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A NUMERICAL INVESTIGATION OF MULTIPLICITY OF EQUILIBRIA

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The emphasis in discussions of uniqueness of equilibrium has traditionally been on what qualitative restrictions on the structure of an economy are sufficient for uniqueness. Unfortunately, these restrictions seem to be extremely stringent. In this paper we ask how common is a particular feature of consumer excess demands required for multiplicity, namely, a violation of the weak axiom of revealed preference, in a very tightly parameterized space of demand functions. A numerical investigation using a random number generator indicates that, at least in the very narrow class of functions that we consider, this feature is very rare.

Key words: General Equilibrium, Uniqueness, Fixed-Point Index, Genericity.

1. Introduction

Recent investigations of conditions that ensure uniqueness of equilibrium in general equilibrium models have made use of the tools of differential topology. Following the introduction of these tools into economic analysis by Debreu (1970), Dierker (1972) has noticed that the concept of fixed point index can be used to count the number of equilibria of a pure exchange economy. Mas-Colell (1984) and Kehoe (1980a) have extended this analysis to economies with production, and Kehoe (1982b) has provided a thorough discussion of the implications of this analysis for the uniqueness question.

The emphasis in this discussion, and in other discussions of the uniqueness question, is on what qualitative restrictions on the structure of an economy are sufficient for uniqueness of equilibrium. Unfortunately, these restrictions seem to be extremely stringent. The emphasis on qualitative, rather than quantitative, restrictions follows a long tradition in economic theory, closely identified with Samuelson (1947). In this paper we ask how common is a particular feature of consumer excess demands required for multiplicity of equilibria, namely a violation of the weak axiom of revealed preference, in a very tightly parameterized space of demand

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functions. A numerical investigation using a random number generator indicates that, at least in the very narrow class of functions that we consider, this feature is very rare. Although the subset of demand functions with this feature contains a nonempty open set, it seems to be very small.

Our results are intended to stimulate more work on the uniqueness question, which is a crucial question for applications of general equilibrium models in comparative statics exercises. Currently, there is no good answer to this question. On one hand, it seems that general conditions that guarantee the uniqueness of equilibrium are impossibly restrictive, especially for models that allow taxes and distortions (see Kehoe (1982a, 1982b)). On the other hand, there is at least some evidence that uniqueness of equilibrium is not uncommon in practice. Kehoe and Whalley (1982), for example, have exploited the special features of two large scale numerical general equilibrium models and carried out exhaustive searches that have verified that both have unique equilibria.

2. The index theorem

The general equilibrium model considered in this paper is highly stylized. Its consumption side is specified by an aggregate excess demand function $\xi:\mathbb{R}^n_{++}\to\mathbb{R}^n$ that assigns any vector of strictly positive prices with a vector of aggregate net trades. We assume that ξ is completely arbitrary aside from the following assumptions:

A.1. ξ is continuously differentiable.

A.2. ξ is bounded from below; there exists $w\in\mathbb{R}^n_{++}$ such that $\xi(\pi)\geqslant -w$ for all $\pi\in\mathbb{R}^n_{++}$.

A.3. If $\pi^0\neq 0$, $\pi^0_j=0$, and $\pi^i\in\mathbb{R}^n_{++}$, $i=1,2,\ldots$, are such that $\pi^i\to\pi^0$, then $\|\xi(\pi^i)\|\to\infty$.

A.4. ξ is homogeneous of degree zero; $\xi(t\pi)\equiv\xi(\pi)$ for all $t>0$.

A.5. ξ obeys Walras's law; $\pi'\xi(\pi)\equiv 0$.

We specify the production side of the model by an $n\times m$ activity analysis matrix A. Kehoe (1983) explains how our analysis can be extended to more general production technologies. We assume that A is arbitrary except for the following assumptions:

A.6. A allows free disposal; in other words, the $n\times n$ matrix $-I$ is a submatrix of A.

A.7. There exists some $\bar{\pi}>0$ for whch $\bar{\pi}'A\leqslant 0$; this is equivalent to the assumption that the only vector $Ay\geqslant 0$, $y\geqslant 0$, is $Ay=0$.

An equilibrium of the economy (ξ, A) is a price vector $\hat{\pi}$ that satisfies the following conditions:

E.1. $\hat{\pi}'A\leqslant 0$.

E.2. $\xi(\hat{\pi})=A\hat{y}$ for some $\hat{y}\geqslant 0$.

