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Gross substitutability and the weak axiom of revealed preference

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This paper investigates the relationship between gross substitutability and the weak axiom of revealed preference in excess demand functions. Gross substitutability implies the weak axiom when the number of goods, n, is less than four. There are robust counterexamples to this proposition when $n \geqq 4$. Nonetheless, Monte Carlo experiments indicate that violations of the weak axiom are extremely rare in functions that satisfy gross substitutability. They also reveal, however, a new class of demand function that violates the weak axiom. These results are of interest because of the close relationship of the weak axiom to uniqueness of equilibrium in production economies.

1. Introduction

The assumptions of gross substitutability and the weak axiom of revealed preference have played major roles in studies of uniqueness in pure exchange economies. In production economies gross substitutability in consumers' demand functions loses much of its relevance: Kehoe (1985b) presents an example with four goods, four Cobb–Douglas consumers, and an activity analysis production technology that has three equilibria. In contrast, it has been known since the time of Wald (1936) that the weak axiom implies uniqueness even in production economies.

In this paper, we investigate the relationship between the gross substitutability and the weak axiom. Kehoe's example demonstrates that the gross substitutability cannot imply the weak axiom since the weak axiom is not compatible with multiplicity of equilibria. Nonetheless, Monte Carlo experiments indicate that violations of the weak axiom are extremely rare in

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functions that satisfy gross substitutability. Since the set of functions that satisfy gross substitutability contains a non-empty open subset of the set of all excess demand functions, these results indicate a new direction for research on the uniqueness question.

In recent years, computer simulation models have become an important research tool for economic theorists, especially in generating numerical examples of their models. Computers can also help a theorist searching for counterexamples before trying to prove a conjecture. Although widespread in econometrics [see, for example, Hendry (1984)], Monte Carlo methods have not been previously employed in economic theory. At this stage, our results are tentative and intended mostly to encourage future research. Even so, extensive computer search for examples of demand functions that satisfy gross substitutability but violate the weak axiom has had an interesting theoretical outcome: it reveals a new class of demand functions that violate the weak axiom.

2. The weak axiom of revealed preference

Consider an excess demand function $\xi: R^n_{++} \to R^n$ that is continuously differentiable, bounded below, and homogeneous of degree zero. We assume that ξ satisfies Walras's law $[\pi'\xi(\pi) \equiv 0]$ and a boundary condition ($\pi^k \to \pi^0$, where $\pi^k \in R^n_{++}$ and $\pi^0 \neq 0$, $\pi^0_i = 0$, implies $\|\xi(\pi^k)\| \to \infty$). Homogeneity implies that $D\xi(\pi)\pi \equiv 0$. Walras's law implies that $\pi' D\xi(\pi) \equiv -\xi(\pi)'$.

An excess demand function ξ satisfies the *weak axiom of revealed preference* if $\xi(\pi^1) \neq \xi(\pi^2)$ and $\pi^{1\prime}\xi(\pi^2) \leqq 0$ imply $\pi^{2\prime}\xi(\pi^1) > 0$. If ξ satifies the weak axiom, then the set of equilibrium prices is convex, and there is almost always a unique equilibrium. If ξ does not satisfy the weak axiom, then a constant-returns production technology can be constructed so that there are multiple equilibria. Consequently, the weak axiom is necessary for uniqueness of equilibirum if the production technology is arbitrary, and is almost always sufficient for uniqueness. See Kehoe (1985b) for details; the sufficiency part was first demonstrated by Wald (1936); the necessity part was pointed out to the writer by Herbert Scarf.

The following result, which characterizes the weak axiom in terms of restrictions on the derivatives of ξ, follows closely ones developed by Kihlstrom, Mas-Colell and Sonnenschein (1976) for compensated demand function. See also Freixas and Mas-Colell (1987) and Kehoe (1985c).

