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COMPARATIVE STATICS AND PERFECT FORESIGHT IN INFINITE HORIZON ECONOMIES

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Does a pure exchange economy with an infinite time horizon have determinate perfect foresight equilibria? When there is a finite number of infinitely lived agents equilibria are generically determinate. This is not true with overlapping generations of finitely lived agents. We ask whether the initial conditions together with the requirement of convergence to a steady state locally determine an equilibrium price path. In this framework there are many economies with isolated equilibria, many with continua of equilibria, and many with no equilibria at all. With two or more goods in every period not only can the price level be indeterminate but relative prices as well. Furthermore, such indeterminacy can occur whether or not there is fiat money and whether or not the equilibria are Pareto efficient.

1. INTRODUCTION

THIS PAPER CONSIDERS whether infinite horizon economies have determinate perfect foresight equilibria. This question is of crucial importance. If instead equilibria are locally indeterminate, not only are we unable to make comparative static predictions, but the agents in the model are unable to determine the consequences of unanticipated shocks. The idea underlying perfect foresight is that agents' expectations should be the actual future sequence predicted by the model; if the model does not make determinate predictions, the concept of perfect foresight is meaningless.

We consider two extreme cases: the first with a finite number of infinitely lived consumers and the second with an infinite number of finitely lived consumers, an overlapping generations model. Both are models of stationary pure exchange economies. No production, including the storage of goods between periods, can occur. These models are unrealistic but are the easiest to study. Extensions of the results of this paper to models with production, infinitely lived assets, and mixtures of the two types of consumers are presented by Muller and Woodford [**29**].

When there is a finite number of infinitely lived consumers, we argue that equilibria are generically determinate. This is because the effective number of equations determining equilibria is not infinite, but equal to the number of agents minus one and must determine the marginal utility of income for all but one agent. Generically, near an equilibrium, these equations are independent and exactly determine the unknowns.

When there are infinitely many overlapping generations, this reasoning breaks down: An infinite number of equations is not necessarily sufficient to determine an infinite number of unknowns. We consider whether the initial conditions

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together with the requirement of convergence to a nearby steady state locally determine an equilibrium price path. We allow two alternative types of initial conditions. In the first the old generation in the initial period has nominal claims on the endowment of the young generation. In the second the old generation has real claims. In the terminology of Samuelson [**31**], the first situation is one with fiat money and the second is one without. This paper provides a catalog of all robust examples of indeterminacy and instability that can occur near a steady state with both types of conditions: In both cases there are many economies with isolated equilibria, many with continua of equilibria, and many with no equilibria at all. With two or more goods in every period not only can the price level be indeterminate but relative prices can be as well. Interestingly, indeterminacy has little to do with Pareto efficiency: Equilibria may be determinate or indeterminate regardless of whether they are Pareto efficient or not.

We also consider an alternative conceptual experiment in which agents use a forecast rule that depends only on current prices to predict next period prices. If the steady state is stable, and if we rule out a certain peculiar case, a perfect foresight forecast rule exists. If there is a continuum of equilibria, there may be a continuum of such forecast rules. Even so, the derivative of such a rule, evaluated at the steady state prices, is locally determinate. This makes it possible to do comparative statics in a neighborhood of the steady state despite the local nonuniqueness of equilibrium. McCallum [**28**] has argued that indeterminacy can often be eliminated in linear rational expectations models by insisting that agents employ a closed-loop forecast rule that utilizes a minimal set of information. This suggestion is in very much the same spirit as our analysis of forecast rules, although differences between the two frameworks are large enough to make a close comparison impossible.

Finally, we contrast the determinacy in the model with a finite number of infinitely lived agents with the indeterminacy in the overlapping generations model. Although the models and conceptual experiments that we perform seem quite different in the two cases, following Barro [**4**], we argue that the infinitely lived model can be thought of as an overlapping generations model with bequests.

We are by no means the first to address the issues analyzed in this paper. Let us therefore briefly discuss the relationship of our results with some of those that have appeared previously:

That an overlapping generations model might have a continuum of equilibria is well known. Samuelson [**31**] himself has noted the problems with counting equations and unknowns in this type of model. Gale [**16**] has provided a complete analysis of the overlapping generations model with a single two-period lived consumer in each generation and one good in each period. In such a model he finds that indeterminacy is always associated with initial conditions that allow nominal claims: If there is no fiat money, then equilibria are always determinate. Even when there is fiat money any indeterminacy is at most one dimensional; in other words, if there is any indeterminacy, the equilibria can be indexed by a single number, for example, the price of fiat money. Balasko and Shell [**2**] have extended these results to a model in which there are many goods in each period but a single two-period lived consumer in each generation with a Cobb-Douglas utility function. In contrast to these results, our analysis indicates that indeterminacy in more general models does not depend on the existence of fiat money nor is it necessarily one dimensional; even if an index of prices relative to the stock of money is exogenously fixed, the relative prices of goods within a period may be indeterminate.

Burmeister, Caton, Dobell, and Ross [**8**] have investigated the possibilities of indeterminacy in growth models with heterogeneous capital goods. Their analysis is similar to that in this paper in that they consider a linearized system near a steady state and ask whether convergence to that steady state ensures determinacy. Their analysis differs from ours in that the savings behavior of consumers is exogenously given rather than the result of utility maximization and perfect foresight. Our analysis indicates that utility maximization and perfect foresight do rule out indeterminacy when there are a finite number of infinitely lived agents. Even in an overlapping generations economy some constraint on potential indeterminacy is imposed by whether or not there are nominal initial claims. Incidentally, although more than one dimension of indeterminacy seems to be possible within the framework of Burmeister et al., they do not mention it: All of the indeterminacy that they discuss is one dimensional.

Calvo [**9**] has constructed simple examples of indeterminacy similar to that discussed by Burmeister et al. His models differ from theirs in that savings behavior arises from utility maximization by overlapping generations consumers with perfect foresight. The indeterminacy in these examples is still one dimensional and is indexed by the price of an asset such as land or capital.

McCallum [**28**] has recently tried to put discussions of indeterminacy into perspective, arguing that indeterminacy "is simply an inescapable aspect of dynamic models involving expectations, one which is not basically attributable to the rationality assumption." In particular, he claims that even the assumption of a finite number of infinitely lived agents does not eliminate the possibility of indeterminacy. He bases this claim on a model constructed by Calvo [**10**] in which real money balances enter into utility functions and production functions, but where the money supply is set by the government in nominal terms. The analysis of the model with a finite number of infinitely lived consumers presented in this paper should cast some doubt on this claim.

Using nonstandard analysis, Brown and Geanakopolos [**7**] have independently investigated the possibility of $n-1$ dimensions of indeterminacy in a nonstationary overlapping generations model without nominal claims. At this stage, however, their approach and results are too different from ours to allow a close comparison.

