the celebrated article of Leontief (1954) which uncovered the apparent paradox that US exports were more labour-intensive than her imports. Leontief's dramatic finding spurred considerable further empirical research motivated by the desire to find a satisfactory explanation. The role of natural resources, the increasing scarcity of which in the US caused capital to be substituted for it in import-competing production, was stressed by Vanek (1963). The role of 'human' capital was stressed by Kenen (1965) and a number of empirical investigators, who found that US exports were considerably more skill-intensive than her imports, even though physical capital-intensity was only weakly correlated with exports and imports. This pointed to the need to reinterpret the simple Heckscher–Ohlin model in terms of skilled and unskilled labour as the two factors, rather than labour of uniform quality and physical capital. Since the formation of skill through education is an endogenous variable, a function of a wage differential that is itself a function of trade, we need a general equilibrium model that can simultaneously handle both these aspects, a task that was attempted in Findlay and Kierzkowski (1983).

Many other extensions of the Heckscher–Ohlin theory are surveyed in Jones and Neary (1984), while Deardorf (1984) gives a very incisive account of the attempts at empirical testing of the theory of comparative advantage in its different manifestations. Ethier is a very helpful guide to comparative advantage with many goods and factors. Finally, the crucial role of increasing returns to scale in specialization and international trade has only recently been rigorously investigated, since it implies departures from perfect competition. Helpman and Krugman (1985) thoroughly examines present knowledge in this area.

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See also HECKSCHER–OHLIN TRADE THEORY; INTERNATIONAL TRADE; RICARDO, DAVID.

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comparative dynamics. *See* MODELS OF GROWTH; NEOCLASSICAL GROWTH MODELS.

comparative statics. Comparative statics is the method of analysing the impact of a change in the parameters of a model by comparing the equilibrium that results from the change with the original equilibrium. The term 'statics' is not usually meant to have descriptive content: although terms like 'comparative dynamics' and 'comparative steady states' are sometimes used, comparative statics analysis can be performed on dynamic economic models. The restrictive aspects of this analysis are that there is no analysis of the historical forces that have brought about the original equilibrium position and no analysis of the transitional process involved in the adjustment from one equilibrium positon to another. The use of comparative statics, of comparing one equilibrium position with another, is as old as economics itself. It was, for example, the method used by Hume (1752) in his analysis of an increase in the stock of gold on prices in an economy.

The neoclassical method of comparative statics analysis was formalized by Hicks (1939) and, most clearly, by Samuelson (1947). This method makes heavy use of differential calculus to analyse the impact of small (infinitesimal) changes in the parameters of model on its equilibrium. Samuelson considered a system of equations involving endogenous variables and exogenous variables, or parameters, of the form:

$$\begin{gathered} f_1(p_1,\ldots,p_n,\alpha_1,\ldots,\alpha_m)=0 \\ \vdots \\ f_n(p_1,\ldots,p_n,\alpha_1,\ldots,\alpha_m)=0. \end{gathered}$$

Here $(\alpha_1,\ldots,\alpha_m)$ is a vector of parameters that specifies the economic environment. It may include the government policy parameters such as tax rates; it may also include completely exogenous factors such as variables describing the state of the weather or consumers' tastes and preferences. The vector $(p_1,\ldots,p_n)$ is a vector of endogenous variables that specifies the state of the economic system. It may include production levels and allocation levels for different goods. For the neoclassical economist, however, the typical endogenous variable is the price of a good. The equations themselves are the equilibrium conditions. A solution to them is taken to specify completely the state of the economic system.

Samuelson's type of model can be written out compactly in vector notation as

$$f(p,\alpha)=0$$

where $f\colon P\times A\to R^n$, $P\subset R^n$, and $A\subset R^m$. The important thing to notice is that the number of endogenous variables and the number of equilibrium conditions are equal. Assume that (p^0,α^0) is a solution to this system of equations, that the function f is continuously differentiable, and that the $n\times n$ matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial f_1}{\partial p_1}(p^0,\alpha^0)\ldots\frac{\partial f_1}{\partial p_n}(p^0,\alpha^0) \\ \vdots \qquad\qquad \vdots \\ \frac{\partial f_n}{\partial p_1}(p^0,\alpha^0)\ldots\frac{\partial f_n}{\partial p_n}(p^0,\alpha^0) \end{bmatrix}$$

