frac{\delta}{2\pi}\right)^k \omega + \left(\frac{\delta}{2\pi}\right)^t w_0^{a_i}$$

if

$$q_{1,1} = q_{1,2} = \frac{q_{1,1} + q_{1,2}}{2} = \frac{2\omega + \delta R}{2(1-\delta)},$$

which implies that $w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = w_{(s',1)}^{a_i} = w_{(s',2)}^{a_i}$ for all states s of length $t \geq 1$ and

$$\lim_{t\to\infty} w_{(s,1)}^{a_i} = \frac{2\pi}{(2\pi - \delta)}\omega = \frac{2\omega + \delta R}{2\omega(1-\delta)}\omega = \frac{\omega + R/2}{(1-\delta)},$$

the wealth distribution converges to a constant value that depends only on ω.

We focus our attention on equilibria with a symmetry on how risk lovers specialize.

Assumption RL. If the risk lovers are indifferent between investing in either event, we assume that half of the agents specialize in event 1 and half in event 2. Moreover, we assume that, for agents with the same level of wealth, half specialize in event 1 and half in event 2.

RL is an assumption that ensures that risk lovers invest symmetrically between the two states. To ensure that this assumption holds, we could assume that, when risk lovers are indifferent between two asset holdings, they randomly choose one with probability 1/2. The advantage of studying equilibrium allocations that satisfy assumption RL is that we can characterize a unique invariant distribution of wealth holdings.

We can now characterize the wealth distribution for risk lovers:

- For $t = 1$, we have that $w_0^{l_i} = (q_{1,1} + R)\theta_{1,0}^{l_i} + q_{2,1}\theta_{2,0}^{l_i}$.
- For $t = 2$, we have that $w_{(1,1)}^{l_i} = \omega$ with measure 1/2 of the risk lovers and $w_{(1,1)}^{l_i} = \omega + \delta w_0^{l_i}/\pi$ with measure 1/2.
- For $t = 3$, we have that, for a state s, $w_s^{l_i}$ has the following distribution:
  - ω with measure 1/2,
  - $\omega + \delta\omega/\pi$ with measure 1/4,
  - $\omega + \delta(\omega + \delta/\pi w_0^{l_i})/\pi$ with measure 1/4.
- For $t = 4$, we have that, for a state s, $w_s^{l_i}$ has the following distribution:
  - ω with measure 1/2,
  - $\omega + \delta\omega/\pi$ with measure 1/4,
  - $\omega + \delta(\omega + \delta/\pi\omega)/\pi$ with measure 1/8,
  - $\omega + \delta\left(\omega + \delta/\pi(\omega + \delta/\pi w_0^{l_i})\right)/\pi$ with measure 1/8,
- Therefore, recursively, for any state s with length $t \geq 1$, $w_s^{l_i}$ has the distribution:
  - $\sum_{k=0}^{m}(\delta/\pi)^k\omega$ with measure $1/2^{m+1}$ for $m = 0, \ldots, t-2$,
  - $\sum_{k=0}^{t-2}(\delta/\pi)^k\omega + (\delta/\pi)^{t-1}w_0^{l_i}$ with measure $1/2^{t-1}$.

Notice that the initial distribution of wealth $w_0^{l_i}$ disappears over time because all dynasties of risk lovers eventually have bad luck with probability that approaches 1. Consequently, the wealth distribution for risk lovers tends to

$$\left(w_\infty^{l_i}\right)_i : \sum_{k=0}^{n}(\delta/\pi)^k\omega \text{ with measure (or proportion) } 1/2^{n+1} \text{ for } n \in \mathbb{N}.$$

Notice that RL implies the convergence to the invariant distribution for any initial wealth distribution and the uniqueness of the equilibrium if the initial wealth distribution of risk lovers is discrete. Also notice that although RL does not imply uniqueness of the equilibrium, it does ensure that all the possible wealth distributions in equilibrium converge to the unique invariant distribution w_∞ .

Proposition 4. Under RL, there is a unique invariant distribution for the agents $(w_\infty^i)_i$ and any initial portfolio distribution converges to an invariant distribution, $(w_\infty^{l_i})_i$.

Intuitively, in any given state, half of the risk lovers have no return from the Lucas trees, which implies that their wealth from that state onwards does not depend on their initial wealth, $w_0^{l_i}$, in any way. Additionally, because of RL, with probability 1, all risk lovers fall to the bottom of the wealth distribution at least once in the long run. Consequently, the proportion the risk lovers that do not follow the distribution $(w_\infty^{l_i})_i$ at date t is bounded by $1/2^t$.

The dynamics of wealth distribution of the risk lovers have a positive reflecting barrier; see Benhabib, Bisin, and Zhu (2011, 2015, 2016). With probability 1, the wealth a poor risk lover can reach any threshold of the wealth process generated by risk lovers that were at the bottom of the distribution at least once. Therefore, due to all these properties, for any given any initial wealth distribution, the wealth distribution converges to an invariant distribution that is also unique.

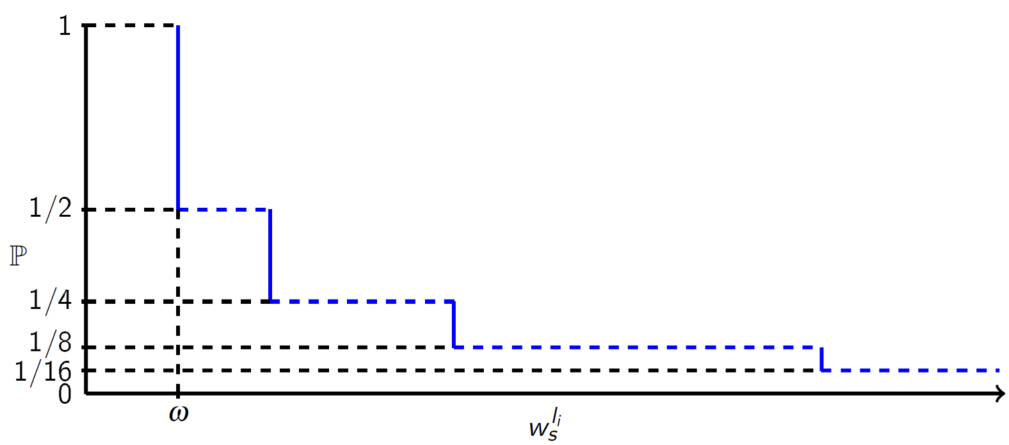

Figure 1. Invariant wealth distribution for risk lovers with high bequest rate ($\delta \geq 2\omega/(4\omega + R)$).

