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COMPUTING ALL OF THE EQUILIBRIA OF ECONOMIES WITH TWO FACTORS OF PRODUCTION

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In general, no method to find all the equilibria of an economy is known that does not involve an exhaustive search. In this paper we argue that for an economy with a generalized input–output structure and two factors of production such a search is feasible, and indeed can be performed graphically. After presenting a search procedure, we apply it to an example with multiple equilibria. It is hoped that study of the simple two factor model will provide insight into the question of non-uniqueness in more general models.

1. Introduction

Scarf (1973) first developed an algorithm that computes an equilibrium for a general equilibrium model. Unfortunately, there are a whole class of equilibria, those with negative index, that it cannot find [see Eaves and Scarf (1976)]. Although such an algorithm can be modified to get around this particular difficulty, it remains true in general that no method to find all the equilibria of an economy is known that does not involve an exhaustive search. In this paper we argue that for an economy with a generalized input–output structure and two factors of production such a search is feasible, and indeed can be performed graphically. We demonstrate the equivalence between the equilibria of an economy with a generalized input–output structure and two factors of production and the equilibria of an appropriately defined two-commodity pure exchange economy. This type of transformation has been used in the past by international trade theorists and more recently by general equilibrium modellers, who have used the dimension reduction to improve computational efficiency [see, for example, Helpman (1976)]. We point out the possibility of using the dimension reduction to find all of the equilibria of an economy. An appealing aspect of the transformation is that the concepts of regularity and fixed point index carry over. We also study extensively the possibility of more activities than produced goods being used at equilibria, a possibility that has not previously

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been explored. After presenting the search procedure, we apply it to an economy with multiple equilibria. This example is interesting in that consumer excess demands exhibit gross substitutability while derived factor demands do not. Finally we discuss what insights the study of the simple two factor model may provide into the question of non-uniqueness in more general models.

2. The model and its equilibria

The model we employ here is a special case of that found in Kehoe (1980). The consumption side of an economy is specified by an excess demand function, $\xi: R^n_+\backslash\{0\}\to R^n$, that assigns any vector of non-negative prices, except the origin, to a vector of aggregate net trades.

A.1. ξ is continuously differentiable.

A.2. ξ is homogeneous of degree zero; $\xi(t\pi)\equiv\xi(\pi)$ for all $t>0$.

A.3. ξ obeys Walras's law; $\pi'\xi(\pi)\equiv 0$.

The production side is specified by an $n\times m$ activity analysis matrix A.

A.4. A allows free disposal; in other words, the $n\times n$ matrix $-I$ is a submatrix of A.

A.5. There exists some $\pi>0$ for which $\pi'A\leqq 0$; this is equivalent to the assumption that the only vector $Ay\geqq 0$ for which $y\geqq 0$ is $Ay=0$.

An equilibrium of (ξ, A) is defined to be a price vector $\hat{\pi}$ that satisfies $\hat{\pi}'A\leqq 0$, $\xi(\hat{\pi})=A\hat{y}$ for some $\hat{y}\geqq 0$, and $\hat{\pi}'e=1$, where e is a vector whose element is unity.

Our two-factor model is similar to models used by international trade theorists in their discussions of factor price equalization [see, for example, McKenzie (1955)]. It satisfies the following additional assumptions:

A.6. There are two non-produced goods; $a_{ij}\leqq 0$ for $i=n-1$, n and $j=1,\ldots,m$.

A.7. There is no joint production; for every $j=1,\ldots,m$ $a_{ij}>0$ for at most one i.

A.8. Production is possible in that there exists a non-negative vector y such that $\sum_{j=1}^{m} a_{ij}y_j>0$, $i=1,\ldots,n-2$.

A.9. Every good that can be produced is actually produced at every equilibrium.

It is easy to check whether an economy (ξ, A) satisfies A.6–A.8; A.9 is more difficult to check. To ensure that it holds, we could require that $\xi_i(\pi) \geq 0$ for the first $n-2$ goods, with strict inequality in one coordinate, for all prices π that are not zero in the final two coordinates by allowing consumers positive initial endowments of only the final two goods. A.9. is more general, however, and we later present an example where this extra generality is essential.

To simplify matters, we impose the following non-degeneracy assumptions on (ξ, A). We can justify them by arguing that they hold for almost all economies.

A.10. No column of A can be expressed as a linear combination of fewer than n other columns.

A.11. Any activity that earns zero profit at equilibrium $\hat{\pi}$ is associated with a positive activity level.