is invertible. Then the inverse function theorem of differential calculus says that, in a small neighbourhood of p^0 in P, this implies that $p^0=f^{-1}(0,\alpha^0)$ is the locally unique solution to the equilibrium conditions. Furthermore, under the same assumptions, the implicit function theorem says that the locally unique vector p that satisfies $f(p,\alpha)=0$ varies continuously with α near α^0. In other words, there exists a continuous function $p(\alpha)$ such that

$$f[p(\alpha),\alpha]\equiv 0.$$

To calculate the impact of small changes in α on p we differentiate with respect to α using the chain rule:

$$\begin{gathered} Df_p[p(\alpha),\alpha]Dp(\alpha)+Df_\alpha[p(\alpha),\alpha]\equiv 0 \\ Df_p(p^0,\alpha^0)Dp(\alpha^0)+Df_\alpha(p^0,\alpha^0)=0 \\ Dp(\alpha^0)=-Df_p(p^0,\alpha^0)^{-1}Df_\alpha(p^0,\alpha^0). \end{gathered}$$

The elements of the $n\times m$ matrix

$$Dp(\alpha^0)=\begin{bmatrix} \frac{\partial p_1}{\partial\alpha_1}(\alpha^0)\ldots\frac{\partial p_1}{\partial\alpha_m}(\alpha^0) \\ \vdots \qquad\qquad \vdots \\ \frac{\partial p_n}{\partial\alpha_1}(\alpha^0)\ldots\frac{\partial p_n}{\partial\alpha_m}(\alpha^0) \end{bmatrix}$$

are called the comparative statics multipliers: for example, when the first parameter changes from α_1^0 to α_1, the equilibrium value of the second endogenous variable changes from p_2^0 to approximately

$$p_2^0+\frac{\partial p_2}{\partial\alpha_1}(\alpha^0)(\alpha_1-\alpha_1^0),$$

at least if $\alpha_1-\alpha_1^0$ is small enough.

The prototypical model to which Samuelson applied such comparative statics analysis was Walras's (1874) model of economic equilibrium. Here the endogenous variables are the prices of the n goods in the economy and the equilibrium conditions are requirements that excess demands (demand minus supply) for these goods be equal to zero. Since the number of endogenous variables and the number of equilibrium conditions are both equal to n, it would seem that the method of analysis described above is immediately applicable. Walras realized, however, that there were two offsetting complications. First, multiplying all prices by any positive constant leaves excess demands unchanged, that is, excess demands are homogeneous of degree zero, since this is merely a change in accounting units. Second, the total value of all demands equals the value of all supplies since all income is spent on goods. This second requirement, known as Walras's Law, can be written out as

$$\sum_{i=1}^{n} p_i f_i(p,\alpha)\equiv 0.$$

Walras considered an economy in which all prices were strictly positive. He used the homogeneity condition to reduce the numbers of endogenous variables by one, setting $p_1=1$ as numeraire. He used Walras's Law to reduce the number of equilibrium conditions by one, ignoring the first one since

$$f_1(p,\alpha)\equiv-\sum_{i=2}^{n} p_i f_i(p,\alpha).$$

Walras gave two arguments for the existence of equilibrium. The first involved the counting of equations and unknowns given above. The second involved a transitional process for adjusting prices when not in equilibrium, which he called