E.3. $\hat{\pi}'e=1$ where $e=(1,\ldots,1)$.

Let $S_A = \{\pi \in \mathbb{R}^n \mid \pi' A \leq 0, \pi' e - 1\}$. A.7 implies that S_A is nonempty. A.6 implies that it is a subset of the unit simplex $S = \{\pi \in \mathbb{R}^n \mid \pi \geq 0, \pi' e = 1\}$. It is obviously compact and convex. To prove the existence of equilibrium for this model we construct a continuous single-valued map of S into itself whose fixed points satisfy the equilibrium conditions E.1–E.3. Let $p^{S_A}: \mathbb{R}^n \to S_A$ be the map that associates a point $q \in \mathbb{R}^n$ with that point in S_A that is closest to q in terms of euclidean distance. Let $g(\pi) = p^{S_A}(\pi + \xi(\pi))$.

Theorem 1. *$\hat{\pi}$ is an equilibrium of (ξ, A) if and only if $\hat{\pi} = g(\hat{\pi})$.*

Proof. $g = g(\pi)$ solves the problem of minimizing $\frac{1}{2}(g - \pi - \xi(\pi))'(g - \pi - \xi(\pi))$ subject to the constraints $g'A \leq 0$ and $g'e = 1$. The Kuhn–Tucker theorem says that g solves this problem if and only if there exist $y \in \mathbb{R}^m_+$ and $\lambda \in \mathbb{R}$ such that $g - \pi - \xi(\pi) + Ay + \lambda e = 0$ and $g'Ay = 0$. At a fixed point $g(\hat{\pi}) = \hat{\pi}$, and these conditions become $-\xi(\hat{\pi}) + A\hat{y} + \hat{\lambda} e = 0$ and $\hat{\pi}' A\hat{y} = 0$. Walras's law now implies that $\hat{\lambda} = 0$. Consequently, $\xi(\hat{\pi}) = A\hat{y}$, and, since we already know that $\hat{\pi}'A \leq 0$ and $\hat{\pi}'e = 1$, $\hat{\pi}$ is an equilibrium if it is a fixed point. Conversely, if $\hat{\pi}$ is an equilibrium, we can set y equal to $\hat{y}$ and λ equal to 0 to demonstrate that it is a fixed point of g. □

The construction of the map g is based on similar least-distance maps used by Eaves (1971) and Todd (1979). There is a minor technical problem that ξ, and possibly g, is not continuous on the boundary of $\mathbb{R}^n_+$. Kehoe (1982b), however, demonstrates that it is possible to replace ξ with another map ξ^* that is continuous, in fact continuously differentiable, on all $\mathbb{R}^n$, that agrees with ξ on some neighborhood of every equilibrium of (ξ, A), and that is such that every equilibrium of (ξ^*, A) is an equilibrium of (ξ, A). Consequently, we are justified in assuming that g is continuous on S.

To demonstrate the existence of equilibrium we appeal to Brouwer's fixed point theorem, which states that any continuous map of a nonempty, compact, convex set into itself has a fixed point. We can motivate this theorem using a graph of the one dimensional case, such as that in Fig. 1. Suppose that $g(\pi)$ is a continuous function from the unit interval into itself: $0 \leq g(\pi) \leq 1$ for any $0 \leq \pi \leq 1$. Brouwer's fixed point theorem says that g must cross the diagonal, where $\pi = g(\pi)$, at least once.

More than this can be said, however. Suppose that all fixed points of g lie in the interior of the interval, that g is continuously differentiable in some neighborhood of every fixed point, and that the graph of g is never tangent to the diagonal. Then the graph of g must first cross the diagonal from above; after that, it crosses once from above for every time it crosses from below. Let us assign a fixed point $\hat{\pi}$ an index $+1$ if the graph of g crosses the diagonal from above at $\hat{\pi}$ and an index -1 if it crosses from below. To compute index($\hat{\pi}$) we need only find the sign of the expression $(1 - (\mathrm{d}g/\mathrm{d}\pi)(\hat{\pi}))$. The index theorem says that the sum of the indices of all equilibria is $+1$. Consequently, there is an odd number of equilibria, and, if