Theorem 1. Suppose that ξ satisfies the weak axiom of revealed preference. Then, for every $\pi \in R^n_{++}$, $D\xi(\pi)$ is negative (quasi) semidefinite on the null space of $\xi(\pi)$; that is, $v' D\xi(\pi) v \leqq 0$ for all $v \in R^n$ such that $v'\xi(\pi) = 0$. Conversely, suppose that if, for every $\pi \in R^n_{++}$ and every $v \in R^n$ that satisfy $v'\xi(\pi) = 0$ where

$v \neq 0$ and where $v/\|v\| \neq \pi/\|\pi\|$, $v'D\xi(\pi)v < 0$. Then ξ satisfies the weak axiom of revealed preference.

Let A be an $n \times n$ matrix, not necessarily symmetric, and let b be an $n \times 1$ vector. A_{kk} denotes the $k \times k$ matrix, $1 \leqq k \leqq n$, formed by the elements in the first k rows and columns of A. b_k denotes the $k \times 1$ vector formed similarly. Let p be a permutation of the first n integers. A^p denotes the matrix obtained from A by performing the permutation p on its rows and columns. b^p denotes the vector obtained from b by performing the permutation p on its elements. A classical result from the theory of constrained optimization says that $v'Av \geqq 0$ for all $v \in R^n$ such that $v'b = 0$ if and only if

$$\det \begin{bmatrix} 0 & b_{ik}^{p'} \\ -b_k^p & 1/2A_{kk}^p + 1/2A_{kk}^{p'} \end{bmatrix} \geqq 0$$

for all $i = 2, \ldots, n$ and all p [see, for example, Debreu (1952)]. Furthermore, $v'Av > 0$ for all $v \in R^n$ such that $v \neq 0$ and $v'b = 0$ if and only if the above inequalities are all strict. In fact, to ensure that these inequalities hold strictly it is sufficient to check only that they hold for the leading principal minors, $p = (1, 2, \ldots, n)$.

Any vector $v \in R^n$ that satisfies $v \neq 0$, $v/\|v\| \neq \pi/\|\pi\|$, and $v'\xi(\pi) = 0$ can be uniquely expressed as $u + t\pi$ where u satisfies $u \neq 0$, $u'\xi(\pi) = 0$, and $u'\pi = 0$. Notice that $(\pi + tu)'D\xi(\pi)(\pi + tu) = u'D\xi(\pi)u$ since $\pi'D\xi(\pi) = -\xi(\pi)'$ and $D\xi(\pi)\pi = 0$. Our necessary and sufficient conditions for the weak axiom can, therefore, be reduced to conditions on the bordered matrix

$$\begin{bmatrix} 0 & 0 & \xi(\pi)' \\ 0 & 0 & \pi' \\ -\xi(\pi) & -\pi & -1/2D\xi(\pi) - 1/2D\xi(\pi)' \end{bmatrix}.$$

Let $\bar{J}_i$ be the $(n-1) \times (n-1)$ matrix formed by deleting row i and column i from the matrix $D\xi(\pi)$. Let $\bar{x}_{-i}$ be the $(n-1) \times 1$ vector formed by deleting element i from $\xi(\pi)$. Arguments using elementary linear algebra imply that $v'D\xi(\pi)v < 0$ for every $v \in R^n$ that satisfies $v \neq 0$, $v/\|v\| \neq \pi/\|\pi\|$ if and only if $\bar{u}'\bar{J}_{-i}\bar{u} < 0$ for some $\bar{J}_{-i}$, $i = 1, \ldots, n$ and every $\bar{u} \in R^{n-1}$ that satisfies $\bar{u} \neq 0$ and $\bar{u}'\bar{x}_{-i} = 0$ [see Kehoe (1985c)]. Furthermore, if these conditions hold for some $\bar{J}_{-i}$, they hold for every $\bar{J}_{-i}$, $i = 1, \ldots, n$. In other words, it makes no difference what row and column we delete from $D\xi(\pi)$ to form $\bar{J}_{-i}$. These same results continue to hold if all strict inequalities are replaced by weak inequalities.