We denote the prices of the first $n-2$ goods by the vector p and the prices of the two non-produced goods by the vector q. We partition the matrix A into

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where A_1 is $(n-2)\times m$ and A_2 is $2\times m$, and similarly partition the vector $\xi(\pi)$ into

$$\begin{bmatrix} \xi^1(\pi) \\ \xi^2(\pi) \end{bmatrix}.$$

We begin with the case where A consists of only $2n-2$ activities, the n disposal activities and an $n\times(n-2)$ matrix B with one activity to produce each of the first $n-2$ goods. A.9 implies that each of these $n-2$ activities is used at a positive level at equilibrium. We partition B into

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

A.8 implies that B_1 is a productive Leontief matrix; A.10 implies that it is indecomposable. Consequently, B_1^{-1} has all its elements strictly positive [see, for example, Debreu and Herstein (1953)]. The equilibrium condition $B_1\hat{y} = \xi^1(\hat{p}, \hat{q})$ implies that $\hat{y} = B_1^{-1}\xi^1(\hat{p}, \hat{q}) > 0$, and the zero profit condition $\hat{p}'B_1 + \hat{q}'B_2 = 0$ implies that $\hat{p} = -(B_2B_1^{-1})'\hat{q}$. Since A.6 and A.10 imply that every element of B_2 is negative, $-(B_2B_1^{-1})$ is a strictly positive matrix. Since all the coordinates of $(\hat{p},\hat{q})$ cannot be equal to zero, both the coordinates of $\hat{q}$ cannot equal zero. Therefore $\hat{p}$ is strictly positive at any equilibrium. We can use these observations to construct a pure exchange model of the two factors whose equilibria are equivalent to those of (ξ, A): We define the 2×1 excess demand function

$$z(q)=\xi^2(-(B_2B_1^{-1})'q,q)-B_2B_1^{-1}\xi^1(-(B_2B_1^{-1})'q,q).$$

It is easy to check that z satisfies the appropriate versions of A.1, A.2, and A.3 since ξ does. Observe that $z(\hat{q})=0$ is equivalent to $\xi^2(\hat{p},\hat{q})=B_2B_1^{-1}\xi^1(\hat{p},\hat{q})=B_2\hat{y}$ and, therefore, that $\hat{q}$ is an equilibrium of $(z, -I)$ if $(\hat{p},\hat{q})$ is an equilibrium of (ξ, A). Unfortunately, A.9 in its present form allows the possibility that $\hat{q}$ is an equilibrium of $(Z,-I)$ but $(\hat{p},\hat{q})$ is not an equilibrium of (ξ, A) because $B_1^{-1}\xi^1(\hat{p},\hat{q})$ contains negative elements. The assumption that $\xi^1(p,q)\geqq 0$ would preclude this possibility. In any case, this possibility is easy to check for.

When the matrix A contains more than $2n-2$ columns, the situation is not so simple. $z(q)$ becomes an upper-semi-continuous, point-to-set correspondence rather than a continuous function. To calculate $z(q)$, we begin by solving the linear programming problem

$$\min\ -q'A_2y \quad \text{subject to} \quad A_1y=e, \quad y\geqq 0.$$

A.8 implies that this problem is feasible. A.5 implies that it has a finite solution. The non-substitution theorem implies that the solution is associated with a feasible basis that does not vary as the right-hand side vector, here e, varies [see Samuelson (1951)]. The solution is associated with a feasible basis

$$\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix},$$

and a vector of commodity prices p such that $p'B_1+q'B_2=0$ and $p'A_1+q'A_2\leqq 0$ [see Gale (1960, pp. 301–306)]. The presence of disposal activities in A_1 ensures that p is non-negative. When the basis is uniquely defined, we can proceed as above.

A problem arises, however, when the basis associated with the solution to the linear programming problem is not unique. There can be $n-1$ activities earning zero profit at prices $(-(B_2B_1^{-1})'q,q)$ and two distinct bases,

$$\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} -q'C_2 \\ C_1 \end{bmatrix},$$

possible in the solution. A.10 implies that there are at most two distinct bases and that they differ by only one activity. At any factor prices q where the linear programming problem is degenerate in this way the final demand for the factors can be either

$$z^B(q)=\xi^2(-(B_2B_1^{-1})'q,q)-B_2B_1^{-1}\xi^1(-(B_2B_1^{-1})'q,q), \quad \text{or}$$

$$z^C(q)=\xi^2(-(C_2C_1^{-1})'q,q)-C_2C_1^{-1}\xi^1(-(C_2C_1^{-1})'q,q).$$

Of course, $p=-(C_2C_1^{-1})'q=-(B_2B_1^{-1})'q$. In fact, for our purposes, $z(q)$ can be any convex combination of the two, $tz^B(q)+(1-t)x^C(q)$ for any $0\leqq t\leqq 1$.