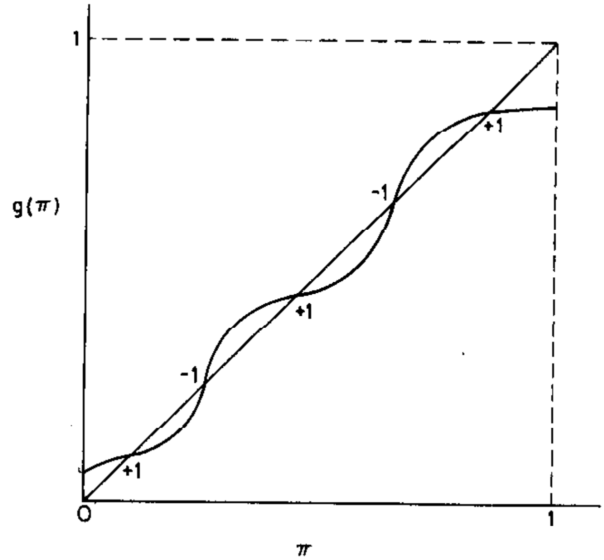

Fig. 1.

index($\hat{\pi}$) = +1 at every equilibrium, then there is a unique equilibrium. If, however, index($\hat{\pi}$) = −1 at any equilibrium, then there are multiple equilibria.

Simple calculus arguments suffice to prove Brouwer's fixed point theorem and the index theorem for the one dimensional case. Although the arguments for the general case involve more complex topological issues, analogous results hold: Suppose that all fixed points of g are interior to S, that g is continuously differentiable in some neighborhood of every fixed point, and that $I - Dg(\hat{\pi})$ is nonsingular at every fixed point. Let the index of a fixed point $\hat{\pi}$ be +1 or −1 depending on the sign of the expression $\det[I - Dg(\hat{\pi})]$. Again the sum of the indices of all equilibria is +1. This implies that there exists at least one equilibrium and that a necessary and sufficient condition for uniqueness of equilibrium is that index($\hat{\pi}$) = +1 at every equilibrium. A proof of this index theorem is given by Kehoe (1980).

To ensure that g is continuously differentiable at its fixed points we impose two additional assumptions on (ξ, A):

A.8. No column of A can be expressed as a linear combination of fewer than n other columns.

A.9. Any column of A that earns zero profit at equilibrium $\hat{\pi}$ is associated with a strictly positive activity level; $\hat{\pi}'a_j = 0$ implies that $\hat{y}_j > 0$.

The following result is an easy application of the implicit function theorem.

Theorem 2. *Suppose that (ξ, A) satisfies A.8 and A.9 and that $\hat{\pi}$ is an equilibrium of (ξ, A). Let B be the $n \times k$ submatrix of A made up of all those columns that satisfy*

$\hat{\pi}'B = 0$, *and let* $C = [B \quad e]$. *Then g is continuously differentiable in an open neighborhood of* $\hat{\pi}$ *and* $Dg(\hat{\pi}) = (I - C(C'C)^{-1}C')(I + D\xi(\hat{\pi}))$.

Kehoe (1980) proves that every economy in an open dense subset of the topological space of economies satisfies A.8, A.9, and the condition that $(I - Dg(\hat{\pi}))$ is nonsingular at every equilibrium. Economies that satisfy these conditions are called regular economies. We focus our attention on regular economies because almost all economies in the topological space of economies are regular.

The topology that we give to the space of economies is a very natural one: We give the space of demand functions ξ, the topology of uniform C^1 convergence on compact sets; two demand functions are close if their values and the values of their derivatives are close at all prices in some compact subset of $\mathbb{R}^n_{++}$. We give the space of activity analysis matrices the standard euclidean topology; two activity analysis matrices are close if their columns are close as vectors in $\mathbb{R}^n$. We find that, when we apply our regularity conditions to appropriately chosen finite dimensional subspaces of the space of all economies, regular economies form an open dense subset of full Lebesgue measure. There is some need to stress this point since Grandmont, Kirman, and Neuefeind (1974) have demonstrated that very restrictive properties can be shown to be generic if the topology given the space of economies is strange enough.

The preceding discussion can be summarized in the following theorem.

Theorem 3. *Suppose that* (ξ, A) *is a regular economy. Let* $\text{index}(\hat{\pi}) = \text{sgn}(\det[C(C'C)^{-1}C'(I + D\xi(\hat{\pi})) - D\xi(\hat{\pi})])$. *Then* Σ $\text{index}(\hat{\pi}) = +1$ *where the sum is over all equilibria.*

To use the index theorem to discuss economic conditions sufficient for uniqueness of equilibrium we need to obtain alternative expressions for $\text{index}(\hat{\pi})$. We can do this by performing elementary row and column operations on the matrix $(I - Dg(\hat{\pi}))$ that do not change its determinant. We present several alternative expressions as corollaries to Theorem 3.