Consequently, we can check for the weak axiom by checking the principal minors of the matrix

$$\bar{D}=\begin{bmatrix} 0 & \bar{x}'_{-1} \\ -\bar{x}_{-1} & -1/2\bar{J}_{-1}-1/2\bar{J}'_{-1} \end{bmatrix}.$$

The problem is that these conditions must hold at every $\pi \in R^n_{++}$ for the weak axiom to hold. In practice, therefore, they are very difficult conditions to check.

3. Gross substitutability

An excess demand function ξ exhibits *gross substitutability* if $(\partial\xi_i/\partial\pi_j)(\pi) > 0$ for all $\pi \in R^n_{++}$ and $i \neq j$. Homogeneity implies that

$$-\sum_{j=2}^{n} \pi_j \frac{\partial \xi_i}{\partial \pi_j}(\pi) = \pi_1 \frac{\partial \xi_i}{\partial \pi_1}(\pi) > 0,$$

$i=2,\dots,n$. Consequently, since $-\bar{J}_{-1}$ has positive diagonal elements and negative off-diagonal elements, it is a P matrix, a matrix with all its principal minors positive. Now any positive definite matrix is a P matrix, but a P matrix is not necessarily positive definite unless it is symmetric.

That gross substitutability implies that the weak axiom holds in comparisons between the equilibrium of a pure exchange economy and any other price vector was first demonstrated by Arrow, Block and Hurwicz (1959). This result is trivial to demonstrate given the conditions that characterize the weak axiom developed in the previous section [see Kehoe (1985a)]. Furthermore, given these conditions, it is easy to argue that gross substitutability implies that the weak axiom holds globally if $n \leqq 3$. [In fact, Kehoe and Mas-Colell (1984) provide a simple proof of this latter result that dispenses with the differentiability requirement.]

Kehoe (1985b), however, presents an example with four goods that satisfies gross substitutability but violates the weak axiom. The following simplified version of that example is due to Mas-Colell. Suppose that $\pi=(1,1,1,1)$ and that

$$D\xi(\pi)=\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Walras's law implies that $\xi(\pi)=(2,1,-2,-1)$. Suppose that we set

$v=(1,2,0,4)$. Then $v'\xi(\pi)=0$ but $v'D\xi(\pi)v=2>0$. Theorem 1 therefore implies that any excess demand function consistent with $D\xi(\pi)$ must violate the weak axiom.

We can perturb this example slightly so that all off-diagonal elements are strictly positive and still have a Jacobian matrix that violates the necessary conditions for the weak axiom. Furthermore, since we can extend this argument to the $n\times n$ matrix $D\xi(\pi)$, $n\geqq 4$, examples consistent with gross substitutability but not the weak axiom can be constructed for any $n\geqq 4$.

4. A Monte Carlo study

In this section we try to answer the question of how common are examples that satisfy gross substitutability but violate the weak axiom. In a sense, such examples are common: In the topology of uniform C^1 convergence on compact sets two functions are close if their values and the values of their derivatives are uniformly close on compact subsets of R^n_{++}. The set of functions that satisfy (strong) gross substitutability, is an open subset of this function space. Furthermore, the set of functions that allow strong violations of the weak axiom, $\pi^{2\prime}\xi(\pi^2)<0$ and $\pi^{2\prime}\xi(\pi^1)<0$, is also an open set. We can easily construct an excess demand function that has a Jacobian matrix $D\xi(\pi)$ similar to the example in the previous section at $\pi=(1,\ldots,1)$. Set

$$\xi_i(\pi)=\sum_{j=1}^{n} d_{ij}(\pi_j/\pi_i)-\sum_{j=1}^{n} d_{ji},$$

where d_{ij} is the corresponding element of $D\xi(\pi)$. It is easy to check that ξ satisfies all of our assumptions on excess demand functions and that $(\partial\xi_i/\partial\pi_j)(\pi)=d_{ij}$ when $\pi=(1,\ldots,1)$. We could, in fact, generate ξ from an economy with n Cobb–Douglas consumers using a procedure illustrated in Kehoe (1985c). Consequently, the example of the previous section, at least when we perturb it to have strictly positive off-diagonal elements, can be viewed as an element in the intersection of two open subsets of the space of demand functions. Since the intersection of two open sets is open, we have demonstrated that, if $n\geqq 4$, there is a non-empty open set of demand functions that satisfy gross substitutability but violate the weak axiom.