A.10 implies that there are only a finite number of prices q in the simplex $S=\{q\in R^2|q_1+q_2=1, q_j\geqq 0\}$ such that $z(q)$ has multiple values. Consequently, $z(q)$ is a single-valued differentiable function at almost all prices $q\in R^2_+$. It is worth pointing out, however, that for $z(\hat{q})$ to be multi-valued at an equilibrium $\hat{q}$ is by no means a degenerate situation. Although in this case the linear programming problem is degenerate, the economy itself need not be. In the next section we demonstrate that it is possible for (ξ, A) to have $n-1$ activities in use at an equilibrium $(\hat{p},\hat{q})$ and to be such that no small perturbation can result in fewer than $n-1$ activities being used at the perturbed equilibrium.

3. Regular economies

Before studying the two-factor model, let us briefly summarize the regularity and index results obtained by Kehoe (1980) for economies that satisfy A.1–A.5. A regular economy is one that satisfies the non-degeneracy assumptions A.10 and A.11 and the condition that the expression

$$(-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e & 0 \\ e' & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right)$$

is non-zero at every equilibrium. [Here $B(\hat{\pi})$ denotes the submatrix of A whose columns are those activities that earn zero profit at equilibrium $\hat{\pi}$.] A regular economy has a finite number of equilibria that vary continuously with the parameters (ξ, A). Furthermore, if we define index($\hat{\pi}$) to be $+1$ or -1, depending on the sign of the above determinantal expression, then we can prove that $\sum$index($\hat{\pi}$)$=+1$, where the sum is over all equilibria. It is this result that is crucial to our study of uniqueness since the condition index($\hat{\pi}$)$=+1$ at every equilibrium is necessary and sufficient for a regular economy to have a unique equilibrium. An alternative formula for index($\hat{\pi}$) can be derived by performing elementary row and column operations on the expression for index($\hat{\pi}$). Strike out one row and column of $D\xi_{\hat{\pi}}$ for which the corresponding $\hat{\pi}_j > 0$. Strike out the corresponding row of $B(\hat{\pi})$. Letting $\bar{J}$ and $\bar{B}$ be the matrices obtained from this process, we can prove that

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right).$$

The price $\hat{\pi}_j$ for which the corresponding row and column are deleted from $D\xi_{\hat{\pi}}$ can be thought of as a numeraire. The appeal of the concept of regularity is enhanced by its genericity in the space of economies: Almost all economies are regular in the sense that regular economies form an open dense subset of a suitable parameterized topological space of economies.

We now extend these results to the two factor model. It is easy to prove that, since regular economies form an open dense subset of the space of economies that satisfy A.1–A.5, they also form an open dense subset of the subspace of economies that satisfy A.1–A.9. To interpret the index theorem in the two factor model we need to develop an alternative expression for index($\hat{\pi}$). We again start with the case where A contains only one production activity for each produced good. We also, for the present, require that both factor prices are strictly positive at every equilibrium $\hat{\pi}$. These restrictions imply that there are exactly $n-2$ activities in use at every equilibrium. The index of an equilibrium $\hat{\pi} = (\hat{p}, \hat{q})$ of a two factor economy (ξ, A) can be expressed as

$$\operatorname{index}(\hat{p}, \hat{q}) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi^1_{\hat{p}} & D\xi^1_{\hat{q}} & B_1 \\ e & D\xi^2_{\hat{p}} & D\xi^2_{\hat{q}} & B_2 \\ 0 & B'_1 & B'_2 & 0 \end{bmatrix}\right)$$

where each of the vectors e is of the appropriate dimension. Performing elementary row and column operations on the matrix in this expression, it is possible to show that

$$\operatorname{index}(\hat{p}, \hat{q}) = \operatorname{sgn}\det\begin{bmatrix} 0 & e' - e'(B_2 B_1^{-1})' \\ e - B_2 B_1^{-1} e & Dz_q \end{bmatrix}, \quad \text{where}$$

$$Dz_{\hat{q}} = D\xi^2_{\hat{q}} - B_2 B_1^{-1} D\xi^1_{\hat{q}} - D\xi^2_{\hat{p}}(B_2 B_1^{-1})' + B_2 B_1^{-1} D\xi^1_{\hat{p}}(B_2 B_1^{-1})'.$$

When we rescale $\hat{q}$ and $Dz_{\hat{q}}$ so that $\hat{q}'e = 1$, this becomes the formula for index $(\hat{q})$ of the two dimensional economy $(z, -I)$. [Recall that in the n dimensional case $\hat{q}'(e - B_2 B_1^{-1} e) = \hat{q}'e + \hat{p}'e = 1$ while in the two dimensional case $\hat{q}'e = 1$.] If neither $\hat{q}_1$ nor $\hat{q}_2$ is equal to zero, then the index of equilibrium $\hat{q}$ can be calculated as $\operatorname{sgn}(-(\partial z_1/\partial q_1)(\hat{q}))$. Alternatively, we can use A.2 to establish that index$(\hat{q}) = \operatorname{sgn}(-(\partial z_1/\partial q_1)(\hat{q}) + (\partial z_1/\partial q_2)(\hat{q}))$. Walras's law implies that, if $z_1(\hat{q}) = 0$, then $z_2(\hat{q}) = 0$. Therefore equilibria of $(z, -I)$, with the possible exceptions of $(1,0)$ and $(0,1)$ are equivalent to zeros of the excess demand function $z_1(q_1, 1-q_1)$. The index of an equilibrium $\hat{q}$ is equal to $+1$ if the graph of $z_1(q_1, 1-q_1)$ crosses the axis from above at $\hat{q}$ and equal to -1 if it crosses from below. Notice that if (ζ, A) is regular, then the graph of $z_1(q_1, 1-q_1)$ cannot become tangent to the axis.