Corollary 1. $\text{index}(\hat{\pi}) = (-1)^n \,\text{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat{\pi}) & B \\ 0 & B' & 0 \end{bmatrix}\right)$.

Proof. Since $C'C$ is positive definite, $\text{index}(\hat{\pi})$ has the same sign as

$$(-1)^n \det\begin{bmatrix} D\xi(\hat{\pi}) - C(C'C)^{-1}C'(I + D\xi(\hat{\pi})) & C \\ 0 & C'C \end{bmatrix}.$$

Postmultiplying the second column of this matrix by $(C'C)^{-1}C'(I + D\xi(\hat{\pi}))$ and adding it to the first column, we do not change its determinant. Premultiplying the first row of the resulting matrix by C' and subtracting it from the second row, we obtain

$$(-1)^n \det\begin{bmatrix} D\xi(\hat\pi) & C \\ C' & 0 \end{bmatrix} = (-1)^n \det\begin{bmatrix} D\xi(\hat\pi) & B & e \\ B' & 0 & 0 \\ e' & 0 & 0 \end{bmatrix}$$

$$= (-1)^n \det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat\pi) & B \\ 0 & B' & 0 \end{bmatrix},$$

which is the desired expression. □

Corollary 2. $\operatorname{index}(\hat\pi) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} D\xi(\hat\pi) - ee' & B \\ B' & 0 \end{bmatrix}\right)$.

Proof. Postmultiplying the first column of the matrix obtained in Corollary 1 by e' and subtracting it from the second column, we can establish that $\operatorname{index}(\hat\pi)$ has the same sign as

$$(-1)^n \det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat\pi) - ee' & B \\ 0 & B' & 0 \end{bmatrix}.$$

We now post-multiply the second column of this matrix by $\hat\pi$ and add it to the first. Using the homogeneity of ξ, which implies that $D\xi(\hat\pi)\hat\pi = 0$, the zero profit condition $\hat\pi' B = 0$, and the price normalization $\hat\pi' e = 1$, we obtain

$$(-1)^n \det\begin{bmatrix} 1 & e' & 0 \\ 0 & D\xi(\hat\pi) - ee' & B \\ 0 & B' & 0 \end{bmatrix} = (-1)^n \det\begin{bmatrix} D\xi(\hat\pi) - ee' & B \\ B' & 0 \end{bmatrix}. \quad \square$$

Let $\bar J$ denote the $(n-1)\times(n-1)$ matrix formed by deleting some row and column i from $D\xi(\hat\pi)$. Let $\bar B$ be the $(n-1)\times k$ matrix formed by deleting the same row from B.

Corollary 3. $\operatorname{index}(\hat\pi) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar J & \bar B \\ -\bar B' & 0 \end{bmatrix}\right)$.

Proof. Assume that $i = 1$. Corollary 2 implies that $\operatorname{index}(\hat\pi)$ has the same sign as

$$\det\begin{bmatrix} -D\xi(\hat\pi) + ee' & B \\ -B' & 0 \end{bmatrix}.$$

We add each column $j = 2, \ldots, n$ multiplied by the corresponding $\hat\pi_j$ to the first column multiplied by $\hat\pi_1$. Again using the homogeneity of ξ, the zero profit condition, and the price normalization, we can write out the expression that we are left with as

$$\hat{\pi}_1^{-1}\det\begin{bmatrix} 1 & -\frac{\partial\xi_1}{\partial\pi_2}+1 & \cdots & -\frac{\partial\xi_1}{\partial\pi_n}+1 & b_{11} & \cdots & b_{1k} \\ 1 & -\frac{\partial\xi_2}{\partial\pi_2}+1 & \cdots & -\frac{\partial\xi_2}{\partial\pi_n}+1 & b_{21} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & -\frac{\partial\xi_n}{\partial\pi_2}+1 & \cdots & -\frac{\partial\xi_n}{\partial\pi_n}+1 & b_{n1} & \cdots & b_{nk} \\ 0 & -b_{21} & \cdots & -b_{n1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & -b_{2k} & \cdots & -b_{nk} & 0 & \cdots & 0 \end{bmatrix}.$$