Let us approach the question of how common are examples like that in the previous section in a different way. We can check for violations of the weak axiom by examining the derivatives of ξ at a price vector π. If we normalize relative qualities of goods so that $\pi=(1,\ldots,1)$, then homogeneity implies that the row sums of $D\xi(\pi)$ are zero, and Walras's law implies that the column sums of $D\xi(\pi)$ are $-\xi(\pi)'$. We can randomly generate examples of matrices with these properties by generating n^2-n numbers d_{ij}, $i=1,\ldots,n$,

$j=1,\ldots,n$, $i\neq j$, and using the homogeneity restriction to compute d_{ii}. We can then use Walras's law to compute $\xi(\pi)$ and check the principal minors of the relevant matrix $\bar{D}$ to see if ξ violates the weak axiom.

By generating a large number of such examples using a pseudo-random number generator on a computer, we can estimate the proportion of all Jacobian matrices that correspond to demand functions that satisfy gross substitutability but violate the weak axiom. To illustrate the procedure followed, consider an example where $n=4$: Suppose that we choose 12 random numbers from the unit interval and get 0.172, 0.800, 0.096, 0.080, 0.320, 0.400, 0.008, 0.004, 0.004, 0.004, 0.008, 0.004, 0.004. There is no restriction involved in choosing all the elements d_{ij}, $i\neq j$, to lie in the unit interval since this corresponds to a one-dimensional rescaling of the absolute quantities of goods. We arrange these elements in order in a 4×4 matrix, filling in the diagonal elements so that row sums are zero:

$$D\xi(\pi)=\begin{bmatrix} -1.068 & 0.172 & 0.800 & 0.096 \\ 0.080 & -0.800 & 0.320 & 0.400 \\ 0.008 & 0.004 & -0.016 & 0.004 \\ 0.008 & 0.004 & 0.004 & -0.016 \end{bmatrix}.$$

Walras's law says that

$$\xi(\pi)=\begin{bmatrix} -0.972 \\ -0.620 \\ 1.108 \\ 0.484 \end{bmatrix}.$$

To test whether the $D\xi(\pi)$ corresponds to an excess demand function that violates the weak axiom, we need only check that

$$\det\begin{bmatrix} 0 & -0.620 & 1.108 & 0.484 \\ 0.620 & 0.800 & -0.162 & -0.202 \\ -1.108 & -0.162 & 0.016 & -0.004 \\ 0.484 & -0.202 & -0.004 & 0.016 \end{bmatrix}$$

is positive because we already know that the determinant of this matrix with the final row and column deleted is positive because it has the right sign pattern. In fact, the weak axiom is violated because the crucial determinant is

-0.00591. [This example is actually the one used by Kehoe (1985b) to construct a production economy with multiple equilibria.]

To estimate the proportion of 4×4 matrices that correspond to excess demand functions that violate the weak axiom, Kehoe (1985c) generates 250,000 12-tuples of random numbers distributed independently and uniformly on the unit interval. Unlike the numbers in the above example, each of the elements d_{ij} is stored as a double precision real number, which has about 15 significant decimal figures. All arithmetic is done in double precision. Diagnostic statistics show that the numbers generated have good random properties, and no cycling occurs. Of the 250,000 examples, exactly 250,000 correspond to demand functions that satisfy the weak axiom and 0 to functions that do not.