Finally, the present authors have written several other papers closely related to this one: Kehoe and Levine [**22**] provide a rigorous regularity analysis of the overlapping generations model; several of the technical results of that paper are used here. An earlier version of this paper, Kehoe and Levine [**19**], contains the regularity analysis for the case of infinitely lived consumers. Kehoe and Levine [**21**] analyze the special case of an overlapping generations model with a two-period lived representative consumer with separable preferences in each generation. Balasko and Shell [3] and Geanakopolos and Polemarchakis [17] have also pointed out that strong determinacy results can be obtained for this case. In another paper, Kehoe and Levine [20] argue that the results obtained for the two-period separable case do not generalize further. There they provide a counter-example with a representative consumer with separable preferences who lives for three periods.

2. THE FINITE AGENT MODEL

We begin by analyzing a pure exchange economy with a finite number of agents who consume over an infinite number of time periods. In each period there are n goods. Each of the m different consumers is specified by a utility function of the form $\sum_{t=0}^{\infty} \gamma_i^t u_i(x_t^i)$ and a vector of initial endowments w^i that is the same in every period. Here $1 > \gamma_i > 0$ is a discount factor. We make the following assumptions on u_i and w^i:

ASSUMPTION a.1 (Differentiability): *$u_i: R^n_{++} \to R$ is C^2.*

ASSUMPTION a.2 (Strict concavity): *$D^2 u_i(x)$ is negative definite for all $x \in R^n_{++}$.*

ASSUMPTION a.3 (Monotonicity): *$Du_i(x) > 0$ for all $x \in R^n_{++}$.*

ASSUMPTION a.4 (Strictly positive endowments): *$w^i \in R^n_{++}$, $i = 1, \ldots, m$.*

ASSUMPTION a.5 (Boundary): *$\|Du_i(x_k)\| \to \infty$ as $x_k \to x$ where some $x^j = 0$, $j = 1, \ldots, n$, $Du_i(x)x$ is bounded, however, for all x in any bounded subset of R^n_{++}.*

It should be possible to extend our analysis to more general types of preferences that do not require additive separability, such as those described by Koopmans, Diamond, and Williamson [23]. We do not attempt to do so here.

Let $p_t = (p_t^1, \ldots, p_t^n)$ denote the vector of prices prevailing in period t. When faced with a sequence $\{p_0, p_1, \ldots\}$ of strictly positive price vectors, agent i chooses a sequence of consumption vectors $\{x_0^i, x_1^i, \ldots\}$ that solves the problem

$$\text{(2.1)} \qquad \max \sum_{t=0}^{\infty} \gamma_i^t u_i(x_t^i)$$
$$\text{subject to } \sum_{t=0}^{\infty} p_t' x_t^i \leqslant \sum_{t=0}^{\infty} p_t' w^i,$$
$$x_t^i \geqslant 0.$$

The purpose of Assumptions a.1–a.5 is to ensure that, for any price sequence, this problem has a solution that is strictly positive and satisfies the budget constraint with equality. Assumption a.5 ensures that consumers' indifference surfaces become parallel to the coordinate hyperplane as we move towards the boundary of the positive orthant. It is this assumption that rules out corner solutions. The necessary and sufficient conditions for $\{x_0^i, x_1^i, \ldots\}$ to solve (2.1) are

$$(2.2) \qquad \gamma_i^t Du_i(x_t^i) = \mu_i p_t' \quad \text{for some} \quad \mu_i > 0 \qquad (t = 0, 1, \ldots),$$

$$(2.3) \qquad \sum_{t=0}^{\infty} p_t' x_t^i = \sum_{t=0}^{\infty} p_t' w^i.$$

A (perfect foresight) *equilibrium* of this economy is defined to be a price sequence $\{p_0, p_1, \ldots\}$ and a sequence of consumption vectors $\{x_0^i, x_1^i, \ldots\}$ for each agent, $i = 1, \ldots, m$, that satisfy the following conditions:

CONDITION e.1: *For each agent i* $\{x_0^i, x_1^i, \ldots\}$ *solves* (2.1).

CONDITION e.2: $\sum_{i=1}^m x_t^i = \sum_{i=1}^m w^i$, $t = 0, 1, \ldots$.

To find the equilibria of this economy we utilize an approach developed by Negishi [**30**] and Mantel [**24**] for a model with a finite number of goods. Letting λ_i, $i = 1, \ldots, m$, be some strictly positive welfare weights, we set up the welfare maximization problem

$$(2.4) \qquad \max \sum_{i=1}^{m} \lambda_i \sum_{t=0}^{\infty} \gamma_i^t u_i(x_t^i)$$

$$\text{subject to} \sum_{i=1}^{m} x_t^i \leqslant \sum_{i=1}^{m} w^i \qquad (t = 0, 1, \ldots).$$

$$x_t^i \geqslant 0.$$

Again a.1–a.5 guarantee that this problem has a solution that is strictly positive and satisfies the feasibility constraint with equality. The necessary and sufficient conditions for a solution are

$$(2.5) \qquad \lambda_i \gamma_i^t Du_i(x_t^i) = p_t', \qquad (i = 1, \ldots, m),$$

$$\text{for some} \quad p_t > 0 \qquad (t = 0, 1, \ldots),$$

$$(2.6) \qquad \sum_{i=1}^{m} x_t^i = \sum_{i=1}^{m} w^i \qquad (t = 0, 1, \ldots).$$

An allocation sequence is Pareto optimal if and only if it solves problem (2.4). Notice that Condition e.2 and (2.6) are the same and, furthermore, if we set $\lambda_i = 1/\mu_i$, that (2.2) and (2.5) are also the same. In other words, a Pareto efficient allocation and associated Lagrange multipliers $\{p_0, p_1, \ldots\}$ satisfy all of our equilibrium conditions except, possibly, (2.3). The problem of finding an equilibrium therefore becomes one of finding the right welfare weights λ_i, $i = 1, \ldots, m$, so that (2.3) is satisfied.

Let $p_t(\lambda)$ and $x_t^i(\lambda)$ be the solutions to (2.5) and (2.6). The strict concavity of u_i ensures that p_t and x_t^i are uniquely defined and continuous. For each agent we define the excess savings function

$$(2.7) \qquad s_i(\lambda) = \sum_{t=0}^{\infty} p_t(\lambda)'(w^i - x_t^i(\lambda)).$$

Using Assumptions a.1 and a.5, we can show that the infinite sum in (2.7) converges uniformly on compact subsets of R_{++}^m and, consequently, that s_i is well defined and continuous.

It is easy to verify that the functions $s_i(\lambda)$ are homogeneous of degree one and sum to zero. In fact, the functions $s_i(\lambda)/\lambda_i$ have mathematical properties identical to those of the excess demand functions of a pure exchange economy with m goods. A standard argument implies the existence of a vector of welfare weights λ such that

$$(2.8) \qquad s(\lambda) = 0.$$

We call this vector λ an equilibrium since our above arguments ensure that when we solve the welfare maximization problem (2.4) using λ for welfare weights the solution is an equilibrium allocation. Conversely, any equlibrium is associated with such a vector λ.