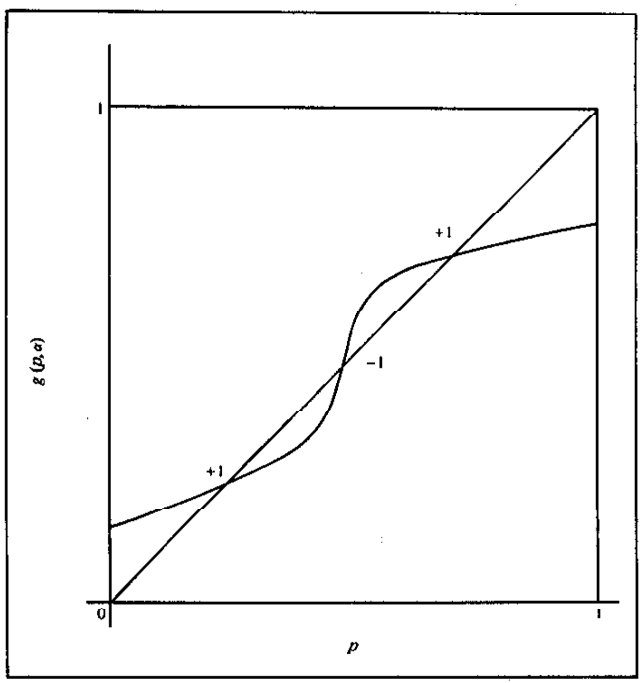

Figure 1

tâtonnement. Samuelson formalized this process in a system of differential equations:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = f_i(p, \alpha), \qquad i = 1, \ldots, n.$$

Near an equilibrium p^0 this system can be linearized as

$$Dp(t) = Df_p(p^0, \alpha)[p(t) - p^0].$$

The requirement that the equilibrium p^0 be locally stable involves restrictions on the eigenvalues of the matrix $Df_p(p^0, \alpha)$. This is the same matrix whose inverse plays the crucial role in determining the comparative statics multipliers. Samuelson's view was that, by making assumptions on the sign pattern of the elements of $Df_\alpha(p^0, \alpha)$, the responses of excess demands to changes in the parameters, and by imposing the requirement of local stability on $Df_p(p^0, \alpha)$, 'meaningful theorems' about the signs of the comparative statics multipliers $Dp(\alpha)$ could be derived. He called this methodology 'The Correspondence Principle'. Unfortunately, however, except for very low dimensional cases ($n = 2, 3$) very few such theorems seem available. The *tâtonnement* process itself is unattractive because it offers no real time interpretation: an auctioneer calls out prices and agents announce their excess demands. The auctioneer then adjusts prices until excess demands equal zero. Until this point is reached no production or consumption takes place; once it is reached all production and consumption take place, and the economy shuts down.

In the 1950s attention turned away from derivation of the comparative statics properties of a model to proofs of the existence of equilibrium. Wald (1936) had pointed out the inadequacy of Walras' existence arguments and had provided existence proofs. The role of differential calculus in providing the mathematical tools was assumed by topology and convexity theory. Using these tools, Arrow and Debreu (1954) and others were able to provide proofs of the existence of equilibrium for very general models. The principal mathematical tool employed was Brouwer's fixed point theorem, which says that any continuous mapping g of a non-empty, compact, convex set into itself has a point that stays fixed under the mapping, that is, where $p^* = g(p^*)$. This theorem is illustrated in Figure 1, where