There are two obvious conjectures to account for this startling result: The first is that gross substitutability coupled with a violation of the weak axiom becomes more and more common as the number of goods increases: it is impossible when $n=3$, rare when $n=4$, but common for large n. The second conjecture is that distributing the numbers d_{ij} independently and uniformly on $[0,1]$ makes the matrix $D\xi(\pi)$ tend to be fairly symmetric. Symmetry would cause $D\xi(\pi)$ to be negative semi-definite, which would preclude violations of the weak axiom. Indeed, both examples of violations of the weak axiom that we have discussed involve substantial asymmetries.

To test the first conjecture, for each integer n, $4\leqq n\leqq 20$, 50,000 random examples have been tested. Each such example involves n^2-n random numbers, and requires the evaluation of $n-2$ determinants. The most efficient method for doing this is to compute the products of the successive pivots involved in the LDU factorization of $\bar{D}$. Since the number of multiplications required to do this factorization is approximately $n^3/3$, testing large numbers of examples becomes expensive for large n. For each n the result has been the same: 50,000 examples correspond to functions that satisfy the weak axiom and 0 to functions that do not.

To see how startling this result is, let us compute the mean of the Bayesian posterior distribution of the proportion of matrices that correspond to violations of the weak axiom for each n. Suppose that our prior distribution of this proportion p is that it is uniformly distributed between 0 and 1. Remember that p cannot be 0 because there is a non-empty open set of matrices parametrized by the n^2-n numbers d_{ij}, $i\neq j$, that correspond to functions that violate the weak axiom, and a non-empty open set cannot have measure zero. We compute the posterior distribution of p as follows:

$$P(p\leqq x\,|\,n=0)=P(n=0\,|\,p\leqq x)P(p\leqq x)/P(n=0)$$

$$=1-P(n=0\,|\,p\geqq x)P(p\geqq x)/P(n=0)$$

$$=1-\left(\int_x^1 (1-z)^{50,000}\,dz\right)\left(\int_x^1 dz\right)\Big/\left(\int_0^1 (1-z)^{50,000}\,dz\right)$$

$$=1-(1-x)^{50,002}.$$

The posterior density is the derivative of this function,

$$f(x|n=0)=50{,}002(1-x)^{50,001}.$$

Consequently, the mean of the posterior distribution is

$$E(p|n=0)=\int_0^1 50{,}002x(1-x)^{50,001}\,dx=1/50{,}003.$$

Notice how small this number is compared to the mean of the prior distribution, 1/2. Notice too how little effect the prior distribution has because of the large numbers of observations.

To test the conjecture that our results are due to tendencies towards symmetry in $D\xi(\pi)$, the same set of 50,000 experiments have been repeated for each n, $4 \leqq n \leqq 20$, with a simple change in the distribution of d_{ij}. Each d_{ij} is still distributed independently on the unit interval, but uniform random numbers are subjected to the change of variable

$$y=\phi(x)=1/2-1/2(1-2x)^{1/4} \quad \text{if} \quad 0 \leqq x \leqq 1/2$$

$$=1/2+1/2(2x-1)^{1/4} \quad \text{if} \quad 1/2 \leqq x \leqq 1.$$

The resulting probability density function is the derivative of the inverse of this change of variable, $\phi^{-1}(y)$, multiplied by the original probability density function, $f(x)=1$. This density function is, therefore,

$$f(y)=4(1-2y)^3 \quad \text{if} \quad 0 \leqq y \leqq 1/2$$

$$=4(2y-1)^3 \quad \text{if} \quad 1/2 \leqq y \leqq 1.$$

It is depicted in fig. 1 along with the original uniform density function. As can be seen, this change of variable has the effect of making very large and very small values of d_{ij} more probable than intermediate values.