PROPOSITION 2.1: *If the economy $((u_i, \gamma_i, w^i),\ i = 1, \ldots, m)$ satisfies Assumptions a.1–a.5, then an equilibrium exists and every equilibrium is characterized by welfare weights λ_i, $i = 1, \ldots, m$, satisfying* (2.8).

We have reduced the equilibrium conditions for the model with a finite number of consumers to a finite number of equations in the same finite number of unknowns: The homogeneity of s implies that one of the variables λ_i is redundant. That the $s_i(\lambda)$ sum to zero, however, implies that we can ignore one of the equations $s_i(\lambda) = 0$. To do regularity analysis we must be able to ensure that s is continuously differentiable. To do this as simply as possible, we impose the following additional assumption on u_i:

ASSUMPTION a.6: *$Du_iD^2u_i^{-1}$ is bounded on bounded subsets of R_{++}^n.*

Suppose, for example, that u_i is homogeneous of degree $0 < \alpha_i < 1$. Then Assumption a.6 is satisfied since $Du_i(x)D^2u_i(x)^{-1} = (\alpha_i - 1)x'$. Notice, however, that Assumption a.6 allows substantially more general preferences. Such an assumption is needed to ensure that the derivatives of u_i are well behaved even as x_t^i approaches 0. A proof of the following proposition can be found in Kehoe and Levine [**19**].

PROPOSITION 2.2: *If the economy $((u_i, \gamma_i, w^i),\ i = 1, \ldots, m)$ satisfies Assumption a.1–a.6, then s is continuously differentiable for all $\lambda > 0$.*

A *regular economy* $((u_i, \gamma_i, w^i),\ i = 1, \ldots, m)$ is defined to be one that satisfies Assumptions a.1–a.6 and the restriction:

ASSUMPTION r.1: *$Ds(\lambda)$ has rank $m-1$ at every equilibrium λ.*

This concept of regular economy is analogous to that developed by Debreu [**12**] for pure exchange economies with a finite number of goods. If an economy is regular, then the inverse function theorem implies that it has a finite number of isolated equilibria. The implicit function theorem implies that these equilibria vary continuously with the parameters of the economy. Furthermore, the topological index theorem introduced into economies by Dierker [**15**] can be used to count the number of equilibria of such economies.

The appeal of the concept of regularity is enhanced by its genericity: Almost all economies are regular. Suppose we parameterize the space of economies $((u_i, \gamma_i, w^i), i=1,\ldots,m)$ by allowing the endowments to vary while keeping their sum, $w=\sum_{i=1}^m w^i$, constant, but fixing the utility functions and discount factors. Allowing endowments to vary while keeping their sum fixed allows us to vary $s_i(\lambda)$ while keeping $P_t(\lambda)$ and $x_t^i(\lambda)$ fixed.

PROPOSITION 2.3: *Regular economies form an open dense set of full measure in the space of economies parameterized by endowments.*

The proof of this Proposition, which is given by Kehoe and Levine [**19**], is a direct application of the transversality theorem of differential topology. It can easily be extended to a proof that regular economies form an open dense subset of the space of economies where the only restrictions are Assumptions a.1–a.6 if we are careful about giving this space a topological structure.

3. THE OVERLAPPING GENERATIONS MODEL

We now analyze an economy with an infinite number of finitely lived agents, a stationary overlapping generations model that generalizes that introduced by Samuelson [**31**]. Again there are n goods in each time period. Each generation $0<t<\infty$ is identical and consumes in periods t and $t+1$. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t and $z(p_t, p_{t+1})$ in period $t+1$. The vector $p_t=(p_t^1,\ldots,p_t^n)$ denotes the prices prevailing in period t. We assume that excess demands satisfy the following assumptions:

ASSUMPTION A.1 (Differentiability): *$y, z: R_{++}^{2n}\to R^n$ are smooth functions.*

ASSUMPTION A.2 (Walras's law): $p_t'y(p_t, p_{t+1})+p_{t+1}'z(p_t, p_{t+1})=0$.

ASSUMPTION A.3 (Homogeneity): *y and z are homogeneous of degree zero.*

ASSUMPTION A.4 (Boundary): *$\|(y(q_k), z(q_k))\|\to\infty$ as $q_k\to q$ where some, but not all, $q^j=0$, $j=1,\ldots,2n$. (y, z) is bounded from below, however, for all $q\in R_{++}^{2n}$.*

Assumption A.1 has been shown by Debreu [**13**] and Mas-Colell [**26**] to entail relatively little loss of generality. Assumption A.2 implies that there is some means of contracting between generations so that each consumer faces an ordinary budget constraint in the two periods of his life. As we show later, this means the economy is one with a constant (possibly zero or negative) stock of fiat money. Assumption A.4 is a standard boundary assumption. It is used to guarantee the existence of interior steady states. Muller and Woodford [**29**] have extended the analysis presented in this paper to allow free goods; we do not attempt to do so here.

Note that we consider only pure exchange economies and two-period lived consumers. We do, however, allow many goods and types of consumers, and the multiperiod consumption case can easily be reduced to the case we consider: If consumers live m periods, we simply redefine generations so that consumers born in periods $1, 2, \ldots, m-1$ are generation 1, consumers born in periods $m, m+1, \ldots, 2m-2$ are generation 2, and so on. In this reformulation each generation overlaps only with the next generation. Notice that the number of goods in each newly defined period, and the number of consumers in each newly defined generation, increase by a factor of $m-1$. See Balasko, Cass, and Shell [**1**] for a detailed description of this procedure.

The economy begins in period 1. The excess demand of old people (generation 0) in period 1 is $z_0(a, p_1)$ where a is a vector of parameters representing the past history of the economy. A (perfect foresight) *equilibrium* of an economy (z_0, y, z) starting at a is defined to be a price sequence $\{p_1, p_2, \ldots\}$ that satisfies the following conditions:

CONDITION E.1: $z_0(a, p_1) + y(p_1, p_2) = 0.$

CONDITION E.2: $z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0, \qquad t > 1.$

Once p_1 and p_2 are determined Condition E.2 acts as a nonlinear difference equation determining all future prices. Our major focus is on the extent to which E.1 determines initial prices p_1 and p_2. The next section studies the role of initial conditions z_0 and a. Let us now ignore E.1, however, and focus attention on the difference equation E.2.

We define a *steady state* of E.2 to be a price vector $(p, \beta p) \in R^{2n}_{++}$ that satisfies

$$(3.1) \qquad z(p, \beta p) + y(\beta p, \beta^2 p) = z(p, \beta p) + y(p, \beta p) = 0.$$

In other words, if the prices p prevail forever and the price level grows by β each period, markets always clear. Here $1/\beta - 1$ is the steady state rate of interest. In the generic case Kehoe and Levine [**22**] show that up to a normalization of prices there are finitely many steady states.