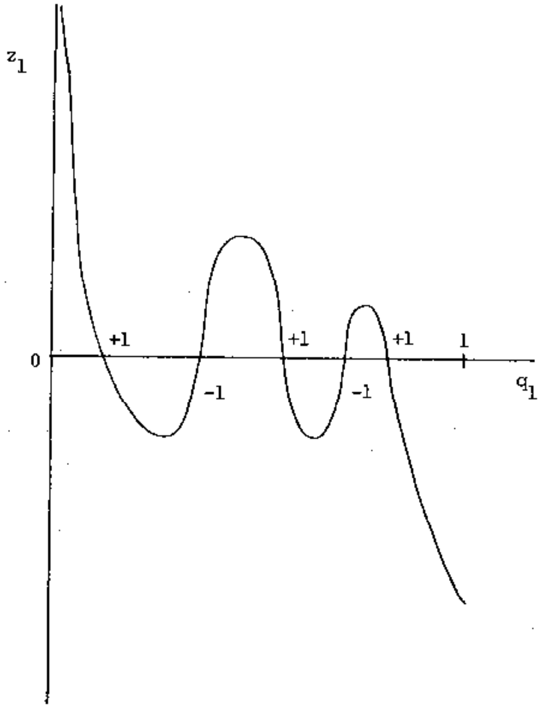

Fig. 1

The index theorem tells us that an economy (ξ, A) has a unique solution if and only if the graph of $z_1(q_1, 1-q_1)$ always crosses the axis from above. In the case where z is single-valued at every equilibrium an economy has a unique equilibrium if $(\partial z_1/\partial q_1)(\hat{q}) < 0$. Walras's law and homogeneity imply that this condition is equivalent to each of three other mutually equivalent conditions, $(\partial z_2/\partial q_2)(\hat{q}) < 0$, $(\partial z_1/\partial q_2)(\hat{q}) > 0$, $(\partial z_2/\partial q_1)(\hat{q}) > 0$. A two factor economy therefore has a unique equilibrium if either factor is normal at every equilibrium or, equivalently, if the two factors are gross substitutes at every equilibrium.

Consider the situation where z is multi-valued at an equilibrium $\hat{q}$. In some neighborhood of $\hat{q}$, as we increase the price of the first factor, say labor, and decrease the price of the second, say capital, one activity goes from being inefficient to efficient. At $\hat{q}$ a change of basis occurs and another activity, which produces the same good, is dropped from the basis. What we shall demonstrate, and what certainly is intuitively plausible, is that, as we increase the price of labor, the activity that comes into the basis is relatively more capital intensive than the vector that drops out. The switch of techniques results in a sudden drop in demand for labor and increase in the demand for capital. Consequently, the interpretation of the index theorem in terms of normality and gross substitutability can be carried over to the case where z is multi-valued.

Suppose now that either one of the factor prices is equal to zero at equilibrium. Then A.11 implies that a disposal activity is used at a positive level. The situation is similar to that where more than one activity is used to produce each produced good. In either case, since A.10 implies that at most $n-1$ activities can earn zero profit at any equilibrium $\hat{\pi}$, there must be exactly $n-1$ activities in use. If there are exactly $n-1$ activities in use at equilibrium $\hat{\pi}$, then $\text{index}(\hat{\pi})=+1$. To see why this is so, let C be the $n\times n$ matrix $[B(\hat{\pi})\,e]$. Since $B(\hat{\pi})$ has full column rank by A.10 and $\hat{\pi}'B(\hat{\pi})=0$ while $\hat{\pi}'e=1$, C is non-singular. Consequently,

$$\text{index}(\hat{\pi})=(-1)^n\,\text{sgn}\left(\det\begin{bmatrix} D\xi_{\hat{\pi}} & C \\ C' & 0\end{bmatrix}\right)=\text{sgn}(\det[C'C])=+1,$$

since $C'C$ is positive definite. We should therefore expect that when there are $n-1$ activities in use at equilibrium that the graph of the excess demand correspondence $z_1(q_1, 1-q_1)$ crosses the axis from above. This result can be viewed as the reason why a one factor input–output model, which always has $n-1$ activities in use at equilibrium, has a unique equilibrium.