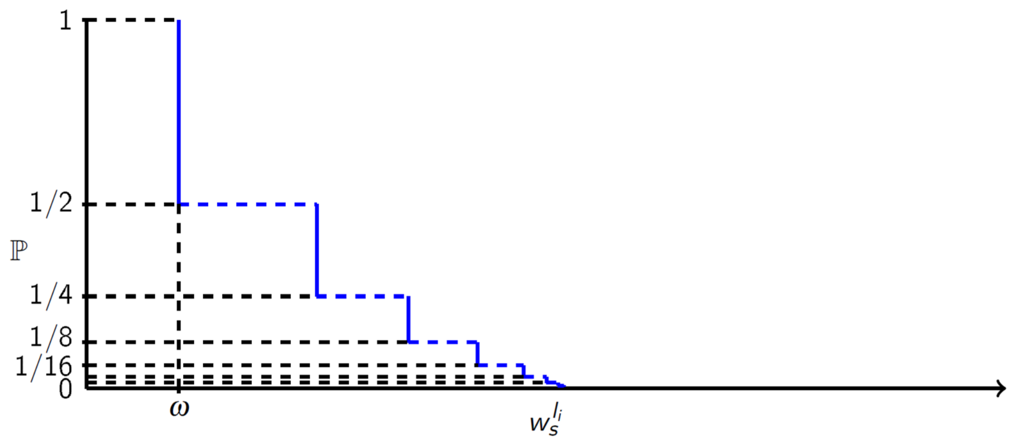

Figure 2. Invariant wealth distribution for risk lovers with low bequest rate ($\delta < 2\omega/(4\omega + R)$).

3.2. Fat tails in the wealth distribution

As we can see in Figure 1 and 2, the shape of the invariant distribution depends on the level of bequest of the agents. For high bequest rates, $\delta \geq \pi$ — that is, $\delta \geq 2\omega/(4\omega + R)$ — gains given by the extreme behavior of risk lovers make it so that the agents whose predecessors were lucky to have invested in the events that in fact occurred become extremely wealthy over time. For low bequest rates, $\delta < 2\omega/(4\omega + R)$, however, the gains from having lucky ancestors have little impact on their successors' wealth. Therefore, when agents invest correctly, their successors' wealth increases by an amount that converges to zero in the long run. For an agent whose dynasty always bets correctly, its wealth converges to

$$\sum_{k=0}^{\infty} \left(\frac{\delta}{\pi}\right)^k \omega = \frac{\pi}{(\pi - \delta)} \omega = \frac{(2\omega + \delta R)}{\big((1 - 2\delta)2\omega - \delta R\big)} \omega.$$

Consequently, a low level of bequests implies bounded levels of wealth even for risk lovers whose predecessors were always lucky.

Comparing the two types of agents, we can observe that bequests affect the risk lovers more than they do the risk averters. The wealth of risk-averse agents increases when the bequest rate increases. Their distribution is still constant, however, and their proportion of the aggregate wealth is also constant. On the other hand, wealth inequality among risk lovers increases as the bequest rate increases. The poorest risk lovers always have the same level of wealth. Furthermore, the wealth of the other risk lovers is strictly higher when the bequest rate increases, and the increase in wealth of the risk lovers increases with the level of wealth. Since the aggregate consumption and wealth proportion of risk lovers are maintained with all levels of bequest rates, a low bequest rate implies a higher share of wealth in the hands of the poorest risk lovers and a lower share of wealth in the hands of the richest risk lovers (see Figure 2).

Notice that the invariant distribution of wealth depends on the return of the Lucas trees, the endowment of the agents, and the bequest rate of the agents. A larger bequest rate implies larger inequality levels. Moreover, if the bequest rate is lower than $2\omega/(4\omega+R)$, inequality is so low that the invariant distribution of wealth is bounded by $(2\omega+\delta R)\omega/\big((1-2\delta)2\omega-\delta R\big)$. Above that threshold, however, the invariant distribution of wealth is unbounded. One of the properties that can be analyzed in this case is the existence of fat tails in the invariant distribution of wealth. The following definition of a fat tail depends on whether the tail of the distribution has an exponential decay. Usually, authors differentiate the tail of the distribution based on the speed of the decay; see Bryson (1974).

Definition 2. Given a random variable W, the distribution F_W has exponential fat tails if there exists $\alpha \geq 0$ such that for every w,

$$\liminf_{w\to\infty} \frac{\log\big(1-F_W(w)\big)}{\log w} = -\alpha.$$

The following proposition ensures that, for high bequest rates, the invariant distribution of wealth of the risk lovers has fat tails.

Proposition 5. If $\delta \geq 2\omega/(4\omega+R)$, the invariant distribution of the risk lovers, $\left(w_\infty^{l_i}\right)_i$, has exponential fat tails, with $\alpha = \log_2 2\beta$, with

$$\beta = \frac{\delta(2\omega+R)}{(2\omega+R)-(1-\delta)R}.$$

Moreover,

$$P\left[i: w_\infty^{l_i} > w\right] = \frac{1}{2^{n+2}} \propto \frac{1}{2^{\log_{2\beta} w}}$$

where $w \in [\sum_{k=0}^{n}(\delta/\pi)^k\omega, \sum_{k=0}^{n+1}(\delta/\pi)^k\omega]$ for all $n \geq 1$.

The proof of Proposition 5 is in the Appendix A.

Note that, if $\delta \geq 2\omega/(4\omega + R)$, the constant β satisfies $\beta \geq 1$. Moreover, β increases with the bequest rate. That is, fat tails become fatter as the bequest rate increases.

Several researchers have shown conditions for the existence of fat tails in wealth distributions based on properties of the wealth transformation process (see Benhabib, Bisin, and Zhu, 2016). Other authors showed that tails of wealth distribution decay exponentially in a heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk (see Beare and Toda, 2017). In our case, the existence of fat tails is due to the existence of risk loving agents who specialize in such a way that generate a large concentration of wealth in the long run.

3.3. Alternative model with idiosyncratic uncertainty

We have specified the model with two types of Lucas trees. To calculate the invariant distribution, we have used Assumption RL to maintain symmetry. An alternative specification of the model that would have maintained the same symmetry would have been to give every risk lover access to a different Lucas tree and to assume that the returns of the different trees are independent. In fact, this is the specification used by Araujo, Gama, and Kehoe (2024).

4. Switching the type of agents: An example

We now present a simple example in which dynasties randomly switch between being risk lovers and risk averters. We make simple assumptions so that economy still has an easy-to-characterize invariant distribution.

We focus on an example in which $\omega = 1$, $\delta = 0.5$, and $R = 1$. In our example a proportion $p \in (0,1)$ of the successor of agents switch types: risk lovers become risk averse, and vice versa. For the case where types do not change from one generation to the next one, the agents give a proportion δ of their wealth as before. To have an easy-to-characterize invariant distribution, we assume that a risk lover who has a successor that is a risk averter leaves to her descendant the average bequest that risk averters receive from their predecessors. We assume that a risk averter who has a successor that is a risk lover leaves no wealth to her descendant. Additionally, we assume that only risk lovers who are sufficiently wealthy — in our example, have been lucky at least twice in a row — can have a risk averse successor.