For each n, $5 \leqq n \leqq 20$, the experiment has produced 50,000 examples that correspond to functions that satisfy the weak axiom and 0 to functions that do not. For $n=4$, however, 49,994 examples correspond to functions that

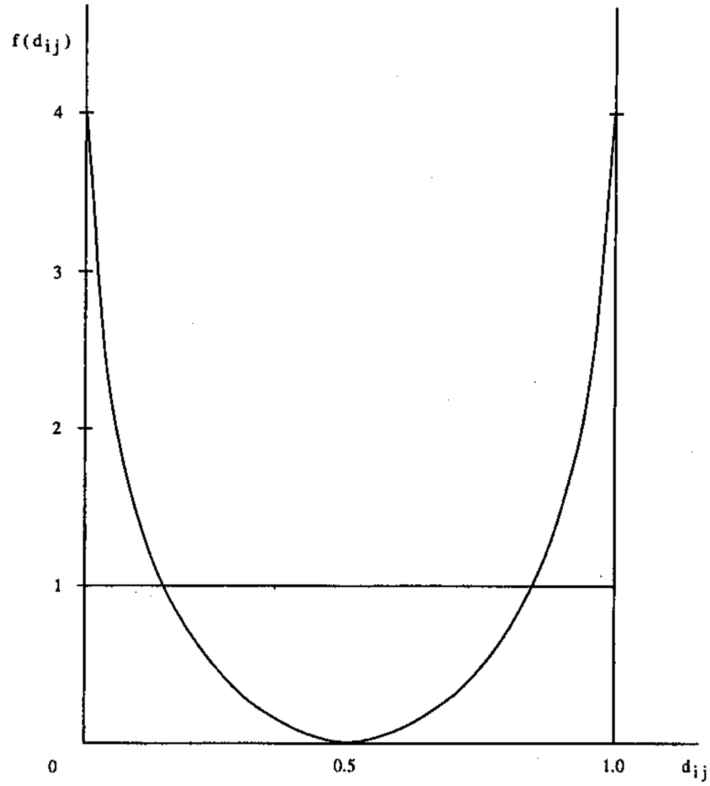

Fig. 1

satisfy the weak axiom and 6 to functions that do not. These six examples are listed below.

$$\begin{bmatrix} -0.939 & 0.886 & 0.033 & 0.020 \\ 0.011 & -0.041 & 0.022 & 0.008 \\ 0.053 & 0.080 & -1.113 & 0.980 \\ 0.002 & 0.038 & 0.137 & -0.177 \end{bmatrix},$$

$$\begin{bmatrix} -0.015 & 0.005 & 0.003 & 0.007 \\ 0.925 & -1.706 & 0.053 & 0.728 \\ 0.032 & 0.017 & -0.186 & 0.137 \\ 0.019 & 0.162 & 0.988 & -1.169 \end{bmatrix},$$

$$\begin{bmatrix} -1.675 & 0.709 & 0.085 & 0.881 \\ 0.109 & -1.100 & 0.975 & 0.016 \\ 0.112 & 0.072 & -0.185 & 0.001 \\ 0.017 & 0.020 & 0.004 & -0.041 \end{bmatrix},$$

$$\begin{bmatrix} -0.963 & 0.904 & 0.028 & 0.031 \\ 0.012 & -0.078 & 0.042 & 0.024 \\ 0.090 & 0.052 & -0.152 & 0.010 \\ 0.024 & 0.009 & 0.906 & -0.939 \end{bmatrix},$$

$$\begin{bmatrix} -0.174 & 0.047 & 0.010 & 0.017 \\ 0.916 & -1.214 & 0.266 & 0.032 \\ 0.022 & 0.012 & -0.136 & 0.102 \\ 0.035 & 0.267 & 0.987 & -1.289 \end{bmatrix},$$

$$\begin{bmatrix} -0.068 & 0.006 & 0.000 & 0.062 \\ 0.970 & -1.124 & 0.036 & 0.118 \\ 0.003 & 0.023 & -0.069 & 0.043 \\ 0.025 & 0.774 & 0.888 & -1.687 \end{bmatrix}.$$

The posterior probability density function of p, given a uniform prior is

$$f(x|n=6)=\frac{50{,}002!}{49{,}995!6!}x^6(1-x)^{49{,}995}.$$

The posterior mean is 7/50,003. The computation of the posterior probability density functions and posterior means for $5 \leqq n \leqq 20$ is the same as previously.