Let us investigate the situation where $\hat{q}$ is an equilibrium of a regular economy $(z, -I)$ and there are two distinct bases possible in the solution to our linear programming problem. There is some neighborhood of $\hat{q}$, $U\subset R^2$. such that the programming problem always results in either the basis

$$\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix} \text{ or } \begin{bmatrix} -q'C_2 \\ C_1 \end{bmatrix}. \quad \text{Let } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

be the activities associated with all those $q\in U\cap S$ such that $q_1>\hat{q}_1$. (Here S denotes the unit simplex in R^2.) We can choose some $v\in R^2$ with $v_1=-v_2>0$ so that both $\hat{q}+v$ and $\hat{q}-v$ are elements of $U\cap S$. That

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

is not associated with the basis at $\hat{q}-v$ implies that $-(\hat{q}-v)'B_2B_1^{-1}C_1+(\hat{q}-v)'C_2 \leqq 0$ with strict inequality holding for the coordinate corresponding to the column in which B_1 and C_1 differ. Since $\hat{q}'B_2B_1^{-1}=\hat{q}'C_2C_1^{-1}=\hat{p}'$, this implies $v'B_2B_1^{-1}C_1-v'C_2 \leqq 0$. Since q is an equilibrium, A.9 implies that $B_1y_B+C_1y_C=\xi^1(\hat{p},\hat{q})$ for some $y_B, y_C>0$. Consequently, $C_1^{-1}>0$ implies $C_1^{-1}\xi^1(\hat{p},\hat{q})-C_1^{-1}B_1y_B=y_C>0$. Multiplying the previous inequality by y_C yields $v'B_2B_1^{-1}C_1y_C<v'C_2y_C$. Similarly, we can prove that $v'B_2y_B<v'C_2C_1^{-1}B_1y_B$. Adding these two inequalities yields $v'B_2(y_B+B_1^{-1}C_1y_C)<v'C_2(y_C+C_1^{-1}B_1y_B)$, which is equivalent to

$$v'B_2B_1^{-1}\xi^1(\hat{p},\hat{q})<v'C_2C_1^{-1}\xi^1(\hat{p},\hat{q}),$$

$$v'\xi^2(\hat{p},\hat{q})-v'B_2B_1^{-1}\xi^1(\hat{p},\hat{q})>v'\xi^2(\hat{p},\hat{q})-v'C_2C_1^{-1}\xi^1(\hat{p},\hat{q}),$$

$$v'z^B(\hat{q})>v'z^C(\hat{q}).$$

Thus, $v_1>0$ implies that $z_1^B(\hat{q})>z_1^C(\hat{q})$, and $v_2<0$ implies that $z_2^B(\hat{q})<z_2^C(\hat{q})$. It follows that whenever the excess demand correspondence is multi-valued at an equilibrium $\hat{q}$ the graph of the correspondence $z_1(q, 1-q_1)$ crosses the axis from above at $\hat{q}$ as in fig. 2.

A.11 rules out such a situation as that depicted in fig. 3. We have demonstrated that, when a change in factor prices leads to a change of basis in the linear programming problem that selects the efficient production activities, there is a vertical section in the graph of $z_1(q_1, 1-q_1)$. If this vertical section passes through the axis, and if A.11 is satisfied, then any small perturbation in (ξ, A) yields an equilibrium where the same vertical section of the perturbed graph passes through the axis, that is, where there are still $n-1$ activities in use at equilibrium.

4. An example

When an economy is regular, we know that its equilibria are finite and odd in number. If we cannot prove that there is a unique equilibrium, this is usually all we can say about the number of equilibria. In this section we present an example of an economy that satisfies A.1–A.11. Using the analysis of the previous sections, we are able to exploit its special characteristics to find all of its equilibria. An interesting feature of this example is that, although the excess demand function ξ exhibits gross substitutability, the derived factor demand function z does not.

In this economy there are four commodities and four consumers. Consumer j has excess demand function $\xi_i^j(\pi)$ for the commodity i given by

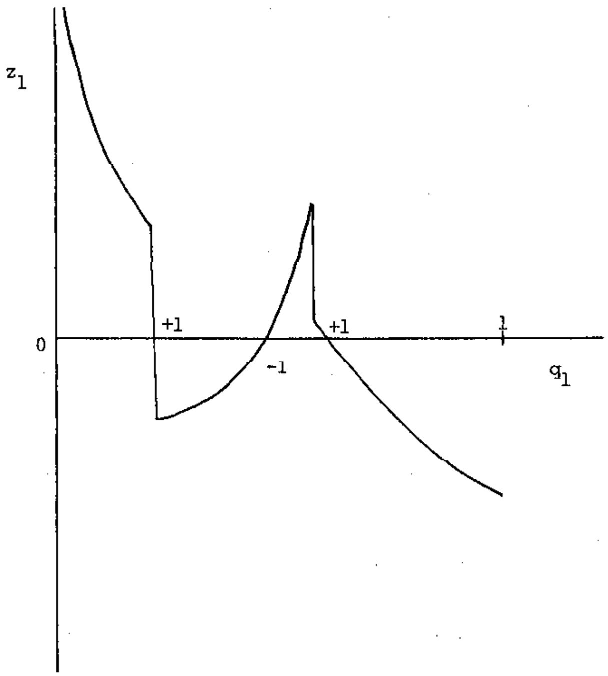

Fig. 2

the rule

$$\xi_i^j(\pi) = \alpha_i^j \left(\sum_{l=1}^{4} \pi_l w_l^j \Big/ \pi_i \right) - w_i^j.$$