We can reduce this expression to the desired one in easy steps: first, we subtract the first column of the above matrix from each column 2 through n. Second we add each row $i=2,\dots,n$ multiplied by the corresponding $\hat{\pi}_i$ and each row $i=n+1,\dots,n+k$ multiplied by the corresponding $\hat{y}_{i-n}$ to the first row multiplied by $\hat{\pi}_1$. Finally, we use Walras's law, which implies that $\hat{\pi}'D\xi(\hat{\pi})=-\xi(\hat{\pi})'=-\hat{y}'B'$, to obtain

$$\hat{\pi}_1^{-2}\det\begin{bmatrix} 1 & 0 & 0 \\ e & -\bar{J} & \bar{B} \\ 0 & -\bar{B}' & 0 \end{bmatrix},$$

which obviously has the same sign as

$$\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}. \quad \square$$

3. Uniqueness of equilibrium

The condition that index$(\hat{\pi})=+1$ at every equilibrium is necessary as well as sufficient for uniqueness of equilibrium in almost all cases. Consequently, it is not surprising to find that most previous theorems dealing with uniqueness of equilibrium are special cases of our index theorem. In this section we discuss two conditions traditionally associated with uniqueness of equilibrium, gross substitutability and the weak axiom of revealed preference.

An excess demand function ξ that satisfies A.1–A.5 is said to satisfy gross substitutability if $(\partial\xi_i/\partial\pi_j)(\pi)>0$ for all $i\neq j$ and all $\pi\in\mathbb{R}^n_{++}$. It is well known that gross substitutability in ξ implies uniqueness of equilibrium in a pure exchange economy where $A=-I$ (see Arrow, Block and Hurwicz (1959)). This result is trivial to demonstrate using our index theorem: The homogeneity assumption, when differentiated, implies that $D\xi(\pi)\pi\equiv 0$. Therefore the matrix $-\bar{J}$ has positive diagonal elements and negative off-diagonal elements. Furthermore multiplying each column $j=1,\ldots,n-1$ of $-\bar{J}$ by the positive scalar $\hat{\pi}_{j+1}$ and adding up produces a strictly positive vector since

$$-\sum_{j=2}^{n} \hat{\pi}_j(\partial \xi_i / / \partial \pi_j)(\hat{\pi}) = \hat{\pi}_1(\partial \xi_i / \partial \pi_1)(\hat{\pi}) > 0, \quad i=2,\ldots,n.$$

Consequently, $-\bar{J}$ has the same form as a productive Leontief matrix; that is, $-\bar{J}$ is a P matrix, a matrix with all its principal minors positive. This implies that index$(\hat{\pi}) = \operatorname{sgn}(\det[-\bar{J}]) = +1$ at every equilibrium and, therefore, that there is a unique equilibrium.

In economies with production gross substitutability in demand does not seem to play a major role in uniqueness theorems. Indeed, in the next section we present an example with four commodities in which the excess demand function satisfies gross substitutability but in which there are multiple equilibria. Such an example cannot be constructed with fewer than four commodities, however: If $n \leq 3$, gross substitutability in demand implies uniqueness. To demonstrate this, we employ the following lemma.

Lemma 1. *Let $\bar{J}$, $\bar{B}$ be defined as previously. If $\bar{B}$ has $n-1$ columns, then* index$(\hat{\pi}) = +1$.

Proof. The $(n-1)\times(n-1)$ matrix $\bar{B}$ is square and, by A.8, nonsingular. Therefore

$$\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix} = \det[\bar{B}'\bar{B}].$$

Since $\bar{B}'\bar{B}$ is positive definite, index$(\hat{\pi}) = +1$. □

First consider the case where $n=2$. Either production takes place at equilibrium or it does not. If it does not, we know index$(\hat{\pi}) = +1$ because $-\bar{J}$ is a P matrix, in this case a positive scalar. If production does take place there must be exactly one activity in use since $\hat{\pi}'B=0$ but A.8 implies that B has full column rank. Therefore index$(\hat{\pi}) = +1$ by Lemma 1. The reasoning for the case where $n=3$ is similar: There are either zero, one, or two activities in use in equilibrium. If there are zero or two, then index$(\hat{\pi}) = +1$ by the above reasoning. Suppose that there is one. Then choose the element to be deleted from the 3×1 vector B to form the 2×1 vector $\bar{B}$ so that the two elements that remain have the same sign. Then the 3×3 matrix

$$\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}$$

has one of the following sign patterns:

$$\begin{bmatrix} + & - & + \\ - & + & + \\ - & - & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} + & - & - \\ - & + & - \\ + & + & 0 \end{bmatrix}.$$

In either case the determinant is unambiguously positive and index$(\hat{\pi}) = +1$.