Our interest in this paper is in what happens near a steady state. Let $(p, \beta p)$ be a steady state, and let $U \subset R^{2n}_{++}$ be an open cone that contains $(p, \beta p)$. It is convenient to define $q_t = (p_t, p_{t+1})$ and view E.2 as the first order difference equation

$$(3.2) \qquad z(q_{t-1}) + y(q_t) = 0, \quad t > 1.$$

We call a path $\{q_1, q_2, \ldots\}$ that satisfies Condition E.1 and E.2 locally stable with respect to $q=(p, \beta p)$ and U if $q_t \in U$ and $\lim_{t\to\infty} q_t/\|q_t\| = q/\|q\|$. The question we are trying to answer is whether or not there is a determinate price path that satisfies Conditions E.1 and E.2 and is locally stable.

One reason for restricting our attention to local stability is that it is the easiest case to study. Stable price paths are also the most plausible perfect foresight equilibria. If prices are converging to a nearby steady state, then traders can compute future prices by using only local information. If prices are not going to the steady state, then traders need global information and very large computers to compute future prices. Note that, if equilibrium is indeterminate in the restricted sense that a continuum of equilibria converge to the steady state, it is indeterminate in the broader sense as well. On the other hand, even if equilibrium is determinate in the restricted sense there may be a continuum of equilibria that leave the neighborhood of the steady state.

We can linearize E.2 around a steady state $(p, \beta p)$ as

$$D_1 z(p_{t-1} - \beta^{t-1} p) + (D_2 z + \beta^{-1} D_1 y)(p_t - \beta^t p) + \beta^{-1} D_2 y(p_{t+1} - \beta^{t+1} p) = 0. \tag{3.3}$$

Here all derivatives are evaluated at the steady state $(p, \beta p)$, and we use (3.1) and the fact that the derivatives of excess demand are homogeneous of degree minus one. Our homogeneity Assumption A.3 allows us to rewrite (3.3) as

$$D_1 z p_{t-1} + (D_2 z + \beta^{-1} D_1 y) p_t + \beta^{-1} D_2 y p_{t+1} = 0. \tag{3.4}$$

If the following regularity restriction is satisfied, then (3.4) defines a second order linear difference equation.

ASSUMPTION R.1: *$D_2 y(p, \beta p)$ is nonsingular at all steady states $(p, \beta p)$.*

Again letting $q_t = (p_t, p_{t+1})$, we can write out (3.4) as the first order equation $q_t = G q_{t-1}$ where

$$G = \begin{bmatrix} 0 & I \\ -\beta D_2 y^{-1} D_1 z & -D_2 y^{-1}(\beta D_2 z + D_1 y) \end{bmatrix}. \tag{3.5}$$

Homogeneity implies that $Gq = \beta q$ where $q = (p, \beta p)$; in other words, G has an eigenvalue equal to β. Walras's law implies that $p'[-\beta D_1 z \; D_2 y] G = p'[-\beta D_1 z \; D_2 y]$; in other words, G has an eigenvalue equal to unity. Let us assume that G also satisfies the following regularity restriction:

ASSUMPTION R.2: G is nonsingular and has distinct eigenvalues; furthermore, eigenvalues have the same modulus if and only if they are complex conjugates.

Consider the difference equation $q_t = (1/\beta) G q_{t-1}$. The steady state price vector q is a fixed point of this difference equation. Let n^s be the number of eigenvalues of $(1/\beta)G$ that lie inside the unit circle, that is, whose moduli are less than unity.

These correspond to eigenvalues of G that lie inside the circle of radius β. A standard theorem on linear difference equations implies that the set of initial conditions q_1 such that $q_t = Gq_{t-1}$ has $\lim_{t\to\infty} q_t/\|q_t\| = q/\|q\|$ is an n^s+1 dimensional subspace V_s of R^{2n} (see Irwin [**18**, pp. 151-154] and Kehoe and Levine [**22**]). The extra dimension shows up because of homogeneity: If q_1 is such that $\lim_{t\to\infty} q_t/\|q_t\| = q/\|q\|$, then so is θq_1 for any $\theta \neq 0$. The subspace V_s is spanned by the n^s eigenvectors of G associated with the eigenvalues that lie inside the circle of radius β and the eigenvector q associated with the eigenvalue β.

The implicit function theorem implies that, if Assumption R.1 is satisfied, then we can solve Condition E.2 to find a nonlinear difference equation $q_t = g(q_{t-1})$ defined for an open cone U that contains q. Naturally, $Dg(q) = G$. Let W_s be the subset of initial conditions $q_1 \in U$ such that $\lim_{t\to\infty} q_t/\|q_t\| = q/\|q\|$. In other words, given (p_1, p_2) we can find a path in U that converges to the ray proportional to $(p, \beta p)$ if and only if $(p_1, p_2) \in W_s$. The relationship between V_s and W_s is given in the following theorem:

PROPOSITION 3.1: *W_s is an n^s+1 dimensional manifold with tangent space at q equal to V_s.*

This result is proven by Kehoe and Levine [**22**]. That V_s is the tangent space of W_s at q justifies our intuition about (3.4) as a linear approximation to E.2: It says that the best linear approximation to W_s at q is affine set $V_s + \{q\}$.

To establish Proposition 3.1 we need the regularity Assumptions R.1-R.2. These can be justified by showing that they hold for almost all economies, in other words, that they hold for an open dense subset of the space of economies. This is done by Kehoe and Levine [**22**]. This means that any regular economy can be approximated by one that satisfies Assumptions R.1-R.2 and that any slight perturbation of an economy that satisfies R.1-R.2 still satisfies them.

We have remarked that G has one root equal to β and one unit root. Are we justified in assuming it satisfies no other restrictions? Might it not be the case, as for example in optimal control, that half the eigenvalues of G lie inside the unit circle and half lie outside? Kehoe and Levine [**19**] show that for any n^s satisfying $2n-1 \geqslant n^s \geqslant 0$, there exists an open set of economies that have a steady state with n^s roots inside the circle of radius β and $2n-n^s-1$ outside the circle with radius β. Furthermore, the work of Mantel [**25**] and Debreu [**14**] shows that for any excess demands (y, z) and any compact subset of R^{2n}_{++} we can find a generation of $2n$ consumers with well behaved preferences whose aggregate excess demands (y^*, z^*) agree with (y, z) on that subset. Mas-Colell [**27**] has shown that we can choose this compact subset and (y^*, z^*) so that all of the steady states of both (y, z) and (y^*, z^*) are contained in the interior of this subset. Since we are interested solely in neighborhoods of steady states, we can therefore regard A.1-A.4 as exhausting all the restrictions placed on excess demands by the assumption of utility maximization by heterogeneous consumers. A formal proof of this point is given by Kehoe and Levine [**22**].

4. DETERMINACY OF EQUILIBRIUM

The excess demand of generation 0 in period 1 is $z_0(a, p_1)$. The vector a represents the history of the system. This is our conceptual experiment: Prior to $t = 1$ the economy is on some price path. Suddenly, after generation 0 makes its savings decisions, but before p_1 is determined, an unanticipated shock occurs. No further shocks occur, and hereafter expectations are fulfilled, although there is no reason why generation 0's expectations of p_1 should be. Do the equilibrium Conditions E.1 and E.2 determine a unique path to the new steady state, at least locally? If so, we can do comparative statics, evaluating the impact of the unanticipated shock. If not, it is questionable that traders could deduce which of the many perfect foresight paths they would be on.