Here the parameters α_i^j and w_i^j are non-negative and such that $\sum_{i=1}^4 \alpha_i^j = 1$, $j = 1, 2, 3, 4$. For each consumer the vector of parameters $w^j = (w_1^j, w_2^j, w_3^j, w_4^j)$, which may be interpreted as initial endowments, is as follows:

		Consumer			
		1	2	3	4
Commodity	1	50	0	0	0
	2	0	50	0	0
	3	0	0	400	0
	4	0	0	0	400

Each individual excess demand function can be derived by maximizing the

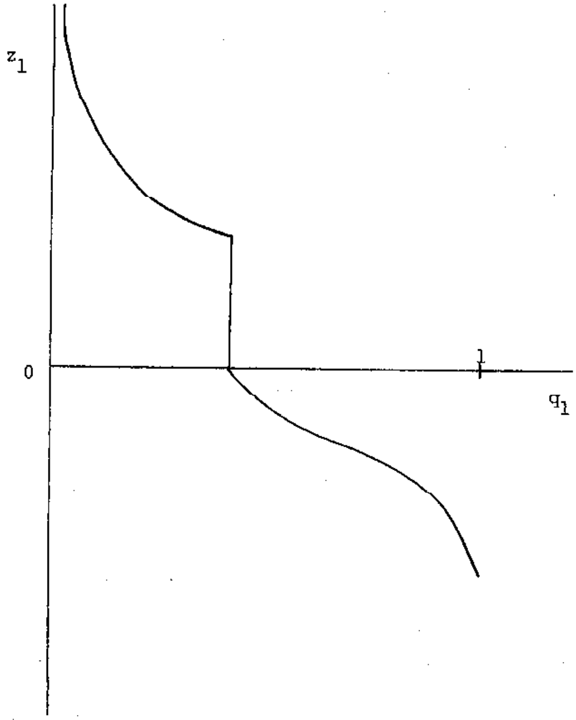

Fig. 3

Cobb–Douglas utility function

$$u^j(x_1, x_2, x_3, x_4) = x_1^{\alpha_1^j} x_2^{\alpha_2^j} x_3^{\alpha_3^j} x_4^{\alpha_4^j},$$

subject to the constraints $\sum_{i=1}^{4} \pi_i x_i \leqq \sum_{i=1}^{4} \pi_i w_i^j$, $x_i \geqq 0$. We specify the parameters α_i^j as follows:

		Consumer			
		1	2	3	4
Commodity	1	0.5200	0.8600	0.5000	0.0600
	2	0.4000	0.1000	0.2000	0.2500
	3	0.0400	0.0200	0.2975	0.0025
	4	0.0400	0.0200	0.0025	0.6875

The aggregate excess demand function $\xi(\pi) = \sum_{j=1}^{4} \xi^j(\pi)$ satisfies A.2 and A.3 and is continuously differentiable for all strictly positive prices. There is a minor technical difficulty in that ξ can become unbounded as some prices tend toward zero. This problem could be handled by constructing a new excess demand function ξ^* that agrees with ξ on some open neighborhood of every equilibrium of (ξ, A) and (ξ^*, A) and that satisfies A.1 [see Kehoe (1982)]. We shall ignore this difficulty. Notice that ξ exhibits gross substitutability,

$$\frac{\partial \xi_i}{\partial \pi_j} = \sum_{l=1}^{4} \frac{\alpha_i^l}{\pi_i} w_j^l > 0 \quad \text{for } j \neq i.$$

The production side of the example is given by a 4×7 activity analysis matrix A,

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 6 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & 4 \\ 0 & 0 & -1 & 0 & -4 & -1 & -3 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

It is trivial to verify that A satisfies A.4–A.8. If we could demonstrate that (ξ, A) satisfies A.9, in other words, that activity a^5 and activity a^6 or a^7 are used at every equilibrium, then we would have demonstrated that (ξ, A) has the two factor structure. A demonstration that (ξ, A) does, in fact, satisfy A.9 is given below. The argument is interesting in its own right because it depends crucially on the index theorem.

