Optimal growth problem:

$\hat{c}_t, \hat{k}_t$ solves

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq \theta k_t^{\alpha}, \; t = 0,1,\ldots$$

$$k_0 \leq \overline{k}_0$$

$$c_t, k_t \geq 0.$$

Sequential markets equilibrium:

Sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t.} \quad c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t + \hat{r}_t^k k_t + \left(1 + \hat{r}_t^b\right) b_t,\ t = 0,1,\ldots$$

$$k_0 = \bar{k}_0,\ b_0 = 0$$

$$b_t \geq -B,\ \ c_t, k_t \geq 0.$$

- $\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1},\ t = 0,1,\ldots$

  $\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha},\ t = 0,1,\ldots.$

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$
- $\hat{b}_t = 0$, $t = 0,1,\ldots$

Proposition: The allocation/production plan in a sequential markets equilibrium is Pareto efficient.

Proof: Suppose that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium. Then

$$\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}, t = 0,1,\ldots$$
$$\hat{k}_0 \le \overline{k}_0,$$

and there exist Lagrange multipliers $p_t \ge 0$, $t = 0,1,\ldots$, such that

$$\frac{\beta^t}{\hat{c}_t} - p_t = 0,\ t = 0,1,\ldots$$
$$-p_t + p_{t+1}\alpha\theta\hat{k}_{t+1}^{\alpha-1} = 0,\ t = 0,1,\ldots$$
$$\lim_{t\to\infty} p_t \hat{k}_{t+1} = 0.$$

The necessary and sufficient conditions for $\tilde{c}_t, \tilde{k}_t$ to be a Pareto efficient allocation/production plan are

$$\tilde{c}_t + \tilde{k}_{t+1} = \theta \tilde{k}_t^{\alpha},\ t = 0,1,\ldots$$
$$\tilde{k}_0 = \bar{k}_0,$$

and that there exist some Lagrange multipliers $\pi_t \geq 0,\ t = 0,\ 1,\ldots,$ such that

$$\frac{\beta^t}{\tilde{c}_t} - \pi_t = 0,\ t = 0,1,\ldots$$
$$-\pi_t + \pi_{t+1} \alpha \theta \tilde{k}_{t+1}^{\alpha-1} = 0,\ t = 0,1,\ldots$$
$$\lim_{t\to\infty} \pi_t \tilde{k}_{t+1} = 0.$$

Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium, we can set $\tilde{c}_t = \hat{c}_t$, $\tilde{k}_t = \hat{k}_t$, and $\pi_t = p_t$ and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

Dynamic programming:

The Bellman equation is

$$V(k) = \max \ \log c + \beta V(k')$$
$$\text{s.t.} \quad c + k' \le \theta k^{\alpha}$$
$$c, k' \ge 0.$$

Guessing that $V(k)$ has the form $a_0 + a_1 \log k$, we can solve for c and k':

$$c = \frac{1}{1+\beta a_1}\theta k^{\alpha}, \ k' = \frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}.$$

We can plug these solutions back into the Bellman equation to obtain

$$a_0 + a_1 \log k = \log\left(\frac{1}{1+\beta a_1}\theta k^\alpha\right) + \beta\left[a_0 + a_1 \log\left(\frac{\beta a_1}{1+\beta a_1}\theta k^\alpha\right)\right].$$

Collecting all the terms on the right-hand side that involve $\log k$, we can solve for a_1:

$$a_1 = \alpha + \alpha\beta a_1$$

$$a_1 = \frac{\alpha}{1-\alpha\beta},$$

which implies that

$$k' = \alpha\beta\theta k^{\alpha}$$
$$c = (1-\alpha\beta)\theta k^{\alpha}.$$

We can also solve for a_0:

$$a_0 = \frac{1}{1-\beta}\left[\log\left(\frac{\theta}{1+\beta a_1}\right) + \beta a_1 \log\left(\frac{\beta a_1 \theta}{1+\beta a_1}\right)\right]$$
$$a_0 = \frac{1}{1-\beta}\left[\log\big((1-\alpha\beta)\theta\big) + \frac{\alpha\beta}{1-\alpha\beta}\log\big(\alpha\beta\theta\big)\right].$$

$k' = g(k) = \alpha\beta\theta k^{\alpha}$ is called the policy function.

To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{t+1} = \alpha\beta\theta k_t^{\alpha}$$

forward, starting at $k_0 = \overline{k}_0$. We set

$$c_t = (1-\alpha\beta)\theta k_t^{\alpha}$$
$$b_t = 0$$
$$r_t^k = \alpha\theta k_t^{\alpha-1}$$
$$r_t^b = \alpha\theta k_t^{\alpha-1} - 1$$
$$w_t = (1-\alpha)\theta k_t^{\alpha}.$$

Notice that this problem actually has an analytical solution:

$$k_t = \alpha\beta\theta k_{t-1}^{\alpha} = \alpha\beta\theta\left(\alpha\beta\theta k_{t-2}^{\alpha}\right)^{\alpha} = \left(\alpha\beta\theta\right)^{\sum_{\tau=0}^{t-1}\alpha^{\tau}} \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t}$$

Convergence to the steady state:

$$\hat{k} = g(\hat{k}) = \alpha\beta\theta\hat{k}^{\alpha} = (\alpha\beta\theta)^{\frac{1}{1-\alpha}}$$

$$\hat{k} = \lim_{t\to\infty}\left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1}{1-\alpha}}.$$