4.1. Example without switching

In our example without switching — that is, $p = 0$ — the wealth distribution for risk lovers is

$$\left(w_\infty^{l_i}\right)_i\colon \sum_{k=0}^{n} (\delta/\pi)^k \omega = \sum_{k=0}^{n} (6/5)^k \text{ with measure (or proportion) } 1/2^{n+1} \text{ for } n \in \mathbb{N},$$

the wealth distribution for the risk averters is

$$\left(w_\infty^{a_i}\right)_i\colon 2\pi/(2\pi - 1/2) = 5/2 \text{ with measure (or proportion) } 1.$$

4.2. Example with switching

Let us now analyze the example in which a proportion of $p = 1/10$ of agents switches from risk lovers to risk averse, and vice versa. A successor of a risk-averse investor has a probability $1/10$ of becoming a risk lover, and the bequests received are equal to zero. A risk lover that has received the high return at least twice has a probability of $4/7$ of having a risk-averse successor. The bequest received by the new risk-averse agent is equal to the average bequest of the risk-averse investors. As we explain below, we have chosen the probability $4/7$ so that $1/10$ of all risk lovers have risk-averse successors.

Let us denote the invariant distribution of wealth by $\left(\hat{w}_\infty^{i}\right)_i$. Note that the invariant distribution for the risk lovers, $\left(\hat{w}_\infty^{l_i}\right)_i$, has a similar form to that in Figure 1 and 2. The proportions of investors at each wealth level are different, however.

We define y^a, y^l as the proportion of risk averters and risk lovers in the invariant distribution, respectively. If the proportion of risk lovers at the bottom $n + 1$ level of wealth is given by $y^{l^{n+1}}$, then the proportions $\left(y^{l^n}\right)_n$ satisfy the following conditions:

- $y^{l^1} = (1/10)y^a + (1/2)\left(y^{l^1} + y^{l^2} + (3/7)\left(2 - y^a - y^{l^1} - y^{l^2}\right)\right).$
- $y^{l^n} = (1/2)y^{l^{n-1}}$ for $n = 2,3,$
- $y^{l^n} = (1/2)\,(3/7)y^{l^{n-1}}$ for $n \geq 4.$

For the proportion of risk averters, we have that

- $y^a = (9/10)y^a + (4/7)\left(2 - y^a - y^{l^1} - y^{l^2}\right)$.

Since $\left(y^{l^n}\right)_n$ and y^a are the proportion of the agents in the invariant distribution, we have that $y^l = y^a = 1$.

Using this notation, we can now easily describe the derivation of the probability $p_l = 4/7$ of sufficiently wealthy risk lovers who has risk-averse successors, given that the probability $p = 1/10$ of risk averters have risk lovers as successors. We have verified that risk lovers that have been lucky twice in a row are sufficiently wealthy to leave the average bequest of risk averters as their bequest to their successors, we have chosen $p_l = 4/7$ to satisfy the equation

$$1 = y^a = (1-p)y^a + p_l\left(2 - y^a - y^{l^1} - y^{l^2}\right). \tag{4.1}$$

Since the bequest rates of agents who have successors of different type are different from $\delta = 0.5$, the aggregate bequest rate is different. In Appendix C, we show that this aggregate bequest rate $\hat{\delta} \approx 0.449$. This implies that the contingent-claims prices are equal to

$$\pi_{\hat{\delta}} = \frac{2\omega + \hat{\delta}R}{2(2\omega + R)} = \frac{2 + \hat{\delta}}{6} \approx 0.408.$$

Then, the invariant distribution for the risk lovers, $\left(\widehat{w}_\infty^{l_i}\right)_i$, is given by:

- 1 with measure (or proportion) $y^{l^1} = 0.55$,
- $1 + \left(1/(2\pi_{\hat{\delta}})\right) \approx 2.224$ with measure (or proportion) $y^{l^2} = 0.275$,
- $1 + \left(1/(2\pi_{\hat{\delta}})\right) + \left(1/(2\pi_{\hat{\delta}})\right)^2 \approx 3.725$ with measure (or proportion) $y^{l^3} = 0.1375$,
- $\sum_{k=0}^{n-1}\left(1/(2\pi_{\hat{\delta}})\right)^k$ with measure (or proportion)

$$y^{l^n} = \frac{11}{80}\left(\frac{3}{14}\right)^{n-3},$$

with $n \geq 4$.

The invariant distribution for the risk averters is

$\left(\widehat{w}_\infty^{a_i}\right)_i$: $2\pi_{\hat{\delta}}/\left(2\pi_{\hat{\delta}} - 1/2\right) \approx 2.580$ for the risk averters with a proportion of $y^a = 1$.

Note that all risk lovers that have a positive probability of becoming risk averse have wealth levels higher than the wealth of the risk averters. Then, the bequest rate of these risk lovers is lower than 0.5. For the other risk lovers, their bequest rate is equal to 0.5. For the risk averters, $9/10$ of them have a bequest rate equal to 0.5, and the remaining $1/10$ has a bequest rate equal to zero. Then, the bequest rate of all agents is at most equal to 0.5, which explains how the aggregate bequest rate $\hat{\delta}$ is lower than 0.5. Therefore, contingent-claims market prices in this example, $\pi_{\hat{\delta}} \approx 0.411$, is a slightly lower than $\pi = 5/12 = 0.41\bar{6}$ in the example without switching.

Since the aggregate bequest rate of the economy is lower than 0.5, the aggregate wealth of the economy is also lower. The aggregate wealth of the risk averters in this example, 2.580, however, is larger — in the example without switching, the aggregate wealth of the risk averters is 2.5. Additionally, all but the lowest risk-lovers' wealth levels are higher than the levels observed in the no switching case. These phenomena are consequences of a higher proportion of risk lovers at the bottom of the distribution and a lower proportion of risk lovers at the top.

Having a lower of proportion of risk lovers at the top, also results in thinner fat tails of the wealth distribution. The invariant distribution of wealth still has exponential fat tails, but now

$$\alpha = \frac{\log_2\left(\frac{3}{2+\hat{\delta}}\right)}{\log_2\left(\frac{14}{3}\right)} \approx 0.132,$$

while, in the example without switching, $\alpha = \log_2(6/5) \approx 0.263$. From this example, we conclude that the slightly larger levels of wealth of the rich risk lovers in the example with switching do not compensate the smaller proportion of risk lovers at those levels of wealth.

We can construct more examples of switching of types like this one as long as we choose the probability p of risk averters having a risk loving successor to be small enough:

Proposition 6. For any economy without switching of types, we can convert the equilibrium to an equilibrium in the economy where dynasties that are risk averters become risk lovers with probability p for all $p \in (0,1/7)$.