Suppose that this assumption does not hold in our example. There are then three possibilities that we need to rule out. First, consider the possibility of there being an equilibrium where no production takes place. If $\hat{\pi}$ is such an equilibrium, then it is also an equilibrium of the pure exchange economy $(\xi, -I)$ formed by deleting a^5, a^6, and a^7 from the activity analysis matrix A. Since ξ exhibits gross substitutability, $-\bar{J}$ has the same structure as a productive Leontief matrix with positive diagonal elements, negative off-diagonal elements, and positive principal minors. Therefore, $\text{index}(\hat{\pi}) = \text{sgn}(\det[-\bar{J}]) = +1$ at every equilibrium, which implies that $(\xi, -I)$ has a unique equilibrium. This equilibrium can be easily located using Scarf's algorithm. It is $\hat{\pi} = (0.65180, 0.32923, 0.00586, 0.01311)$. At $\hat{\pi}$, however, activities a^5, a^6, and a^7 all earn positive profits. It follows that $\hat{\pi}$ is not an equilibrium of (ξ, A) and, consequently, that there is no equilibrium of (ξ, A) where no production takes place.

Second, consider the possibility of there being an equilibrium where only one activity, a^5, a^6, or a^7, is used. We proceed as in the previous case. If some $\hat{\pi}$ is an equilibrium where only one activity is used, then it is also an equilibrium of the economy $(\xi, \bar{A})$ formed by deleting the other two productive activities from A. At such an equilibrium,

$$\text{index}(\hat{\pi}) = \text{sgn}\left(\det\begin{bmatrix} -\bar{J} & -\bar{b} \\ \bar{b}' & 0 \end{bmatrix}\right),$$

where we can choose the numeraire so that every element of the vector $\bar{b}$ is negative. Since $-\bar{J}$ is a productive Leontief matrix, there is some positive linear combination of the columns of $-\bar{J}$ equal to the positive vector $-\bar{b}$. Consequently,

$$\text{index}(\hat{\pi})=\text{sgn}\left(\det\begin{bmatrix}-\bar{J} & 0\\ \bar{b}' & -\bar{b}'x\end{bmatrix}\right)=\text{sgn}(-\bar{b}'x\det[-\bar{J}])=+1,$$

since $x=\bar{J}^{-1}\bar{b}>0$. We can find the unique equilibria of the three economies corresponding to each of the three productive activities and verify that none are equilibria of (ξ, A): The unique equilibrium of the economy with only activity a^6 is (0.14676, 0.83096, 0.00910, 0.01318). At these prices, however, both a^6 and a^7 earn positive profits. The unique equilibrium of the economy with only a^6 is (0.72917, 0.25000, 0.00629, 0.01454). Here a^5, but not a^7, earns positive profits. The unique equilibrium of the economy with only a^7 is (0.77473, 0.20283, 0.00708, 0.01536). Here both a^5 and a^6 earn positive profits.

Third, consider the possibility that there is some equilibrium where only a^6 and a^7 are used. It would be an equilibrium of the economy $(\xi, \bar{A})$ formed by deleting a^5 from A. We have already found one equilibrium of $(\xi, \bar{A})$, $\hat{\pi}=$ (0.72917, 0.25000, 0.00629, 0.01454). There are two alternative methods for demonstrating that $\hat{\pi}$ is the unique equilibrium of $(\xi, \bar{A})$. The first, and more general, approach is to use the two zero profit conditions to reduce the search for equilibria to a one dimensional line search. The procedure would be similar to that of section 3. The second approach depends on calculating index$(\hat{\pi})$ at an equilibrium of $(\xi, \bar{A})$ where both a^6 and a^7 are used. The index has the same sign as

$$\det\begin{bmatrix}d_{11} & d_{13} & d_{14} & 1 & 1\\ d_{31} & d_{33} & d_{34} & 1 & 3\\ d_{41} & d_{43} & d_{44} & 1 & 1\\ -1 & -1 & -1 & 0 & 0\\ -1 & -3 & -1 & 0 & 0\end{bmatrix},$$

where $d_{ij}=-(\partial\xi_i/\partial\pi_j)(\hat{\pi})$. Using elementary row and column operations, we can reduce this expression to

$$4\det\begin{bmatrix}d_{11} & d_{14} & 1\\ d_{41} & d_{41} & 1\\ -1 & -1 & 0\end{bmatrix}.$$

The index therefore has the same sign as the determinant of a matrix with sign pattern

$$\begin{bmatrix} + & - & + \\ - & + & + \\ - & - & 0 \end{bmatrix},$$

which is unambiguously positive. Since $\sum \text{index}(\bar{\pi}) = +1$, and since every equilibrium is such that $\text{index}(\hat{\pi}) = +1$, $(\xi, \bar{A})$ has a unique equilibrium. We have already found it, and it is not an equilibrium of (ξ, A).