It would seem that the probability of violating the weak axiom decreases as n increases. This is certainly not the case when the elements of $D\xi(\pi)$ are not constrained to satisfy the gross substitutability restrictions: When the elements d_{ij}, $i \neq j$, are uniformly distributed between -1 and 1, for example, the proportion of matrices that violate the weak axiom restrictions increases sharply as n increases.

As a final measure of how common violations of the weak axiom conditions are, all of the possible ways of arranging 0.001, 1, and 2 among the 12 off-diagonal elements of a 4×4 matrix $D\xi(\pi)$ have been computed and checked for violations of the weak axiom. Some examples result in situations where $\xi(\pi)=0$, for example, when $D\xi(\pi)$ is symmetric. Here the weak axiom is automatically satisfied since $-\bar{J}_{-1}$ is positive definite in these cases. These situations could safely be ignored in the case where d_{ij} is continuously distributed. Of the $3^{12}=531{,}441$ possible arrangements, 528,284 satisfy the weak axiom conditions and 3,144 violate them. The proportion of violations is, therefore, 0.00592.

To test our Monte Carlo procedure, 50,000 examples have been generated for $n=4$, with the change of variable

$$\begin{aligned} y &= 0.001 \quad \text{if} \quad 0 \leqq x \leqq 1/3, \\ &= 1 \quad \text{if} \quad 1/3 < x \leqq 2/3, \\ &= 2 \quad \text{if} \quad 2/3 < x \leqq 1. \end{aligned}$$

Of these, 315 violate the weak axiom conditions; the expected number of violations is 296. With a uniform prior, the posterior mean of the proportion of violations is $316/50{,}003 = 0.00632$.

This final experiment provides us with an opportunity to compare our results with those of Grodal and Hildenbrand (1989). They argue that, if $D\xi(\pi)$ has the form

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix},$$

where A is square and B has rank greater than or equal to two, then ξ must violate the weak axiom. [See also Hildenbrand (1989) and Eber (1989).] In the case where $n=4$ both A and B are 2×2. Notice that Mas-Colell's example has this form. The probability of choosing a matrix of this form at random is very small. however: Only $3^6=729$ of the possible arrangements of off-diagonal elements have the final six all equal to 0.001, and of these, 279 have B either singular or approximately singular. Since this type of example continues to violate the weak axiom even with its rows permuted, and there are six possible ways to permute the rows, $2{,}700 = 6(729-279)$ of the 3,144 violations of weak axiom have this form. Of the 444 matrices that violate the weak axiom but do not have this form, two more or less typical ones are of the forms

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix}.$$

5. Decomposable demand functions and the weak axiom

This final example of a Jacobian matrix that violates the weak axiom conditions suggests a general result: its rows and columns can be permuted to produce

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$

Notice that this matrix has a pattern that would be generated by an economy with two groups of consumers, the first group with preferences for and endowments of the first two goods, and the second group with preferences for and endowments of the final two goods. The only connection between these two subeconomies would be through the production sector. An international trade model where trade is modeled as a production activity would have this property.

That decomposable demand functions violate the weak axiom is a general result, which, to the writer's knowledge, has not previously been noticed.

Theorem 2. Suppose that $\xi: R^n_{++} \to R^n$ can be decomposed into $l \geqq 2$ excess demand functions $z^i: R^{n_i}_{++} \to R^{n_i}$, $i=1,\dots,l$, $n_1+\cdots+n_l=n$, that satisfy continuity, boundedness from below, the boundary condition, homogeneity, and Walras's law. Suppose that there are two of these functions, say z^1 and z^2, such that $q^{1'}z^1(p^1)<0$, $p^{1'}z^1(q^1)>0$ for some $p^1, q^1 \in R^{n_1}_{++}$ and $q^{2'}z^2(p^2)>0$, $p^{2'}z^2(q^2)<0$ for some $p^2, q^2 \in R^{n_2}_{++}$. Then ξ violates the weak axiom of revealed preference.