5. Concluding remarks

We have developed an overlapping generations model with risk-averse and risk-loving dynasties in an economy with Lucas trees and no aggregate uncertainty. The simplicity of our specification has allowed us to analytically characterize the invariant equilibrium wealth distribution. The specialization of risk lovers implies that, in this distribution, a large proportion of them will be at the bottom of the wealth distribution, while others of them will be the wealthiest agents of the economy. In contrast, in this distribution, risk-averse agents will be all concentrated at the average initial wealth of risk-averse agents, which we have assumed to equal the average wealth of the economy.

In any given state, a large proportion of the risk lovers have no return from the Lucas trees, which implies that their wealth from that state onwards does not depend on their initial wealth in any way. Additionally, the proportion the risk lovers that have no return in each state converges to 0.5 since, with probability one, all risk lovers will be at the bottom of the wealth distribution. Moreover, the dynamics of wealth distribution of the risk lovers have a positive reflecting barrier which is the initial endowment received in each period by the new generation. Furthermore, every risk-loving dynasty has a positive probability of becoming arbitrarily wealthy. Consequently, for any given any initial wealth distribution, the wealth distribution converges to an invariant distribution, which is also unique.

The invariant distribution of wealth depends on the return to the Lucas trees, the endowment of the agents, and the bequest rate of the agents. Larger bequest rates imply larger inequality levels. Moreover, if the bequest rate is lower than some threshold, inequality is so low that the invariant distribution of wealth is bounded. Above that threshold, however, the invariant distribution of wealth is a Pareto distribution.

When dynasties have a positive probability of switching their attitudes towards risk, the wealth distribution converges to an invariant distribution with a Pareto distribution for the risk lovers investors as long as the model without switching has a Pareto distribution for the risk lovers.

Bibliography

Acemoglu, D., and **J. A. Robinson** (2015), "The rise and decline of general laws of capitalism," *Journal of Economic Perspectives*, 29(1), 3–28.

Alesina, A., and **D. Rodrick** (1994), "Distributive politics and economic growth," *Quarterly Journal of Economics*, 109(2), 465–490.

Andreoni, J. (1989), "Giving with impure altruism: Applications to charity and Ricardian equivalence," *Journal of Political Economy*, 97(6), 1447–1458.

Aoki, S., and **M. Nirei** (2017), "Zipf's law, Pareto's law, and the evolution of top incomes in the United States," *American Economic Journal: Macroeconomics*, 9(3), 36–71.

Araujo, A., **A. Chateauneuf**, **A.**, **J. P. Gama**, and **R. Novinski** (2018), "General equilibrium with uncertainty seeking preferences," *Econometrica*, 86(5), 1859–1871.

Araujo, A., J. P. Gama, and T. J. Kehoe (2024), "Growth, and Redistribution with Heterogeneous Attitudes toward Risk," Unpublished manuscript, Federal Reserve Bank of Minneapolis.

Araujo, A., J. P. Gama, and **C. E. Suarez** (2022), "Lack of prevalence of the endowment effect: An equilibrium analysis," *Journal of Mathematical Economics*, 102: 102763.

Bargain, O., **M. Dolls, H. Immervoll, D. Neumann, A. Peichl, N. Pestel,** and **S. Siegloch** (2015), "Tax policy and income inequality in the United States, 1979-2007," *Economic Inquiry*, 53(2), 1061–1085.

Beare, B. K., and **Toda, A. A**. (2017), "Geometrically stopped Markovian random growth processes and Pareto tails," Working paper, University of California, San Diego.

Benhabib, J., **A. Bisin,** and **S. Zhu** (2011), "The distribution of wealth and fiscal policy in economies with finitely lived agents," *Econometrica*, 79(1), 123–157.

Benhabib, J., **A. Bisin,** and **S. Zhu** (2015), "The wealth distribution in Bewley economies with capital income risk," *Journal of Economic Theory*, 159, 489–515.

Benhabib, J., **A. Bisin,** and **S. Zhu** (2016), "The distribution of wealth in the Blanchard-Yaari model," *Macroeconomic Dynamics*, 20(2), 466–481.

Bryson, M. C. (1974), "Heavy-tailed distributions: Properties and tests," *Technometrics*, 16(1), 61–68.

Bühlmann, H. (1980), "An economic premium principle," *ASTIN Bulletin: The Journal of the IAA*, 11(1), 52–60.

De Gregorio, J. (1992), "Economic growth in Latin America," *Journal of Development Economics*, 39(1), 59–84.

Gabaix, X., **J. M. Lasry, P. L. Lions,** and **B. Moll** (2016), "The dynamics of inequality," *Econometrica*, 84(6), 2071–2111.

García-Peñalosa, C. and **J. F. Wen** (2008), "Redistribution and entrepreneurship with Schumpeterian growth," *Journal of Economic Growth*, 13(1), 57–80.

Gouin-Bonenfant, E., and **A. A. Toda** (2023), "Pareto extrapolation: An analytical framework for studying tail inequality," *Quantitative Economics*, 14(1), 201–233.

Hernandez, A., and **M. S. Santos** (1996), "Competitive equilibria for infinite-horizon economies with incomplete markets," *Journal of Economic Theory*, 71(1), 102-130.

Jappelli, T., and **M. Pagano** (1994), "Saving, growth, and liquidity constraints," *Quarterly Journal of Economics*, 109(1), 83–109.

Jones, C. I. (2015), "Pareto and Piketty: The macroeconomics of top income and wealth inequality," *Journal of Economic Perspectives*, 29(1), 29–46.

Kehoe, T. J., and **D. K. Levine** (1993), "Debt constrained asset markets," *Review of Economic Studies*, 60(4), 865–888.

Kehoe, T. J., and **D. K. Levine** (2001), "Liquidity constrained markets versus debt constrained Markets," *Econometrica*, 69(3), 575–598.

Kehoe, T. J., and **D. K. Levine** (2008), "Bankruptcy and collateral in debt constrained markets," in Roger E. A. Farmer, editor, *Macroeconomics in the Small and the Large: Essays on Microfoundations, Macroeconomic Applications, and Economic History in Honor of Axel Leijonhufvud*, Edward Elgar, 99–114.

Kihlstrom, R. E., and **J. J. Laffont** (1979), "A general equilibrium entrepreneurial theory of firm formation based on risk aversion," *Journal of Political Economy*, 87(4), 719–748.

Kuznets, S., (1955), "Economic growth and income inequality," *American Economic Review*, 45(1), 1–28.

Lindert, P. H., and **J. G. Williamson** (2016), "Unequal gains: American growth and inequality since 1700," *Juncture*, 22(4), 276–283.

Papanek, G. F. (1973), "Aid, foreign private investment, savings, and growth in less developed countries," *Journal of Political Economy*, 81(1), 120–130.

Perotti, R. (1996), "Growth, income distribution, and democracy: What the data say," *Journal of Economic Growth*, 1(2), 149–187.

Persson, T., and **G. Tabelini** (1994), "Is inequality harmful for growth?" *American Economic Review*, 84(3), 600–621.