Note that this is not the only question we could ask. We might enquire whether for a given perfect foresight path stretching back to minus infinity there is a unique extension to plus infinity. We believe that the answer to this question is in general yes. Or we might ask whether the price paths $\{\ldots, p_{-1}, p_0, p_1, \ldots\}$ that are perfect foresight paths are locally unique. We believe that there is a large set of economies for which the answer to this question is yes, and an equally large set for which it is no. We feel that the question we have posed is the most interesting one, however, and, of these questions, the only one relevant for applied work. Another relevant question is, of course, how to handle price paths that are not near steady states. As we have mentioned, however, it is not clear that perfect foresight is a good hypothesis in such cases.

With this conceptual experiment in mind, we can now see the role played by the vector a: It represents the claims on current consumption owed to old people based on their savings decisions made in period 0. Define the money supply $\mu = p_1' z_0(a, p_1)$ to be the nominal claims of old people. Observe that in equilibrium $p_1' y(p_1, p_2) = -\mu$, by Walras's law $p_2' z(p_1, p_2) = \mu$, in equilibrium $p_2' y(p_2, p_3) = -\mu$, and so forth. Consequently, μ is the fixed nominal net savings of the economy for all time; that is, we assume that there is no government intervention in money markets.

In the steady state we have $\beta^t p' z(p, \beta p) = \mu$ and $\beta^t p' y(p, \beta p) = -\mu$. There are two cases of interest. The *nominal* case has $\mu \neq 0$. In this case it must be that $\beta = 1$. Gale [16] calls steady states of this type *golden rule* steady states. This is because for excess demand functions derived from utility maximization nominal steady states maximize a weighted sum of individual utilities subject to the constraint of stationary consumption over time. Alternatively, in the *real* case $\mu = 0$. Gale refers to steady states of this type as *balanced* steady states. In this case if $\beta = 1$ then $y(p, p) + z(p, p) = 0$ and $p' y(p, p) = 0$, which are typically n equations in the $n - 1$ unknowns p, and $\beta = 1$ is merely coincidental. Thus, when $\mu = 0$ the most interesting case is $\beta \neq 1$. Using an index theorem, Kehoe and Levine [22] prove that there is generically an odd number of steady states of each type, which, of course, implies the existence of a steady state of each type.

We suppose first that claims are denoted in nominal terms. We cannot assume that excess demand by the old $z_0(a, p_1)$ is homogeneous of degree zero in p_1. We do assume, however, that a is an element of an open subset A of a finite dimensional vector space and that the following assumptions hold:

ASSUMPTION I.1 (Differentiability): *$z_0: A \times R^n_{++} \to R^n$ is a smooth function.*

ASSUMPTION I.2 (Homogeneity): *z_0 is homogeneous of degree zero in a and p_1.*

Let $q = (p, \beta p)$ be the steady state after the shock. We make the following assumption.

ASSUMPTION I.3 (Steady state): *There exists $a_0 \neq 0$ such that $z_0(a_0, p) + y(p, \beta p) = 0$.*

In other words, when $a = a_0$ we are at a steady state. Our goal is to analyze what happens when $\|a - a_0\|$ is small. Various interpretations of this assumption are possible: Prior to $t = 0$ the economy was at or near a steady state and a temporary shock displaced it. Alternatively, a permanent shock occurred and the steady state itself was slightly displaced. All that is necessary is that there be some steady state nearby.

To analyze the impact of the shock, observe that prices (p_1, p_2) are determined by Condition E.1. Using the homogeneity of z_0, we can linearize E.1 around the steady state to find

$$(D_2 z_0 + D_1 y) p_1 + D_1 z_0 a + D_2 y p_2 = 0. \tag{4.1}$$

R.1 implies that we can solve (4.1) for p_2 as

$$p_2 = D_2 y^{-1}(D_1 z_0 + D_1 y) p_1 + D_2 y^{-1} D_2 z_0 a, \tag{4.2}$$

or, introducing, as before, $q_1 = (p_1, p_2)$,

$$q_1 = L\begin{bmatrix} a \\ p_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D_2 y^{-1} D_1 z_0 & -D_2 y^{-1}(D_2 z_0 + D_1 y) \end{bmatrix} \begin{bmatrix} a \\ p_1 \end{bmatrix}. \tag{4.3}$$

Let $U_1 \subset R^n_{++}$ denote the natural projection of U onto its first n coordinates. The implicit function theorem implies that in a neighborhood of the steady state we get a corresponding solution to the nonlinear equation E.1, $q_1 = l(a, p_1)$, defined for $p_1 \in U_1$, $a \in A$, with $Dl(a_0, p) = L$. We ask whether, for given $a \in A$, there is a unique initial $q_1 = (p_1, p_2)$ that satisfies E.1 and has an extension to a price path $\{q_1, q_2, \ldots\}$ in U that satisfies E.2 and converges to some point on the steady state ray. The results of the last section imply that the corresponding mathematical question is whether, for given a, there is unique p_1 such that $l(a, p_1) \in W_s$.

Let us consider the linear problem first. For fixed $a \in A$ (4.3) defines an n dimensional affine subspace of R^{2n}. The linearized version of W_s is V_s, which is $n^s + 1$ dimensional. We would expect, in general, that these spaces intersect in an $n + (n^s + 1) - 2n = n^s + 1 - n$ dimensional linear space. Suppose, in fact, that L satisfies the following assumption:

ASSUMPTION IR.1: *L has rank 2n.*

Note that this requires that A be at least n dimensional, in other words, that there are at least n independent ways to shock the economy. The transversality theorem of differential topology can be translated into the following result:

PROPOSITION 4.1: *Let S_a denote the set of $p_1 \in U_1$ such that $l(a, p_1) \in W_s$. For almost all $a \in A$ the set S_a, if it is nonempty, has dimension $n^s + 1 - n$.*

In other words, what we expect in general of the linear system is almost always true of the nonlinear system. Here we use almost all to mean an open dense subset of A whose complement has measure zero. If $n^s + 1 - n < 0$, this means there is no $p_1 \in U_1$ with $l(a, p_1) \in W_s$. If $n^s + 1 - n > 0$, however, S_a can either have this dimension or be the empty set. I.2 implies that S_0 is nonempty. If we can ensure that l is transversal to W_s at q, then the structural stability of transversality would imply that S_a is nonempty for all a close enough to a_0. We make the following assumption:

ASSUMPTION IR.2: *The $2n \times (n + 1 + n^s)$ matrix*

$$\left[\begin{array}{c|cccc} I & & & & \\ -D_2 y^{-1}(D_2 z_0 + D_1 y) & q & v_1 & \dots & v_{n^s} \end{array}\right]$$

has full row rank whenever $n + 1 + n^s \geqslant 2n$ where $q, v_1, \dots, v_{n^s}$ are the eigenvectors of G that span V_s.