Proof. Choose $p^i \in R^{n_i}_{++}$, $i=3,\dots,l$ arbitrarily. Consider the price vectors $p=(p^1, \alpha p^2, p^3, \dots, p^l)$ and $q=(\beta q^1, q^2, p^3, \dots, p^l)$ where α and β are positive scalars. Then

$$\begin{aligned} q'\xi(p) &= \beta q^{1'}z^1(p^1) + q^{2'}z^2(\alpha p^2) \\ &= \beta q^{1'}z^1(p^1) + q^{2'}z(p^2) \end{aligned}$$

by Walras's law and homogeneity. Similarly

$$p'\xi(q) = p^{1'}z^1(q^1) + \alpha p^{2'}z^2(q^2).$$

Choosing α and β large enough, we have $q'\xi(p)<0$ and $p'\xi(q)<0$. □

Violations of the weak axiom in the strong form hold for an open set of demand functions in the topology of uniform C^0 convergence on compact sets. Consequently, any perturbation of a decomposable demand function that is suitably small in this topology also violates the weak axiom.

6. Concluding remarks

Although there are robust examples of demand functions that satisfy gross substitutability but violate the weak axiom when $n \geqq 4$, such functions are extremely rare. Our results do not allow us to say much more than this. There is obviously more work to be done: Uniqueness of equilibrium is a desirable property of an economic model. Applied economists seem to regard non-uniqueness as a theoretical curiosum, while theorists seem to accept it as commonplace. Perhaps non-uniqueness is rare in practice, but, at present, the easy-to-check-for conditions that guarantee uniqueness are impossibly restrictive. Kehoe and Whalley (1985), for example, have conducted exhaustive searches to verify that two fairly typical large-scale numerical general equilibrium models have unique equilibria; yet neither model satisfies any known condition that implies uniqueness of equilibrium.

It should be stressed that our comparisons involve Jacobian matrices evaluated at a point and not excess demand functions themselves. Verifying that the weak axiom is satisfied everywhere seems to be an impossibly difficult task except in very special cases, such as where there exists a representative consumer, or conditions like Hildenbrand's (1983) restrictions on income distribution are satisfied. To offset this limitation in our approach we make three comments: First, checking that gross substitutability is satisfied globally by an individual excess demand function is fairly easy [see Fisher (1972)]. Furthermore, gross substitutability aggregates in the sense that the sum of functions that satisfy gross substitutability also satisfy it. Second, if a function violates the weak axiom, then it must violate the derivative conditions at some point, and that violation could always be turned into one of our examples by rescaling prices and quantities. Third, uniqueness of equilibrium require only that the weak axiom is satisfied locally at equilibrium price vectors. This is precisely our condition on derivatives.

There are three warnings worth making about possible interpretations of our results: First, gross substitutability itself is a very restrictive condition to impose on consumer demand functions. If consumers have constant-elasticity-of-substitution demand functions, for example, gross substitutability fails to hold at all prices unless the elasticity of substitution is greater than or equal to one. This is particularly restrictive in dynamic models since the bulk of empirical evidence suggests that the elasticity of substitution in consumption over time is substantially less than one [see, for example, Mankiw, Rotemberg and Summers (1985)]. Second, even the weak axiom is not sufficient for uniqueness of equilibrium in models that allow distortionary taxes and subsidies, which are important features of most applied models. See Kehoe (1985a) for a summary of what is known about uniqueness of equilibrium in such models. Third and finally, the results of Monte Carlo experiments in high dimensions should be treated with caution. Suppose, for example, we look at n-tuples of random numbers independently and uniformly distributed between -1 and 1. The proportion of such vectors that lie in the positive orthant is 2^{-n}, a number that becomes very small as n grows. Does this mean that vectors in the positive orthant are not interesting or important? Certainly not. Analogously, although we would rarely choose at random a Jacobian matrix with several rows all zero as in Mas-Colell's example, we would naturally end up with such a matrix in a production economy in which several goods are inelastically supplied.

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