Piketty, T. (1997), "The dynamics of the wealth distribution and the interest rate with credit rationing," *Review of Economic Studies*, 64(2), 173–189.

Piketty, T., and **E. Saez** (2014), "Inequality in the long run," *Science*, 344(6186), 838–843.

Saltz, I. S. (1999), "An examination of the causal relationship between savings and growth in the Third World," *Journal of Economics and Finance*, 23(1), 90–98.

Santos, M. S., and **M. Woodford** (1997), "Rational asset pricing bubbles," *Econometrica*, 65(1), 19-57.

Shane, S., E. A. Locke, and **C. J. Collins** (2003), "Entrepreneurial motivation," *Human Resource Management Review*, 13(2), 257–279.

Temple, J. (1999), "The new growth evidence," *Journal of Economic Literature,* 37(1), 112–56.

van der Weide, R., and **B. Milanovic** (2014), "Inequality is bad for growth of the poor (but not for that of the rich)," Policy Research Working Paper 6963, World Bank.

Appendix A. Proofs

Proof of Lemma 1. The proof is a direct consequence of Proposition 7 in Appendix B.

Proof of Lemma 2. Let us assume that there is an equilibrium with complete markets $\left(q_s, \left(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i}\right)_i, \left(c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i}\right)_i\right)_s$ such that the state contingent prices satisfy that $p_{(s,1)} < p_{(s,2)}$ for some state s. Because of the optimality conditions of risk-averse investors and the absence of arbitrage opportunities, we know that the state contingent prices and the equilibrium allocation satisfy Equations 2.1, 2.2, 2.3, and 2.4 for all states. Therefore, $c_{(s,1)}^{a_i} > c_{(s,2)}^{a_i}$ for all risk averse a_i. Since $p_{(s,1)} < p_{(s,2)}$, risk lovers invest all their wealth in state $(s,1)$ and none in state $(s,2)$ (see Equation 2.5), which implies that $c_{(s,1)}^{l_i} > c_{(s,2)}^{l_i} = 0$. Therefore, the aggregate consumption in state $(s,1)$ is strictly larger than in state $(s,2)$, which contradicts aggregate in each state being equal to the same aggregate endowment $2\omega + R$.

Now, let us prove that

$$p_s = \frac{2\omega + \delta R}{2(2\omega + R)}, q_{k,s} = \frac{2\omega + \delta R}{2(1-\delta)} \quad \forall s, k$$

are the state-contingent prices and asset prices at equilibrium, respectively. First note that

$$p_{(s,1)} q_{(s,1),1} + p_{(s,2)}\left(q_{(s,2),1} + R\right) = \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{2(1-\delta)}\right) + \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{2(1-\delta)} + R\right) =$$

$$= \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{(1-\delta)} + R\right) = \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + R}{1-\delta}\right) = \frac{2\omega + \delta R}{2(1-\delta)} = q_{s,1},$$

and

$$p_{(s,1)}\left(q_{(s,1),2} + R\right) + p_{(s,2)} q_{(s,2),2} = \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{2(1-\delta)} + R\right) + \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{2(1-\delta)}\right) =$$

$$= \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + \delta R}{(1-\delta)} + R\right) = \frac{2\omega + \delta R}{2(2\omega + R)}\left(\frac{2\omega + R}{1-\delta}\right) = \frac{2\omega + \delta R}{2(1-\delta)} = q_{s,2}.$$

To see that $(p_s)_s$ are equilibrium state contingent prices, we must prove that the consumption plan described above is optimal for every agent and that the prices implement an equilibrium. We first check that proposed consumption plans satisfy the investors' budget constraints and first-order conditions. The budget constraint for a risk lover l_i in a state s is

$$q_{1,s}\theta_{1,s}^{l_i} + q_{2,s}\theta_{2,s}^{l_i} = \frac{2\omega + \delta R}{2(1-\delta)}\left(\frac{2R + 2\omega - \delta R}{R(2\omega + \delta R)}\left(\omega + b_s^{l_i}\right) - \frac{\left(\omega + b_s^{l_i}\right)}{R}\right)$$
$$= \frac{2\omega + \delta R}{2(1-\delta)}\left(\omega + b^{l_i}\right)\left(\frac{2R + 2\omega - \delta R - (2\omega + \delta R)}{R(2\omega + \delta R)}\right)$$
$$= \frac{2\omega + \delta R}{2(1-\delta)}\left(\omega + b^{l_i}\right)\left(\frac{2(1-\delta)R}{R(2\omega + \delta R)}\right) = \omega + b^{l_i}.$$

If the risk lover decides to invest in state $(s,1)$, we have, in state $(s,1)$, that

$$\left(q_{1,(s,1)} + R\right)\theta_{1,s}^{l_i} + q_{2,(s,1)}\theta_{2,s}^{l_i} = \left(\omega + b^{l_i}\right)\left(1 + \frac{2R + 2\omega - \delta R}{2\omega + \delta R}\right) = \left(\omega + b^{l_i}\right)\left(\frac{4\omega + 2R}{2\omega + \delta R}\right)$$
$$= \frac{\left(\omega + b^{l_i}\right)}{\pi},$$

and, in state $(s,2)$,

$$q_{1,(s,2)}\theta_{1,s}^{l_i} + \left(q_{2,(s,2)} + R\right)\theta_{2,s}^{l_i} = \omega + b^{l_i} + R\left(-\frac{\left(\omega + b_s^{l_i}\right)}{R}\right) = 0.$$

If, however, the risk lover decides to invest in state $(s,2)$, we have, in state $(s,1)$, that

$$\left(q_{1,(s,1)} + R\right)\theta_{1,s}^{l_i} + q_{2,(s,1)}\theta_{2,s}^{l_i} = 0,$$

and, in state $(s,2)$,

$$q_{1,(s,2)}\theta_{1,s}^{l_i} + \left(q_{2,(s,2)} + R\right)\theta_{2,s}^{l_i} = \left(\omega + b^{l_i}\right)\left(1 + \frac{2R + 2\omega - \delta R}{(2\omega + \delta R)}\right) = \frac{\left(\omega + b^{l_i}\right)}{\pi}.$$

Consequently, the consumption plans

$$c_{(s,1)}^{l_i} = \left((1-\delta)/p_{(s,1)}\right)\left(\omega + b_s^{l_i}\right), b_{(s,1)}^{l_i} = \left(\delta/p_{(s,1)}\right)\left(\omega + b_s^{l_i}\right), c_{(s,2)}^{l_i} = b_{(s,2)}^{l_i} = 0$$

and

$$c_{(s,1)}^{l_i} = b_{(s,1)}^{l_i} = 0, c_{(s,2)}^{l_i} = \left((1-\delta)/p_{(s,2)}\right)\left(\omega + b_s^{l_i}\right), b_{(s,2)}^{l_i} = \left(\delta/p_{(s,2)}\right)\left(\omega + b_s^{l_i}\right)$$

satisfy the budget constraint.