The first n columns of this matrix span the tangent space of the manifold of vectors q_1 that satisfy $q_1 = l(a, p_1)$. The final $n^s + 1$ columns span V_s, which is the tangent space W_s. For $n^s + 1 - n \geqslant 0$ this says that l is transversal to W_s at q.

Like our previous regularity conditions, Assumptions IR.1 and IR.2 are generic: Given a y that satisfies R.1, these conditions can easily be shown to hold for almost all z_0. Under Assumptions IR.1 and IR.2, we can distinguish three cases:

(i) $n^s < n - 1$. In this case, for almost all a, S_a is empty. In other words, there are no stable paths locally. We call such a $(p, \beta p)$ an *unstable* steady state. For most initial conditions the asymptotic behavior of the system is to not reach the steady state. Such steady states are not very interesting; they are unreachable.

(ii) $n^s = n - 1$. In this case, locally stable equilibrium paths are locally unique and, in a small enough neighborhood actually unique. This is the case where we can do comparative statics and in which perfect foresight is a plausible description of behavior. This is called a *determinate* steady state.

(iii) $n^s > n - 1$. In this case there is a continuum of locally stable paths. The steady state is *indeterminate.* Comparative statics is impossible and perfect foresight implausible.

There are large sets of economies (nonempty open sets of economies) that have steady states of any desired type: unstable, determinate, or indeterminate. Thus, none of these possibilities is in any way degenerate.

Let us consider the argument that we get indeterminacy because we ask too much: Because z_0 is not homogeneous we demand that the price level be determined by initial conditions. Is it possible that this one dimensional indeterminacy is the only possible form of indeterminacy? No. If $n^2+1-n>1$, S_a has two or more dimensions, implying that there is relative price indeterminacy.

Now we turn to the case of real initial conditions. The change in conceptual experiment lies in z_0: It is homogeneous of degree zero in p_1 and satisfies Walras's law $p_1' z_0(a, p_1)=0$. Since $\mu=0$, the initial price vector must satisfy $p_1' y(p_1, p_2)=0$. This restriction defines a $2n-1$ dimensional manifold in some neighborhood of the steady state $(p, \beta p)$ if $(p, \beta p)$ is a regular point of $p_1' y(p_1, p_2)$, in other words, if $(y'+p'D_1 y, p'D_2 y)$ does not vanish at $(p, \beta p)$. This, however, follows immediately from R.1. We call this manifold the *real* manifold and denote it Q_r. Its tangent space at $(p, \beta p)$ is made up of the vectors (p_1, p_2) that satisfy $(y'+p'D_1 y)p_1+p'D_2 y p_2=0$. Differentiating Walras's law with respect to p_1, we establish that $y'+p'D_1 y+\beta p'D_1 z=0$ at $(p, \beta p)$. Consequently, the condition defining the tangent space of Q_r can be expressed

$$p'[-\beta D_1 z \quad D_2 y]\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}=0. \tag{4.4}$$

The stability of the system is determined by the roots of $(1/\beta)G$. Recall that Walras's law implies that $p'[-\beta D_1 z \quad D_2 y]G=p'[-\beta D_1 z \quad D_2 y]$. In other words, $(1/\beta)G$ has an eigenvalue $1/\beta$ associated with a (left) eigenvector that is orthogonal to the tangent space of Q_r. Consequently, the root $1/\beta$ has no effect on the behavior of the system on Q_r. Outside of Q_r, however, the root $1/\beta$ determines the behavior of the system. In particular, if $\beta<1$, no path with initial conditions that do not satisfy $p_1' y(p_1, p_2)=0$ can ever approach the real steady state.

We let $\bar{n}^s$ be the number of roots of $(1/\beta)G$, excluding the root $1/\beta$, that lie inside the unit circle. Because of homogeneity, including that of z_0, the price level is indeterminate and we can reduce everything by one dimension by a price normalization. In this reduced space Q_r has $2n-2$ dimensions, while the initial condition $z_0(a, p_1)+y(p_1, p_2)=0$ generically determines an $n-1$ dimensional submanifold. The intersection of stable manifold W_s with Q_r has dimension $\bar{n}^s$. Consequently, the intersection of the initial condition submanifold and W_s has dimensional $(n-1)+\bar{n}^s-(2n-2)$. Thus, there are the same three possibilities in the real case as in the nominal case, although in the real case $0 \leqslant \bar{n}^s \leqslant 2n-2$ while in the nominal case $0 \leqslant n^s \leqslant 2n-1$. In particular, notice that, if $\beta>1$ and $\bar{n}^s=n-1$, then the steady state is determinate for real initial conditions but has a one dimensional indeterminacy for nominal initial conditions.

So far we have assumed that $D_2 y$ is nonsingular at every steady state. Suppose instead that $D_2 y$ has rank k, $0 \leqslant k<n$, on an open neighborhood of the steady state $(p, \beta p)$. In this situation linearizing E.2 produces an $n+k$, rather than a $2n$, dimensional first order difference equation to replace (3.5). Otherwise our analysis stays the same. In the nominal case the determinacy condition remains $n^s=n-1$

where now $0 \leq n^s \leq n+k-1$. In the real case the determinacy condition remains $\bar{n}^s = n+1$ where now $0 \leq \bar{n}^s \leq n+k-2$. In particular, if $k=1$, only a one dimensional indeterminacy is possible in the nominal case, and no indeterminacy is possible in the real case.

That $D_2 y$ has rank one at a steady state where $n \geq 2$ is true only for a closed, nowhere dense set of economies; it is a degenerate situation. Yet, if each generation consists of a single, two-period lived consumer who has an intertemporally separable utility function, then both $D_2 y$ and $D_1 z$ have at most rank one. That $D_1 z$ has rank one implies that $n-1$ of the $n+k=n+1$ eigenvalues are zero. The determinacy conditions for an economy of this type are therefore the same as for an economy with only one good in every period. This has been noted by previous authors: Balasko and Shell [**2**], who assume consumers with Cobb-Douglas preferences, and Geanakopolos and Polemarchakis [17]. A more complete discussion of these issues can be found in Kehoe and Levine [**21**].

It might be conjectured that in the case where excess demand is derived from consumer optimization over well behaved preferences that the Pareto inefficiency of paths is related to the indeterminacy of equilibrium. A moment's reflection on the real case shows this is not true. If $\beta < 1$, prices along paths converging to the steady state decline exponentially in the limit; this means that the value of every agent's endowment is finite, and, by a standard argument due to Debreu [**11**], all these paths are efficient. But $\beta < 1$ implies only that no path with $\mu \neq 0$ ever approaches the real steady state; it places no restriction on $\bar{n}^s$. Thus if $n > 1$ indeterminacy is possible. Conversely, if $\beta > 1$, then an argument due to Balasko and Shell [**2**] implies that all convergent paths are inefficient, but there is still no restriction on the possible types of steady states.