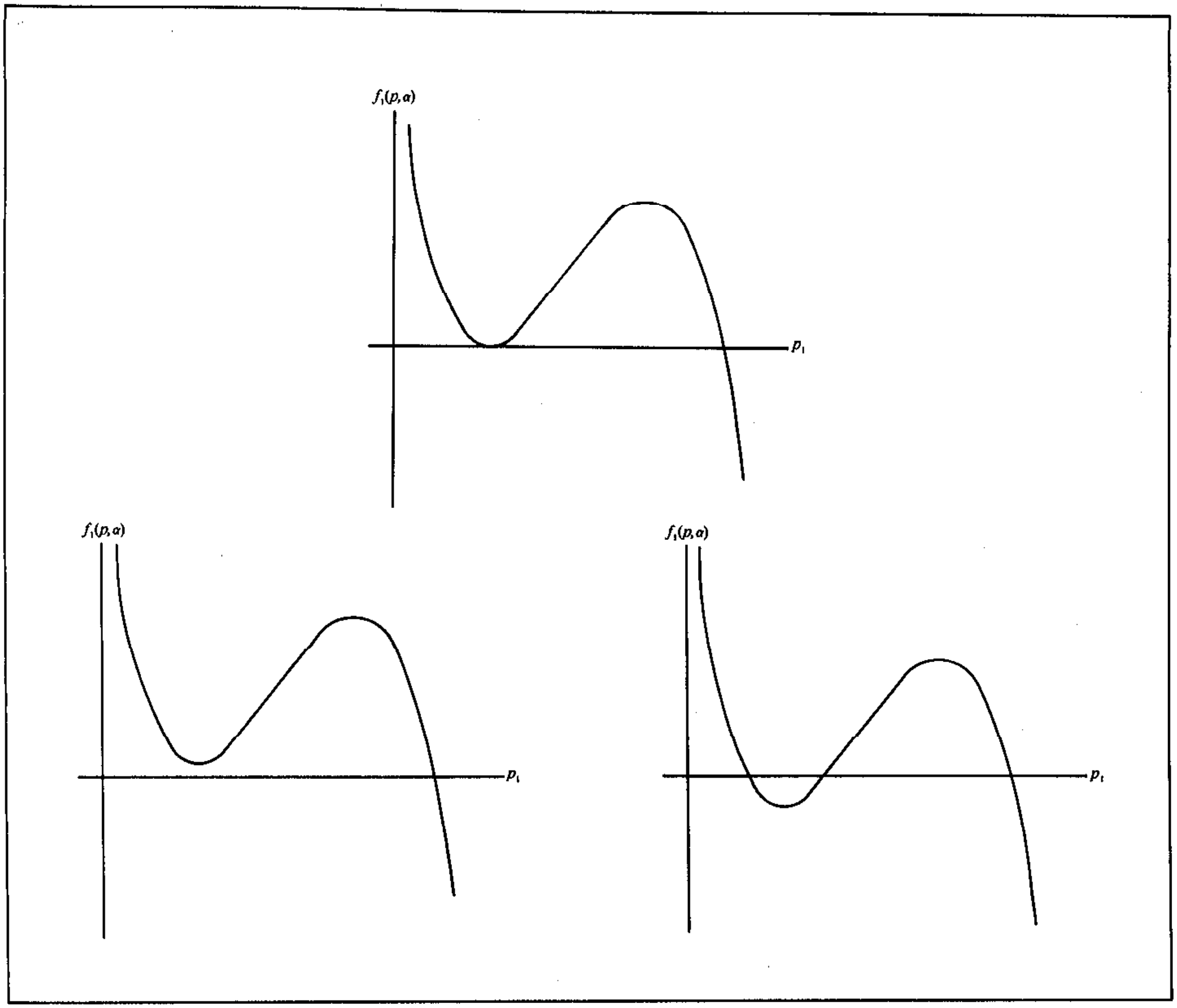

Figure 2

the non-empty, compact convex set is the interval (0, 1). Notice that there are three fixed points.

To use Brouwer's fixed point theorem it is necessary to convert equilibrium conditions of the form $f(p, \alpha) = 0$ into the form $p - g(p, \alpha) = 0$, where the mapping g has the right properties. The price domain is often taken to be the simplex

$$S = \left\{ p \in R^n \mid \sum_{i=1}^{n} p_i = 1, p_i \geq 0 \right\},$$

which is a non-empty, compact, convex set. The mapping g is constructed by manipulating $p + f(p, \alpha)$ until it only takes on values in S.

Scarf (1973) developed a computational algorithm for finding the fixed points of mappings that satisfy the conditions of Brouwer's fixed point theorem. As our discussion above indicates, this algorithm can be used to compute equilibria of economic models. Following this development, a number of economists have employed large-scale numerical general equilibrium models to analyse the impact of policy changes such as changes in tax systems and tariff rates on individual countries or groups of countries. The methodology is that of comparative statics: The parameters of the model are calibrated or econometrically estimated to match a historically given state of the economy. A change in some of the parameters is then introduced and a new equilibrium position is calculated. Finally, the new equilibrium is compared with the original equilibrium. Such models have been developed to incorporate such phenomena as fixed prices, rationing, government intervention, and simple dynamic considerations. Shoven and Whalley (1984) offer a survey. An earlier tradition in large-scale use of comparative statics started with the work of Johansen (1960), who used calculus techniques to analyse the impact of small changes in parameters.