Now we show that these consumption plans satisfy the risk lovers' first-order conditions. Depending on the state in which she specializes, the first-order conditions are

$$b_{(s,k)}^{l_i} = c_{(s,k)}^{l_i}\frac{\delta}{(1-\delta)} = \frac{\delta}{p_{(s,k)}}\left(\omega + b_s^{l_i}\right) = \frac{\delta}{\pi}\left(\omega + b_s^{l_i}\right)$$

or

$$b^{l_i}_{(s,k)} = c^{l_i}_{(s,k)} \frac{\delta}{(1-\delta)} = 0,$$

which implies that the consumption plans specified above are optimal.

That the consumption plan

$$c^{a_i}_{(s,k)} = \left((1-\delta)/(2p_{(s,k)})\right)\left(\omega + b^{a_i}_s\right), b^{a_i}_{(s,k)} = \left(\delta/(2p_{(s,k)})\right)\left(\omega + b^{a_i}_s\right)$$

for $k = 1{,}2$ for a risk averter a_i satisfies the budget constraint is straightforward. The risk averters' first order conditions imply that

$$b^{a_i}_{(s,k)} = c^{a_i}_{(s,k)} \frac{\delta}{(1-\delta)} = \frac{\delta}{2p_{(s,k)}}\left(\omega + b^{a_i}_s\right) = \frac{\delta}{2\pi}\left(\omega + b^{a_i}_s\right) \quad \forall k = 1{,}2.$$

We have shown that the consumption plans for risk lovers and risk averters solve their maximization problems. Now, we check aggregate feasibility. If risk lovers specialize symmetrically between the states — that is, if they satisfy Assumption RL — we have

$$\int_i c^i_s di = 2\omega + R,$$

$$\int_i b^i_s di = \frac{\delta}{(1-\delta)}(2\omega + R),$$

$$\begin{aligned}
\int_0^1 \theta^{a_i}_{j,s} di + \int_0^1 \theta^{l_i}_{j,s} di &= \int_0^1 \frac{(1-\delta)\left(\omega + b^{a_i}_s\right)}{2\omega + \delta R} di + \int_0^1 \left(\frac{1-\delta}{2\omega + \delta R}\right)\left(\omega + b^{l_i}_s\right) di \\
&= \frac{(1-\delta)}{2\omega + \delta R}\left(2\omega + \int_0^1 b^{a_i}_s di + \int_0^1 b^{l_i}_s di\right) = \frac{(1-\delta)}{2\omega + \delta R}\left(2\omega + \frac{\delta}{(1-\delta)}(2\omega + R)\right) \\
&= \frac{(1-\delta)}{2\omega + \delta R}\left(\frac{2\omega(1-\delta) + \delta(2\omega + R)}{(1-\delta)}\right) = 1 \quad \forall j = 1{,}2,
\end{aligned}$$

for all states s, which concludes the proof.

Proof of Lemma 3. If there is an equilibrium with complete markets such that $p_{(s,1)} = p_{(s,2)}$ for some state s, all risk lovers are indifferent between investing in either of the Lucas trees. If we do not restrict allocations to satisfy Assumption RL, there is at least a continuum of family of sets $\{A_1, A_2\} \in \mathcal{P}([0,1])$ such that $\lambda(A_1) = \lambda(A_2) = 1/2$ and that $\int_{A_1} w^{l_i} di = \int_{A_2} w^{l_i} di$ where λ is the Lebesgue measure, which concludes the proof.

Proof of Proposition 4. Notice that any invariant distribution of wealth of the economy must satisfy the condition that the poorest risk lovers are half of the risk lovers with ω as wealth. Using the same recursive process used to determine the invariant distribution, we have that any invariant distribution for the risk lovers coincides with the one presented in Subsection 3.1.

Also notice that, for risk-averse agents, any invariant distribution is constant since all of them have riskless returns and constant endowments. Additionally, since all agents have the same endowment $\omega > 0$, we have that $\overline{w}_{\infty}^{l} = \overline{w}_{\infty}^{a}$, which concludes the proof.

Proof of Proposition 5. Notice that the invariant wealth distribution of the risk lovers $\left(w_{\infty}^{l_i}\right)_{i\in[0,1]}$

$$\sum_{k=0}^{n}\left(\frac{\delta}{\pi}\right)^{k}\omega=\sum_{k=0}^{n}\left(\frac{2\delta(2\omega+R)}{(2\omega+R)-(1-\delta)R}\right)^{k}\omega$$

with measure (or proportion) $1/2^{n+1}$ (for $n\in\mathbb{N}$). Given that

$$w\in\left(\sum_{k=0}^{n}\left(\frac{2\delta(2\omega+R)}{(2\omega+R)-(1-\delta)R}\right)^{k}\omega,\sum_{k=0}^{n+1}\left(\frac{2\delta(2\omega+R)}{(2\omega+R)-(1-\delta)R}\right)^{k}\omega\right],$$

we have that $\lambda\left[i: w_{\infty}^{l_i}>w\right]=1/2^{n+2}$ where $\lambda[\cdot]$ is the Lebesgue measure in [0,1]. Let us define

$$\beta=\frac{\delta(2\omega+R)}{(2\omega+R)-(1-\delta)R}.$$

Since, $\delta\geq 2\omega/(4\omega+R)$, $\beta\in[1/2,1)$. Let us first suppose that $\delta>2\omega/(4\omega+R)$. Then, $\beta>1/2$, which implies that

$$\sum_{k=0}^{n}(2\beta)^{k}\omega\leq w\leq\sum_{k=0}^{n+1}(2\beta)^{k}\omega,$$

$$\frac{(2\beta)^{n+1}-1}{2\beta-1}\omega\leq w\leq\frac{(2\beta)^{n+2}-1}{2\beta-1}\omega,$$

$$\frac{(2\beta)^{n+1}-1}{2\beta-1}\leq\frac{w}{\omega}\leq\frac{(2\beta)^{n+2}-1}{2\beta-1},$$

$$(2\beta)^{n}\leq\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\leq(2\beta)^{n+1},$$

$$n \leq \log_{2\beta}\left(\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\right) \leq n+1$$

Then,

$$\frac{1}{2^{\log_{2\beta}\left(\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\right)}} \geq \lambda\left[i: w_{\infty}^{l_i} > w\right] = \frac{1}{2^{n+1}} \geq \frac{1}{2^{\log_{2\beta}\left(\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\right)+1}}.$$

Therefore,

$$\frac{\log \lambda\left[i: w_{\infty}^{l_i} > w\right]}{\log(w)} \propto \frac{-\log_{2\beta}\left(\frac{w}{\omega}\right)}{\log_2 w} \propto -\log_2(2\beta),$$

which concludes the proof for $\beta > 1/2$. Let us now suppose that $\beta = 1/2$. Then,

$$(n+1) \leq \frac{w}{\omega} \leq (n+2),$$

which implies that,

$$\frac{2}{2^{w/\omega}} \geq \lambda\left[i: w_{\infty}^{l_i} > w\right] = \frac{1}{2^{n+1}} \geq \frac{1}{2^{w/\omega}},$$

which concludes the proof.