Our arguments have yielded the following theorem.

Theorem 4. *If the economy* (ξ, A) *is such that* ξ *exhibits gross substitutability, then it has a unique equilibrium if either* $A = -I$, *that is,* (ξ, A) *is a pure exchange economy, or* $n \leqslant 3$.

That ξ exhibits gross substitutability does not, in general, imply that (ξ, A) has a unique equilibrium. If, however, ξ satisfies the weak axiom of revealed preference, then (ξ, A) does have a unique equilibrium. In fact, this is the only condition that can be imposed on ξ independently of A that implies uniqueness: If ξ does not satisfy the weak axiom, then a matrix A can be constructed so that (ξ, A) has multiple equilibria.

ξ is said to satisfy the weak axiom of revealed preference if, for every $\pi^1, \pi^2 \in \mathbb{R}^n_{++}$, $\pi^1/\|\pi^1\| \neq \pi^2/\|\pi^2\|$ and $\pi^{1\prime}\xi(\pi^2) \leqslant 0$ imply $\pi^{2\prime}\xi(\pi^1) > 0$. That this condition is sufficient for uniqueness of equilibrium in production economies was first demonstrated by Wald (1951).

Theorem 5. *If an economy* (ξ, A) *is such that* ξ *satisfies the weak axiom of revealed preference, then it has a unique equilibrium.*

Proof. Suppose instead that (ξ, A) has multiple equilibria and let π^1 and π^2, $\pi^2 \neq \pi^2$, be two of them. Then $\xi(\pi^1) = Ay^1$ and $\pi^{2\prime}A \leqslant 0$ imply that $\pi^{2\prime}\xi(\pi^1) \leqslant 0$. Similarly, $\xi(\pi^2) = Ay^2$ and $\pi^{1\prime}A \leqslant 0$ imply that $\pi^{1\prime}\xi(\pi^2) \leqslant 0$. This, however, contradicts the assumption that ξ satisfies the weak axiom. □

That the weak axiom is actually necessary for uniqueness of equilibrium if the production technology is arbitrary was shown to the writer by Herbert Scarf.

Theorem 6. *Suppose that the excess demand function* ξ *violates the weak axiom of revealed preference. Then there exists an activity analysis matrix* A *that satisfies* A.6–A.7 *and is such that* (ξ, A) *has multiple equilibria.*

Proof. If ξ violates the weak axiom, then there exist π^1, $\pi^2 \in \mathbb{R}^n_{++}$, $\pi^1/\|\pi^1\| = \pi^2/\|\pi^2\|$, such that $\pi^{1\prime}\xi(\pi^2) \leqslant 0$ and $\pi^{2\prime}\xi(\pi^1) \leqslant 0$. Let $a^1 = \xi(\pi^1)$, $a^2 = \xi(\pi^2)$, and $A = [-I \;\; a^1 \;\; a^2]$. A obviously satisfies A.6 and A.7. Since both $\pi^1/(e'\pi^1)$ and $\pi^2/(e'\pi^2)$ satisfy the equilibrium conditions, (ξ, A) has multiple equilibria. □

To understand the relationship between these two theorems concerning the weak axiom and uniqueness with the index theorem, we need to characterize the weak axiom in terms of the derivatives of ξ.

Theorem 7. *Suppose that* ξ *satisfies the weak axiom of revealed preference on some open set* $U \subset \mathbb{R}^n_{++}$ *that contains* π. *Then* $D\xi(\pi)$ *is negative semi-definite on the null space of* $\xi(\pi)$; *that is,* $v'D\xi(\pi)v \leqslant 0$ *for all* $v \in \mathbb{R}^n$ *such that* $v'\xi(\pi) = 0$.