Proofs of the existence of an equilibrium leave many important questions unanswered: for example, is the equilibrium price vector unique? If not, is it locally unique? Does it vary continuously with the parameters of the model? Answers to such questions are essential in any application of comparative statics analysis. To answer such questions, economists have made heavy use of the tools of differential topology, which combines those of calculus and topology (see Mas-Colell, 1985). Debreu (1970) developed the concept of a regular economy to answer the questions of local uniqueness and continuity. A regular economy is one for which the matrix of partial derivatives of $p - g(p, \alpha)$ with respect to p, $I - Dg_p(p, \alpha)$, where I is $n \times n$ identity matrix, is invertible at every equilibrium p^*. Not surprisingly, this turns out to be equivalent to the condition that the $(n-1) \times (n-1)$ matrix formed by deleting the first row and column from $Df_p(p, \alpha)$ is invertible. That this latter matrix is invertible is, of course, what is meant by the claim that the equilibrium conditons involve equal numbers of independent equations and unknowns once one price has been fixed as numeraire and one equation deleted. Debreu proved that almost all models, in a strict mathematical sense, are regular. This result provides some justification for Samuelson's comparative statics methodology. It is illustrated in Figure 2, where small perturbations to a model that is not regular result in it becoming a regular economy.

Dierker (1972) and Varian (1975) realized that the tools of differential topology could also be used to answer the uniqueness question. Consider again the illustration of Brouwer's fixed point theorem in Figure 1. Suppose that neither 0 or 1 is a fixed point of g and that the graph of g never becomes tangent to the diagonal. Then the graph of g must cross the diagonal one more time from above than it does from below. Associate an index

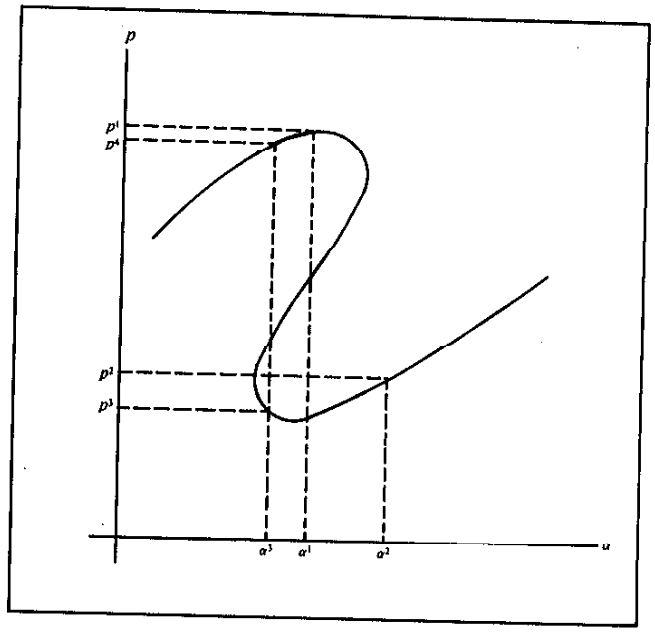

Figure 3

$+1$ with a fixed point p^* if the graph of g crosses the diagonal from above and an index -1 if it crosses from below. This index can be computed by finding the sign of the expression $1 - (\partial g/\partial p)(p^*, \alpha)$. The sum of the indices of all fixed points is $+1$. Consequently, if index $(p^*) = +1$ at every equilibrium, then there is only one equilibrium, but, if index $(p^*) = -1$ at any equilibrium, there must be multiple equilibria. The fixed point index theorem says that these results hold in higher dimensions when the index of a fixed point is computed by finding the sign of the expression $\det[I - Dg_p(p^*, \alpha)]$. We have already seen that, if this expression is non-zero, then the equilibrium p^* is unique and varies continuously with α. The fixed point index theorem says that this expression is also crucial for conditions that guarantee uniqueness of equilibrium.

Unfortunately such conditions appear to be extremely restrictive. For Walrasian models with production, for example, Kehoe (1985b) argued that the only two such conditions with economic interpretations are (i) that the consumer side of the economy behaves as though there were only one consumer, and (ii) that the conditions of the non-substitution theorem are met, so that prices are determined by production conditions alone. In applied models where taxes and other distortions play significant roles not even these conditions suffice (see Kehoe, 1985a). In the presence of multiple equilibria, the value of comparative statics analysis becomes problematical. We must pay attention to the historical forces that have brought about the original equilibrium position and to the transitional process involved in the adjustment from one equilibrium position to another. Even if we start with a historically given equilibrium that is locally stable with respect to some adjustment process, we can run into problems in the face of mathematical catastrophes of the sort depicted in Figure 3. Here the change in the value of the parameter α from α^1 to α^2 brings about a discontinuous jump from p^1 to p^2. Another danger that non-uniqueness presents is the possibility that a researcher might compute p^4 as the new equilibrium after changing the parameter from α^1 to α^3, without even knowing that p^4 exists.