Appendix B. Incomplete market case

In calculating the equilibrium in Lemma 1, we implicitly assume that any sequential market equilibrium corresponds to an Arrow-Debreu equilibrium. To justify this assumption, we need to rule out the case in which the returns $\left(q_{1,(s,1)} + R, q_{1,(s,2)}\right)$ and $\left(q_{2,(s,1)}, q_{2,(s,2)} + R\right)$ are identical. We refer to this case as the incomplete market case because, if it occurs, the sequential markets equilibrium does not correspond to an Arrow-Debreu equilibrium. Notice that, because of the absence of arbitrage in the economy, we know that, in this case, there is a sequence of prices $(p_s)_s$ such that

$$q_{1,s} = p_{(s,1)}\left(q_{1,(s,1)} + R\right) + p_{(s,2)} q_{1,(s,2)},$$
$$q_{2,s} = p_{(s,2)}\left(q_{2,(s,2)} + R\right) + p_{(s,1)} q_{2,(s,1)}.$$

Notice that the optimal consumption and bequests of a risk averter a_i are

$$c_{(s,1)}^{a_i} = \frac{(1-\delta)\left(q_{1,(s,1)}+R\right)}{2q_{1,s}}\left(\omega + b_s^{a_i}\right), c_{(s,2)}^{a_i} = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}}\left(\omega + b_s^{a_i}\right),$$

$$b_{(s,1)}^{a_i} = \frac{\delta\left(q_{1,(s,1)}+R\right)}{2q_{1,s}}\left(\omega + b_s^{a_i}\right), \text{and } b_{(s,2)}^{a_i} = \frac{\delta q_{1,(s,2)}}{2q_{1,s}}\left(\omega + b_s^{a_i}\right),$$

and the optimal consumption and bequests of a risk lover l_i are

$$c_{(s,1)}^{l_i} = \frac{(1-\delta)\left(q_{1,(s,1)}+R\right)}{2q_{1,s}}\left(\omega + b_s^{l_i}\right) \text{ and } c_{(s,2)}^{l_i} = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}}\left(\omega + b_s^{l_i}\right),$$

$$b_{(s,1)}^{l_i} = \frac{\delta\left(q_{1,(s,1)}+R\right)}{2q_{1,s}}\left(\omega + b_s^{l_i}\right) \text{ and } b_{(s,2)}^{l_i} = \frac{\delta q_{1,(s,2)}}{2q_{1,s}}\left(\omega + b_s^{l_i}\right).$$

Since there is no aggregate uncertainty, in equilibrium

$$\int_i c_{(s,1)}^i di = \frac{(1-\delta)\left(q_{1,(s,1)}+R\right)}{2q_{1,s}}\left(\int_i \left(\omega + b_s^i\right)di\right) = 2\omega + R,$$

$$\int_i c_{(s,2)}^i di = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}}\left(\int_i \left(\omega + b_s^i\right)di\right) = 2\omega + R.$$

Consequently,

$$\frac{q_{1,(s,1)}+R}{q_{1,s}} = \frac{q_{1,(s,2)}}{q_{1,s}} = \frac{q_{2,(s,1)}}{q_{2,s}} = \frac{q_{2,(s,2)}+R}{q_{2,s}} = \frac{2\omega+R}{2\omega+\delta R}, \qquad \text{(B.1)}$$

and

$$p_{(s,1)} = p_{(s,2)} = p = \frac{2\omega+\delta R}{2(2\omega+R)}.$$

Proposition 7. There is no equilibrium with incomplete markets. Moreover, in any equilibrium the asset prices are given by

$$q_{j,s} = \frac{2\omega+\delta R}{2(1-\delta)} \quad \forall j = 1,2, \text{and } s.$$

Proof. Let us suppose that there is an equilibrium with incomplete markets in state s with date $t \geq 0$. Without loss of generality, we assume that

$$q_{1,(0)} \leq \frac{2\omega+\delta R}{2(1-\delta)}.$$

We will show by induction on t that the price of asset 1 is negative for some state that is a successor of s. If the market is incomplete in $t = 0$, we know from Equation B.1 that

$$q_{1,(0,1)} = \left(\frac{2\omega+R}{2\omega+\delta R}\right)q_{1,(0)} - R \leq \frac{2\omega - R + 2\delta R}{2(1-\delta)} = \frac{2\omega+\delta R}{2(1-\delta)} - \frac{R}{2} < \frac{2\omega+\delta R}{2(1-\delta)}.$$

Since $q_{1,s} = p\big(q_{1,(s,1)} + R + q_{1,(s,2)}\big)$ for all state s, then

$$\min\{q_{1,(s,1)}, q_{1,(s,2)}\} \leq \frac{2\omega + R}{2\omega + \delta R} q_{1,s} - R/2.$$

Let us consider $\{s^n\}_n$ the sequence of states where s^{n+1} is the immediate successor of s^n such that $q_{1,s^{n+1}}$ is the minimum price of asset 1 between $q_{1,(s^n,1)}$ and $q_{1,(s^n,2)}$, and $s^1 = (0)$. Since

$$q_{1,(0,1)} \leq \frac{2\omega + \delta R}{2(1-\delta)} - R/2,$$

for n large enough we know that

$$q_{1,(s^n,1)} \leq \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2}\left(\frac{2\omega + R}{2\omega + \delta R}\right)^n < 0,$$

which is a contradiction.

Let us assume that the market is incomplete in state $s = (0, s_1, \dots, s_{t-1}, s_t)$ where $t \geq 1$. Let us consider $\{s^n\}_n$ the sequence of states where s^{n+1} is the immediate successor of s^n such that $q_{1,s^{n+1}}$ is the minimum price of asset 1 between $q_{1,(s^n,1)}$ and $q_{1,(s^n,2)}$, and $s^1 = s$. Since

$$q_{1,(s,1)} \leq \left(\frac{2\omega + R}{2\omega + \delta R}\right) q_{1,s} - R \leq \left(\frac{2\omega + R}{2\omega + \delta R}\right) q_{1,(0,s_1,\dots,s_{t-1})} - R \leq \cdots \leq \left(\frac{2\omega + R}{2\omega + \delta R}\right) q_{1,(0)} - R$$
$$\leq \frac{2\omega - R + 2\delta R}{2(1-\delta)} = \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2} < \frac{2\omega + \delta R}{2(1-\delta)}.$$

For n large enough, we know that

$$q_{1,(s^n,1)} \leq \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2}\left(\frac{2\omega + R}{2\omega + \delta R}\right)^n < 0,$$

which is a contradiction.