Perhaps the case $\beta < 1$ is the most puzzling of all: Here if $n \geq 2$ we can have indeterminacy among equilibria converging to the steady state, yet all these paths are Pareto efficient and all mimic the finite dimensional case in that Walras's law is satisfied even by the initial generation.

We conclude this section by noting that there are six possible types of steady states: real or nominal, each of which may be unstable, determinate, or indeterminate. If there are two or more goods each period then there are open sets of economies with each possible combination. The case with one good each period, which has been studied most extensively, is exceptional however: Instability is impossible and, in the real case, indeterminacy is also impossible.

5. FORECASTING

In this section we examine the case of nominal initial conditions in more detail. We again focus on the neighborhood of a stable steady state $(p, \beta p)$ with $n^s \geq n-1$, and we assume that all regularity conditions are satisfied. Our focus is on how agents forecast future prices. One possibility is that they use the dynamic equation E.2; equivalently, they forecast $q_{t+1} = g(q_t)$. Note that unless $n^s = 2n-1$ this is actually an unstable dynamical system: Small perturbations can cause the path to depart from the steady state.

We now investigate the alternative possibility that traders forecast future prices solely as a function of current prices. This type of closed-loop forecasting leads to convergence to the steady state. Surprisingly, it also is locally determinate: This restriction on forecasting rules is sufficient to eliminate much of the indeterminacy we found in the previous section, making local comparative statics possible. Not surprisingly, such forecasting is impossible when the steady state is unstable. Here we only examine nominal initial conditions to keep the presentation as simple as possible; an analogous analysis can be done for real initial conditions.

A closed-loop forecast rule is a function $p_{t+1} = f(p_t)$ that gives prices next period as a function of current prices. We assume that f satisfies the following assumptions:

ASSUMPTION F.1 (Differentiability): *f is a smooth function defined on an open cone $U_1 \subset R^n_{++}$ that contains the steady state relative prices p.*

ASSUMPTION F.2: *$f(p) = \beta p$.*

ASSUMPTION F.3 (Homogeneity): *f is homogeneous of degree one.*

ASSUMPTION F.4 (Perfect foresight): $z(p_t, f(p_t)) + y(f(p_t), f^2(p_t)) = 0$.

ASSUMPTION F.5 (Convergence): $\lim_{t\to\infty} f^t(p_1)/\|f^t(p_1)\| = p/\|p\|$ *for all* $p_1 \in U_1$.

Here, for example, $f^2(p)$ denotes $f(f(p))$. Assumption F.2 insists that at the steady state the forecast rule pick out the steady state. F.4 is the perfect foresight assumption: If forecasts are realized, markets indeed clear. Assumption F.5 says we are interested only in forecast rules that permit convergence to the steady state, in other words, are stable.

We begin by asking whether, for $n^s \geq n-1$, there actually exists a forecast rule that satisfies Assumptions F.1–F.5. As before, we consider the linearized problem first. To construct a forecast rule we choose $v_1, \ldots, v_{n-1}, q$ to be independent eigenvectors in V_s, the stable subspace of the linearized system. It is important that we be able to choose $v_1, \ldots, v_{n-1}$ so that complex vectors appear in conjugate pairs. This can always be done if $n-1$ is even. It can also always be done if $n^s = n-1$ since $v_1, \ldots, v_{n-1}$ includes all of the eigenvectors corresponding to eigenvalues inside the circle of radius β and such eigenvectors necessarily show up in complex conjugates. In the peculiar case where $n-1$ is odd and there are no real eigenvalues inside the circle of radius β, and hence no real eigenvectors in V_s, we cannot make this choice of $v_1, \ldots, v_{n-1}$. This is no accident: In this case there are no stable perfect foresight forecast rules.

Let V_* be the real vector space spanned by $v_1, \ldots, v_{n-1}, q$; because complex vectors come in conjugate pairs, it is n dimensional. What we suggest is, for given p_t, to choose p_{t+1} so that $(p_t - \beta^t p, p_{t+1} - \beta^{t+1} p)$ is an element of V_*. From the structure of g there exists a unique choice of p_{t+1} provided that

ASSUMPTION FR.1: *$v_1^1, \ldots, v_{n-1}^1$, p are independent vectors where v_i^1, $i = 1, \ldots, n-1$, consists of the first n components of the v_i.*

If FR.1 holds, we can find a unique matrix F, which depends on $v_1, \ldots, v_{n-1}$, so that

$$(5.1) \qquad (p_{t+1} - \beta^{t+1} p) = F(p_t - \beta^t p)$$

is our linear forecast rule.

First we check that the linearized system (5.1) satisfies the linearized versions of F.2–F.5. Since $q \in V_*$, $(p, p) \in V_*$ and, consequently, $Fp = \beta p$. Since $v_1, \ldots, v_{n-1}, q$ are eigenvectors of G, V_* is invariant under the dynamical system G, which means that if $q_t \in V_*$ then $Gq_t \in V_*$. Finally, since $V_* \subset V_s$ and $(p_t - \beta^t p, p_{t+1} - \beta^{t+1} p) \in V_*$, we must have $\lim_{t\to\infty} p_t/\|p_t\| = p/\|p\|$.

It is natural to conjecture that we can thus find an f with $Df(p) = F$ that satisfies Assumption F.1–F.5; this follows from Hartmann's smooth linearization theorem in Irwin [**18**, p. 117]. Because g is homogeneous of degree one, f may also be chosen to be homogeneous of degree one. If $n^s = n-1$, then f is unique. This is well known when f is linear (see, for example, Blanchard and Kahn [**6**]). If, however, $n^s > n-1$, f may not be unique nor even locally unique. Furthermore, in the case where $n-1$ is odd and all the eigenvalues of G that lie inside the circle of radius β are complex, f does not even exist. The derivative $Df(p) = F$ at the steady state is locally unique, however; there are only finitely many possibilities. To see this write Assumption F.4 as $(f(p_t), f^2(p_t)) = g(p_t, f(p_t))$. Differentiating this at p we see that

$$(5.2) \qquad \begin{bmatrix} F \\ F^2 \end{bmatrix} = G \begin{bmatrix} I \\ F \end{bmatrix}.$$

Writing F in Jordan canonical form as $F = H\Lambda H^{-1}$, we see that

$$(5.3) \qquad \begin{bmatrix} H\Lambda \\ H\Lambda^2 \end{bmatrix} = G \begin{bmatrix} H \\ H\Lambda \end{bmatrix}.$$

R.2 implies that Λ is diagonal with diagonal entries equal to eigenvalues of G and that the columns of

$$\begin{bmatrix} H \\ H\Lambda \end{bmatrix}$$

are the corresponding eigenvectors of G. Since G has only finitely many eigenvalues, there are only finitely many choices of F; indeed, our original construction is the only way to get solutions that satisfy the stability requirement F.5.