The problem of non-uniqueness of equilibrium is even more acute in dynamic models (such as Samuelson's, 1958 overlapping generations model) that involve an infinite number of consumers and goods. Such models need not have even locally unique equilibria, and no current version of comparative statics analysis seems to be applicable (see Kehoe and Levine, 1985).

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See also ENVELOPE THEOREM; LE CHATELIER PRINCIPLE.

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compensated demand. I. HICKS COMPENSATION – DEFINITION; II. BASIC PROPERTIES; III. UNCOMPENSATED DEMAND; IV. SLUTSKY COMPENSATION; V. HISTORICAL NOTES.

I. HICKS COMPENSATION – DEFINITION

When a consumer faces a price change under a given nominal income, his utility (or real income) level as well as his demand vector changes. Suppose, however, that his income level is simultaneously changed as the price is changed so as to keep his utility at the initial level. This operation may be regarded as a compensation for the price change, and we call the resulting demand vector the *compensated demand* for the new price.

Thus the compensated demand is a function of the price vector and the utility level, and we may write it as

$$x = h(p, \mu), \tag{1}$$

where x and p are the consumption and price vectors, while μ is the utility level. We call h the *compensated* (or *Hicks*) *demand function*. Formally, it may be defined as the solution function of the following minimization problem:

$$\min_x p'x \text{ subject to } \mathrm{u}(x) = \mu, \tag{2}$$

where u is the utility function.

II. BASIC PROPERTIES

(*a*) *Hicksian Demand Rules.* The Jacobian matrix of h with respect to p, denoted as h_p, is nothing but the Hicks substitution matrix. It has well-known properties:

$$p'h_p(p, \mu) = 0. \tag{3}$$

$$yh_p(p, \mu)y \leqslant 0 \quad \text{for all } y. \tag{4}$$

$$h_p(p, \mu) = h_p'(p, \mu). \tag{5}$$

Condition (3) is called the homogeneity condition, since it shows that the function h is homogeneous of degree zero with respect to p. Conditions (4) and (5) are called the negative semi-definiteness and the symmetry conditions, respectively. We will call these three conditions the *Hicksian Demand Rules*.

(*b*) *Shephard–Samuelson Lemma.* The minimized expenditure value of problem (2) is a function of p and μ. This is called the *expenditure function*. Formally, we define it by

$$e(p, \mu) \equiv \min_x \{p'x \,|\, \mathrm{u}(x) = \mu\}.$$

By definition, we obviously have

$$e(p, \mu) \equiv p'h(p, \mu). \tag{6}$$

There is a less obvious, but extremely useful, relationship between the compensated demand and the expenditure functions:

$$e_p(p, \mu) \equiv h(p, \mu). \tag{7}$$

This identity usually referred to as the *Shephard–Samuelson Lemma* was obtained by Hicks (1946, p. 331), Samuelson (1947, p. 68; 1953–54, pp. 15–16), Shephard (1953).

To prove the Shephard–Samuelson Lemma, let x^* be an expenditure-minimizing vector that yields μ at the price p^*, i.e.,

$$x^* \equiv h(p^*, \mu). \tag{8}$$

Define the *gain function* g by

$$g(p) \equiv e(p, \mu) - p'x^*. \tag{9}$$

This and the definition of e imply that $g(p) \leqslant 0$. Also from (6) and (8), we have $g(p^*) = 0$. Hence the function g takes its minimum value of 0 when $p = p^*$. Therefore, the first and the second order minimization conditions yield

$$g_p(p^*) = 0 \tag{10}$$

and

$$y'g_{pp}(p^*)y \geqslant 0 \quad \text{for all } y \neq 0. \tag{11}$$

Equation (10) immediately proves (7).

To demonstrate the usefulness of the Shephard–Samuelson Lemma, let us prove the Hicksian Demand Rules from this Lemma. From (7) we have

$$e_{pp}(p, \mu) \equiv h_p(p, \mu).$$

This immediately yields (5). In view of (9) and (11), this also proves (4). On the other hand, (6), (9) and (10) yield

$$h(p^*, \mu) + p^{*\prime}h_p(p^*, \mu) - x^* = 0.$$

From (8), therefore, we obtain (3).