Following the procedure outlined in the previous section, we can find all of the equilibria of (ξ, A) using a one-dimensional line search. The accuracy of the three equilibria reported below was improved using Newton's method. The prices reported are based on calculations that resulted in an equality of supply and demand accurate to twelve significant figures for all commodities.

Equilibrium 1

$$\pi^1 = (0.19643,\ 0.25000,\ 0.12500,\ 0.42857),$$

$$y^1 = (0,\ 0,\ 0,\ 47.653,\ 72.425,\ 3.130).$$

		Consumer 1	2	3	4
Commodity	1	26.000	54.727	127.273	52.364
	2	15.714	5.000	40.000	171.429
	3	3.143	2.000	119.000	3.429
	4	0.917	0.583	0.292	275.000
	u^i	17.088	36.820	97.481	219.797

$\text{index}(\pi^1) = +1.$

Equilibrium 2

$$\pi^2 = (0.25000,\ 0.25000,\ 0.25000,\ 0.25000),$$

$$y^2 = (0,\ 0,\ 0,\ 0,\ 52.000,\ 69.000,\ 0).$$

		Consumer 1	2	3	4
Commodity	1	26.000	43.000	200.000	24.000
	2	20.000	5.000	80.000	100.000
	3	2.000	1.000	119.000	1.000
	4	2.000	1.000	1.000	275.000
	u^i	19.067	29.832	140.802	181.909

$\text{index}(\pi^2) = -1.$

Equilibrium 3

$$\pi^3 = (0.27514,\ 0.25000,\ 0.30865,\ 0.16621),$$

$$y^3 = (0,\ 0,\ 0,\ 0,\ 53.180,\ 65.148,\ 0).$$

		Consumer			
		1	2	3	4
Commodity	1	26.000	39.072	224.362	14.499
	2	22.011	5.000	98.768	66.485
	3	1.783	0.810	119.000	0.539
	4	3.311	1.504	1.857	275.000
	u^i	20.123	27.581	155.794	159.115

index $(\pi^3) = +1$.

5. Concluding remarks

The part of this paper most interesting to a general equilibrium modeller is probably the example of non-uniqueness. Merely looking at the parameters of the economy reveals nothing pathological. The consumer excess demand function, for example, exhibits gross substitutability. Yet the model has three very different equilibria. Using comparative statics to do policy analysis with this model, we would surely derive very different results depending on which equilibria we started with. Even more disturbing is the possibility of jumping from one equilibrium to another without knowing it, which would render any results meaningless.

These are, of course, possibilities that are always present as long as we are not sure that a specific model has a unique equilibrium. Conditions that ensure uniqueness, however, seem to be too restrictive to have general applicability [see, for example, Kehoe (1983a)]. Some researchers have attempted to remedy this problem by using different starting values for their computational algorithms and then verifying that they all lead to the same equilibrium. The unwary should be warned against putting much faith in such *ad hoc* tests. Using a version of Scarf's algorithm developed by Merrill (1972), which allows variable starting points, the writer applied such a test to the example of the previous section. The first eleven starting values chosen randomly from the price simplex in R^4 all led to Equilibrium 3. It was not until the twelfth try that Equilibrium 1 was located. A conventional fixed point algorithm could never locate Equilibrium 2. There is nothing so special about Equilibrium 3, however, that we would be justified in studying it alone.

Applied economists seem to view non-uniqueness of equilibrium as pathological. Theorists, on the other hand, seem to have accepted it as commonplace. Perhaps the models used for policy analysis do usually have unique equilibria. They may, in fact, have enough structure so that this can be checked. The two-factor case seems to be a good place to start a study of such structure.

Our results could, of course, be easily extended to economies with smooth production functions [see Kehoe (1983b)]. The only complication would be in making sure that the problem of computing commodity prices from factor prices using the zero profit conditions was easy to solve. An increase in the dimension of the factor space would cause more of a problem, however. An exhaustive search over a two-dimensional simplex, although feasible, would be much more difficult to carry out than a line search. In a model with three factors, moreover, either $n-3$, $n-2$, or $n-1$ activities could be used at equilibrium. The possibility of $n-3$ and $n-1$, of course, presents no problems. If there are $n-2$ activities in use, however, the derived factor demands are multi-valued, but we have no information about the local properties, such as the index, of the equilibrium. Nonetheless, addition of more equations and unknowns to the model may not necessarily render our analysis useless. Suppose, for example, that we are working with a model in which there is a government that taxes and spends [see, for example, Shoven (1974)]. It may well be that for every vector of factor prices there is a unique level of government spending that balances with the taxes it receives. If this is the case, as it probably is in most applied models, then by computing this level of spending as we move along a one-dimensional simplex of factor prices we could still carry out our line search. Kehoe and Whalley (1982) have applied this method to verify that the model developed by Fullerton et al. (1981) has a unique equilibrium. They have also verified that the model developed by Kehoe and Serra-Puche (1983), which has three factors of production, has a unique equilibrium.

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