Overlapping generations

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$

$$\text{s.t} \ \ \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m$$

$$c_1^0 \geq 0.$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\max \ c_t^t + \log c_{t+1}^t$$

$$\text{s.t.} \ \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2$$

$$c_t^t, c_{t+1}^t \geq 0.$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1, \ t = 1, \ 2, \ \ldots.$

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_2^1, \hat{s}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$

$$\text{s.t } \ c_1^0 \leq w_2 + m$$

$$c_1^0 \geq 0.$$

- Given $\hat{r}_{t+1}$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_{t+1}^t$ to solve

$$\max \ c_t^t + \log c_{t+1}^t$$
$$\text{s.t. } c_t^t + s_{t+1}^t \leq w_1$$
$$c_{t+1}^t \leq w_2 + (1 + \hat{r}_{t+1}) s_{t+1}^t$$
$$c_t^t, c_{t+1}^t \geq 0.$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

- $\hat{s}_2^1 = m,\ \hat{s}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1+\hat{r}_\tau)\right]m,\ t = 2,3,\ldots.$

The two equilibria are, in a precise sense, equivalent:

Proposition: Suppose that $\hat{p}_1, \hat{p}_2, \ldots$ and $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that are an Arrow-Debreu equilibrium. Then $\hat{r}_2, \hat{r}_3, \ldots$, $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and $\hat{s}_2^1, \hat{s}_3^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = ?$$
$$\hat{s}_{t+1}^t = ?$$

Proposition: Suppose that $\hat{r}_2, \hat{r}_3, \ldots$, $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and $\hat{s}_2^1, \hat{s}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_1, \hat{p}_2, \ldots$ and $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that are an Arrow-Debreu equilibrium where

$$\hat{p}_t = ?$$

Can we do the same with infinitely lived consumers? Yes but there are two complications:

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2,3$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t.} \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \ge 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 = w_t^1 + w_t^2 + w_t^3$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;\ \hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots; \hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \ldots$ such that

• Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

$$\text{s.t.} \quad c_0^i + b_1^i \leq w_0^i$$

$$c_t^i + b_{t+1}^i \leq w_t^i + \left(1 + \hat{r}_t\right) b_t^i,\ t = 1, 2, \ldots$$

$$c_t^i \geq 0.$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 = w_t^1 + w_t^2 + w_t^3,\ t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 + \hat{b}_t^3 = 0,\ t = 0,1,\ldots$

First problem: The consumer's problem

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

$$\text{s.t.} \quad c_0^i + b_1^i \leq w_0^i$$

$$c_t^i + b_{t+1}^i \leq w_t^i + \left(1 + \hat{r}_t\right) b_t^i,\ t = 1,2,\ldots$$

$$c_t^i \geq 0$$

does not have a solution.

We need to impose a no-Ponzi (scheme) condition:

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

$$\text{s.t.} \quad c_0^i + b_1^i \leq w_0^i$$

$$c_t^i + b_{t+1}^i \leq w_t^i + \left(1 + \hat{r}_t\right) b_t^i, \; t = 1, 2, \ldots$$

$$b_t^i \geq -B$$

$$c_t^i \geq 0.$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

Second problem: To prove the proposition that a sequential markets equilibrium can be made into an Arrow-Debreu equilibrium, we need to prove that

$$\lim_{T\to\infty} \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$

converges.