Notice that, if $n^s > n+1$, there are in general many possible choices of $v_1, \ldots, v_{n-1}$ and, consequently, of F. The important fact is that there are only a finite number of choices. Furthermore, under our regularity assumptions, F varies smoothly with small changes in the parameters of (y, z). When doing comparative statics faced with a choice of finitely many forecast rules, we choose the unique F that corresponds to the forecast rule being used before the shock.

Finally, let us check the initial condition; it is now

$$z_0(a, p_1) + y(p_1, f(p_1)) = 0. \tag{5.4}$$

We can locally solve for p_1 as

$$p_1 = -(D_2 z_0 + D_1 y + D_2 y F)^{-1} D_1 z_0 a. \tag{5.5}$$

6. CONCLUDING REMARKS

We conclude by summarizing our results and indicating some possible directions for future research. When there are finitely many infinitely lived consumers we have shown equilibria are generically determinate. In the overlapping generations case we have argued that determinacy, indeterminacy, and instability are all possible for a wide range of economies.

There are a number of obvious differences in the specifications of these two models and in the conceptual experiment that we have performed on them. We should not, however, let these differences, which are actually more superficial than substantive, obscure the striking contrast in the two sets of results. One obvious difference in the specification of the two models is that in the first consumer behavior is specified in terms of utility functions and endowments while in the second it is specified in terms of aggregate excess demand. As we have explained, however, the results of Mantel [**24**], Debreu [**14**], and Mas-Collell [**27**] justify using the concept of aggregate excess demand in the overlapping generations model: As long as the number of consumers in each generation exceeds the number of goods, in this case $2n$, then all that utility maximization implies about aggregate excess demand is given by Assumptions A.1–A.4. In other words, we could have derived the same results for the overlapping generations model if we had specified it in terms of utility functions and endowments; we have employed the excess demand formulation only because it is more convenient. In the model with a finite number of infinitely lived consumers, however, the Mantel-Debreu theorem does not hold: There are more goods than consumers. Reducing the dimension of the problem of characterizing equilibria to the number of consumers, a finite number, is the crucial step in our arguments. It is, in fact, an open question whether a model with an infinite number of infinitely lived consumers generically has determinate equilibria.

A very restrictive aspect of our specification of the model with infinitely lived consumers is that utility is additively separable and discounted at a constant rate. This implies, for example, that, if all consumers do not have the same discount factor γ_i, then those with discount factors less than the maximum asymptotically consume nothing. We conjecture, however that our results carry over to models with far more general preferences; it would be worthwhile to verify this.

We should point out that the determinacy results in the model with infinitely lived consumers are in no way related to the determinacy results obtained for the overlapping generations model with a single consumer with additively separable utility in each generation: The latter results depend crucially on the assumption of a representative consumer while the former do not. It would certainly be useful, however, to know that indeterminacy in the overlapping generations model can arise not only for an open set of economies, but for economies with reasonable preferences. One step in this direction is taken by Kehoe and Levine [**20**], who illustrate all the possibilities of determinacy, indeterminacy, and instability in an economy where each generation is a single three-period lived consumer with constant-elasticity-of-substitution utility. A more general characterization of the eigenvalues in terms of assumptions on preferences would also be worthwhile.

Another obvious difference in the specifications of the two models is the concept of equilibrium: In the model with infinitely lived consumers we allow general price paths while in the overlapping generations model we restrict our attention to paths that converge to a steady state. This should not weaken the contrast in the results for the two models, however, since indeterminacy of the more restricted concept of equilibrium implies indeterminacy of the general concept. Our analysis of equilibria in the overlapping generations model can be applied to equilibrium price paths that converge to cycles if we redefine generations and time periods: A cycle of k periods can be viewed as a steady state of a model where each newly defined generation consists of k original generations and each newly defined time period includes k of the original time periods. If we were to analyze the model with infinitely lived consumers using the requirements of convergence to steady states, we would, of course, find that any equilibrium price path is either determinate or unstable. Studying a model more general than ours, Bewley [**5**] has demonstrated that, if the largest discount factor γ_i is sufficiently close to unity, then all equilibria do, in fact, converge to steady states. Yet another difference in the two models is the nature of the two conceptual experiments: In the overlapping generations model the initial old generation may hold claims on the endowments of the young, either nominal or real, positive or negative. In the model with infinitely lived consumers, in contrast, all income is generated from the sale of own initial endowments. Having nominal claims is impossible with infinitely lived consumers. Real claims, however, are easily accommodated as (possibly nonstationary) changes in the structure of endowments. As long as these changes do not result in any consumer's income becoming nonpositive at equlibrium, our analysis follows through as before.

It might be thought that the differences in the determinacy results comes from the requirement in the infinitely lived consumer model for some kind of transversality conditions, which the overlapping generations model does not have. This is not the case, however. All that the transversality conditions would guarantee in the overlapping generations model is that $\lim_{t\to\infty} p_t = 0$. This obviously precludes equilibria with fiat money. It also guarantees Pareto efficiency, as Balasko and Shell [**1**] have demonstrated. Since it is possible to have indeterminate equilibrium price paths converging to a steady state with $\beta < 1$, however, it cannot guarantee determinacy.

To make the differences between the two models appear even smaller, let us point out that the model with infinitely lived agents can be thought of as an overlapping generations model where consumers leave bequests to their offspring or give gifts to their progenitors. Barro [4] has argued that, if the lifetime utility levels of offspring and progenitors enter into the utility functions of each member of an infinitely lived family of consumers, then that family acts as if it were a single infinitely lived consumer. Although there is no general reason to suspect that this infinitely lived consumer would have a utility function with a constant discount factor and additive separability, it is certaintly possible to think of specifications in which he would. One obvious problem with this kind of specification is that, if all discount factors are not equal, then some families asymptotically consume nothing: Consumers use almost all of their income to service the debt of their progenitors, which they, in turn, pass on to their offspring. We have already conjectured, however, that the determinacy results hold for more general specifications that avoid this kind of problem.

What are the properties of a model in which some consumers leave bequests and some do not, in other words, a model with some infinitely lived agents and some finitely lived agents? Muller and Woodford [29] have addressed this question using the approach we have developed here. They find that, although the presence of infinitely lived consumers rules out equilibria that are Pareto inefficient or include fiat money, it does not rule out indeterminacy. They also extend our analysis to include economies with production and infinitely lived assets. They are able to identify a number of cases where equilibria are determinate, although, as in the model we have analyzed here, there are open sets of economies with indeterminate equilibria.

Our results raise many interesting questions: Does the determinacy versus indeterminacy result depend on the finite versus infinite number of agents or on the infinite versus finite lifetimes? What mathematical properties do models with an infinite number of infinitely lived agents possess? In the overlapping generations model we have studied the behavior of equilibrium price paths near steady states. Is it possible to say much about their behavior away from steady states? Throughout the paper we have assumed perfect foresight expectations. What theoretically attractive alternatives exist? How far do we have to depart from the perfect foresight assumption to get determinacy?

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