To conclude the proof, notice that, if $q_{j,(0)} < (2\omega + \delta R)/\big(2(1-\delta)\big)$ for some asset j, we know that there is a state s at date t large enough such that $q_{j,s} < 0$. The proof of this fact is a direct consequence of the first part of the proof. Therefore, the only possible asset price is

$$q_{j,s} = \frac{2\omega + \delta R}{2(1-\delta)} \quad \forall j = 1,2, \forall s.$$

In this case, markets are complete, which concludes the proof.

Appendix C. Switching type of agents

Let us suppose that the probability of a risk averter of having a risk lover successor is $p \in (0,1/7)$. Let us denote p_1 the probability of a wealthy risk lover of having a risk averse successor. Let us analyze how the proportion of risk lovers and risk averter changes. To do so, we define y_t^a, y_t^l as the mass of the risk averters and the risk lovers at date t, respectively, then $y_t^a = y_t^l = 1$. Note that for the mass of the risk lovers at the bottom of the distribution in date t, $y_t^{l^1}$, then the mass if risk lovers at the bottom of the distribution in date $t+1$ satisfies that

$$y_{t+1}^{l^1} = py_t^{l^1} + \frac{1}{2}\left(y_t^{l^1} + y_t^{l^2} + (1-p_1)\left(2 - y_t^a - y_t^{l^1} + y_t^{l^2}\right)\right).$$

where consider $y_t^{l^k}$ as the mass of risk lovers that are at the bottom k level of wealth at date t. Then, the mass of the second poorest group of risk lovers we have that

$$y_{t+1}^{l^2} = \frac{1}{2} y_t^{l^1}, \qquad y_{t+1}^{l^3} = \frac{1}{2} y_t^{l^2},$$

and for the other risk lovers we have that

$$y_{t+1}^{l^k} = \frac{y_t^{l^{k-1}}(1-p_1)}{2},$$

For $k \leq t$. For the mass of risk averters we have that

$$y_{t+1}^a = (1-p)y_t^a + p_1\left(2 - y_t^a - y_t^{l^1} - y_t^{l^2}\right) = 1.$$

when $t \to \infty$, we have that

$$y_\infty^{l^1} = \frac{1+p}{2}, y_\infty^{l^2} = \frac{1+p}{4}, \text{and } y_\infty^a = 1 = (1-p)y_\infty^a + p_1\left(2 - y_\infty^a - y_\infty^{l^1} - y_\infty^{l^2}\right).$$

Then,

$$y_\infty^{l^{n+1}} = \frac{(1+p)}{4}(1-p_1)^n = \frac{(1+p)}{4}\left(\frac{1-7p}{2(1-3p)}\right)^n$$

for $t \geq 0$. Then, substituting $y_\infty^{l^1}, y_\infty^{l^2}$ in the formula of y_∞^a, we find that

$$p_1 = \frac{4p}{(1-3p)}. \qquad \text{(C. 1)}$$

Since $p \in (0,1/7)$, $p_1 = 4p/(1-3p) \in (0,1)$.

Finally, we can construct recursively the invariant distribution as in Proposition 1. In this case, the wealth of each group of risk lovers and the wealth of the risk averters in the invariant distribution follows the argument of Proposition 4. The reason for this is that any risk averter that the predecessor is a risk lover receives the average bequest of the risk averters. However, the aggregate bequest rate is different in this case. Let us consider $\pi_{\hat{\delta}}$ as

$$\pi_{\hat{\delta}} = \left(\frac{2\omega + R - (1-\hat{\delta})R}{2(2\omega + R)}\right)$$

where $\hat{\delta}$ is the aggregate bequest rate of the economy. Note that $\pi_{\hat{\delta}}$ is the analogous of π of the model without switching. Then

$$\frac{1-\hat{\delta}}{\pi_{\hat{\delta}}}\left(\int_i w_\infty^{a_i} di + \int_i w_\infty^{l_i} di\right) = \frac{1-\hat{\delta}}{\pi_{\hat{\delta}}}\left(\int_i \left(b_\infty^{a_i} + \omega\right) di + \int_i \left(b_\infty^{l_i} + \omega\right) di\right) = 2\omega + R,$$

that is, $\hat{\delta}$ is the solution of

$$\frac{1-\hat{\delta}}{\pi_{\hat{\delta}}}\left(\left(\frac{\pi_{\hat{\delta}}}{2\pi_{\hat{\delta}} - \delta}\omega\right) + \sum\nolimits_{n=0}^{\infty} \frac{(1+p)}{4}\left(\frac{1-7p}{2(1-3p)}\right)^n \left(\sum\nolimits_{k=0}^{n+2}\left(\frac{\delta}{\pi_{\hat{\delta}}}\right)^k \omega\right) + \left(\frac{1+p}{2}\right)\right.$$

$$\left. + \left(\frac{1+p}{4}\right)\left(\omega + \left(\frac{\delta}{\pi_{\hat{\delta}}}\right)\omega\right)\right) = 2\omega + R. \qquad \text{(C. 2)}$$

Notice that since, for all $d \in [0,1]$, we have that

$$\left(1 + \frac{2\delta(2\omega + R)}{2\omega + dR} + \left(\frac{2\delta(2\omega + R)}{2\omega + dR}\right)^2 - \frac{2\omega + dR}{2(1-\delta)\omega + (d-\delta)R}\right) > 0,$$

in particular

$$\omega + \frac{2\delta(2\omega + R)}{2\omega + \hat{\delta}R}\omega + \left(\frac{2\delta(2\omega + R)}{2\omega + \hat{\delta}R}\right)^2 \omega > \frac{2\omega + dR}{2(1-\delta)\omega + \left(\hat{\delta} - \delta\right)R}\omega,$$

which implies that the wealth of the risk lovers who receives the highest return twice in a row have a larger wealth than the risk averters. Therefore, $\hat{\delta} < \delta$.

In the numerical example, we have that $p = 0.1$, we estimate the proportion of agents in each wealth level using the formulas of $y_\infty^{l^n}$ mentioned above, $\hat{\delta}$ using Equation C.1. Finally, the level of wealth of each group are given by

$$w_\infty^{a_i} = \frac{\pi_{\hat{\delta}}}{2\pi_{\hat{\delta}} - \delta}\omega,$$

$$\left(w_\infty^{l_i}\right)_i : \sum\nolimits_{k=0}^{n}\left(\frac{\delta}{\pi_{\hat{\delta}}}\right)^k \omega \text{ with measure (or proportion) } y_\infty^{l^n} \text{ for } n \geq 1.$$

Proof of Proposition 6. Notice that the calculation of p_1 as a function of p in Equation C.1 does not depend on the parameters δ, ω, R. The calculation of $\hat{\delta}$ in Equation C.2 does depend on these parameters, but we can do the calculation and find $\hat{\delta}$ such that $0 < \hat{\delta} < \delta$.