Econ 8602, Fall 2024

Homework 1

Due Tues. Feb. 6

Question 1

Consider the Logit Model of Product Differentiation (For background related to some of the tasks for this question, you can look at Anderson and De Palma, "The Logit as a Model of Product Differentiation," *Oxford Economic Papers* 44 (1992), 51-57.)

Suppose there are n firms plus an "outside good" labeled by 0. Each firm has constant marginal cost equal to c. The is a measure M of consumers. Let i index an individual consumer and suppose the utility of consumer i from purchasing good j is

$$\begin{aligned} U_{i,j} &= \xi - \alpha p_j + \varepsilon_{i,j} \text{ for } j = 1, 2, ...n \\ &= \varepsilon_{i,0} \text{ for good } 0. \end{aligned}$$

Note the parameters ξ and α are constant across the n firms and across consumers, so the firms are symmetric. It is convenient to write the utility has having two parts

$$U_{i,j} = \delta_j + \varepsilon_{ij}$$

(where $\delta_j = \xi - \alpha p_j$ for $j \geq 1$ and $\delta_0 = 0$). The first part δ_j is common to all consumers. The second part is idiosyncratic, capturing random reasons why one consumer i might get value product j. Assume the ε_{ij} are drawn i.i.d. from the type 1 extreme value distribution. It can be shown that the probability of drawing a vector $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}, ...\varepsilon_{in})$ so that

$$U_{i,j} \geq U_{i,k}, \text{ for } k \neq j \tag{1}$$

is

$$S_j(p_1, p_2, ...p_n) = \frac{\exp(\delta_j)}{1 + \sum_{k=1}^{n} \exp(\delta_k)}, \tag{2}$$

where the δ_j are implicitly functions of the prices. The event (1) is the event that good j provides the consumer the highest utility over of all the choices. Given the continuum of consumers, this is the share of consumers that will select option j. Hence, the quantity of sales of firm j, given the vector of price is

$$q_j(p_1, p_2, ...p_n) = M \times S_j(p_1, p_2, ...p_n)$$

(a) Calculate the slope $\frac{\partial S_j}{\partial p_j}$ and write it in convenient way in terms of S_j.

(b) Suppose the n firms compete in a Bertrand fashion. Set up the problem of firm 1 given the choices of the remaining firms $p_2, p_3, ...p_n$. Derive the first-order necessary condition.

(c) Define a symmetric Bertrand equilibrium.

(d) There exists a symmetric Bertrand price equilibrium $p^e(n)$ that depends upon the number of firms. Derive the equation characterizing this price. Show a price solving this equation exists.

(e) Consider the numerical example where $\alpha = 1$, $c = 1$, $\xi = 1$ and $n = 5$. Plot on the same graph the following functions of price:

$$\begin{aligned} f_1(p) &= p - c \\ f_2(p) &= \frac{1}{\alpha}\frac{1}{1-\tilde{S}(p)} \end{aligned}$$

where $\tilde{S}(p) = S_1(p, p, p, ..p)$ (the representative firm share when all price the same.) What does this graph tell you about existence and uniqueness of the Bertand price equilibrium?

(f) Now make the number of firms n endogenous. Suppose there is a fixed cost ϕ to enter the industry. Suppose there is a two stage game. In stage 1, $n \geq 0$ firms enter the industry. In stage 2 the n firms play a simultaneous move Bertrand price game. We are interested in subgame perfect Nash equilibrium. Suppose n^e is an equilibrium entry level for this game. What condition must it solve?

(g) Set $M = 1$. Determine the interval of fixed costs $[\underline{\phi},\bar{\phi}]$ such that $n^e = 5$ is the equilibrium with free entry in the numerical example of part (f).

(h) Using the δ_j notation above, the formula for consumer surplus for the logit model is (Small and Rosen, *Econometrica,* 1981)

$$CS = M \ln \sum_{j=0}^{n} \exp(\delta_j)$$

Consider the following social planner problem. The social planner picks an integer n in the first stage. Then in stage 2, the firms engage in Bertrand competition to maximize profits. Suppose the social planner chooses n to maximize the sum of CS plus total profit (where profit nets out the fixed cost). Over what range of fixed costs $[\underline{\phi},\bar{\phi}]$ is the social planner's solution equal to $n^* = 5$? How does this compare with the range of fixed cost for $n^e = 5$ in the market allocation that you determined in part (g). (Note: the social planner is picking an integer, so your solution should not include differentiating with respect to n.)

Question 2

Take the dynamic industry model discussed in class. Assume the parameterization

$$\begin{aligned} c(q) &= \frac{q^2}{2} \\ D(p) &= p^{-\varepsilon_D} = p^{-2} \\ P(Q) &= Q^{-\frac{1}{2}} \end{aligned}$$

where ε_D is the elasticity of demand. Assume $\beta = .5$ and $\sigma = 1 - \delta = .5$. Following the class notes:

$$\begin{aligned} q^* &= \frac{1}{\sigma} = 2 \\ p_C^* &= (1-\beta)c'(q^*) + \beta\sigma c(q^*) \\ &= .5q^* + .25\frac{q^{*2}}{2} = 1.5 \\ p_M^* &= \frac{\varepsilon_D}{\varepsilon_D - 1} p_C^* = 3 \\ Q_C^* &= p_C^{*-2} \\ Q_M^* &= p_M^{*-2} \\ K_C^* &= \sigma Q_C^* \\ K_M^* &= \sigma Q_M^* \end{aligned}$$

(a) Use value function iteration to solve for the Markov-perfect equilibrium. It is up to you to chose a method to approximate the function.

One possibility is to use Chebyshev approximation (see page 223 in Judd) to calculate the equilibrium value function for the monopoly problem. If you do this, try using $n = 5$ (the order of the polynomials) and $m = 10$ (the number of grid points). Let $a = .5K_M^*$ and $b = 1.5K_M^*$; these are the endpoints of the grid using Judd's notation. Iterate on the vector $(a_0, a_1, ..., a_n)$ which is the vector determining the approximation of $w(K)$ (Sorry for the awkward notation where a denotes two things; this is Judd's fault). Start with $a_i = 0$ for all i and stop when

$$\max_{i \in \{0,n\}} |a_i^{t+1} - a_i^t| < .000001$$

where t denotes a particular iteration.

After the value function converges approximate the policy function $q(K)$. Let the initial capital level be $K_0 = a = .5K_M^*$ and calculate for periods 1-25 the following variables: K_t, q_t, P_t and $w_t(K_t)$. Make a table with this information. Compare with K_M^*, q^*, P_M^* and w_M^*, the stationary monopoly levels.

(b) Let $(a_0, ...a_n)$ be the coefficient vector for the value function $v_1(K_1, K_2)$ approximation and $(b_0, ..., b_n)$ the coefficient vector for the policy function $q_1(K_1, K_2)$ approximation. You will need to approximate a function defined on R^2 (see Judd page 238). Note you need to iterate on q_1 as well as v_1 since firm 1 takes firm 2's action as given in the problem (and $q_2(x, y) = q_1(y, x)$).

Let $a = .25K_M^*$ and $b = K_C^*$ be the end points of the grid.

Solve for the equilibrium path for the first 25 periods starting at $K_{1,0} = b$ and $K_{2,0} = a$. Print out $q_{1,t}, q_{2,t}, K_{1,t}$, $K_{2,t}$, and P_t. Again, put this information in a table. What happens to market share over time?