Proof. Much of our argument follows one given by Kihlstrom, Mas-Colell, and Sonnenschein (1976). See also Freixas and Mas-Colell (1983) and Kehoe (1980b). Letting π and $\pi+v$ be points in U such that $\pi/\|\pi\| \neq (\pi+v)/\|\pi+v\|$, we define $\pi(t)=\pi+tv$. Since each ξ_i is continuously differentiable, we can use the definition of partial derivative to establish that

$$0=\lim_{t\to 0}\left|\xi_i(\pi(t))-\xi_i(\pi)-\sum_{j=1}^{n}(\pi_j(t)-\pi_j)\frac{\partial\xi_i}{\partial\pi_j}(\pi)\right|/t\|v\|.$$

Since $tv_i=\pi_i(t)-\pi_i$, we can multiply this expression by $(\|v\|/t)(\pi_i(t)-\pi_i)$ to obtain

$$0=\lim_{t\to 0}\left|(1/t^2)(\pi_i(t)-\pi_i)(\xi_i(\pi(t))-\xi_i(\pi))-\sum_{j=1}^{n}v_iv_j\frac{\partial\xi_i}{\partial\pi_j}(\pi)\right|.$$

Summing over $i=1,\dots,n$ yields

$$0=\lim_{t\to 0}\left|(1/t^2)(\pi(t)-\pi)'(\xi(\pi(t))-\xi(\pi))-\sum_{i=1}^{n}\sum_{j=1}^{n}v_iv_j\frac{\partial\xi_i}{\partial\pi_j}(\pi)\right|.$$

Walras's law implies that

$$(\pi(t)-\pi))'(\xi(\pi(t))-\xi(\pi))=-\pi(t)'\xi(\pi)-\pi'\xi(\pi(t)).$$

Now suppose that $v'\xi(\pi)=0$. Then $\pi(t)'\xi(\pi)=\pi'\xi(\pi)+tv'\xi(\pi)=0$. The weak axiom therefore implies that $\pi'\xi(\pi(t))>0$. Consequently,

$$v'D\xi(\pi)v=\sum_{i=1}^{n}\sum_{j=1}^{n}v_iv_j\frac{\partial\xi_i}{\partial\pi_j}(\pi)\leq 0.\quad\Box$$

Let A be an $n\times n$ matrix, not necessarily symmetric, and let B be an $n\times k$ matrix, $k\leq n$. A_{ij} denotes the $i\times j$ matrix, $1\leq i\leq n$, $1\leq j\leq n$, formed by keeping only the elements in the first i rows and j columns of A. B_{ij} denotes the matrix formed similarly. Let p be a permutation of the first n integers. A^p denotes the matrix obtained from A by performing the permutation p on its rows and columns. B^p denotes the matrix obtained from B by performing the permutation p on its rows. The following three lemmas are classical results from the theory of constrained optimization (see, for example, Debreu (1952)).

Lemma 2. *If* $v'Av>0$ *for all* $v\in\mathbb{R}^n$ *such that* $v\neq 0$ *and* $v'B=0$ *and if* B *has full column rank, then*

$$\det\begin{bmatrix} A^p_{ii} & B^p_{ik} \\ -B^{p'}_{ik} & 0 \end{bmatrix}>0$$

for all $i=k+1,\dots,n$ *and all* p.

Lemma 3. *If* $v'Av\geq 0$ *for all* $v\in\mathbb{R}^n$ *such that* $v'B=0$, *then*

$$\det\begin{bmatrix} A^p_{ii} & B^p_{ik} \\ -B^{p\prime}_{ik} & 0 \end{bmatrix} \geq 0$$

for all $i = k+1, \ldots, n$ *and all* p.

Lemma 4. *If*

$$\det\begin{bmatrix} \frac{1}{2}A_{ii} + \frac{1}{2}A'_{ii} & B_{ik} \\ -B'_{ik} & 0 \end{bmatrix} > 0$$

for all $i = k+1, \ldots, n$, *then* $v'Av > 0$ *for all* $v \in \mathbb{R}^n$ *such that* $v \neq 0$ *and* $v'B = 0$.

Suppose that ξ satisfies the weak axiom. Then $-v'D\xi(\hat{\pi})v \geq 0$ for all v such that $v'[B\ e] = 0$ since $v'B\hat{y} = v'\xi(\hat{\pi})$ implies that $v'\xi(\hat{\pi}) = 0$. Consequently, Lemma 4 implies that

$$(-1)^n \det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat{\pi}) & B \\ 0 & B' & 0 \end{bmatrix} = \det\begin{bmatrix} -D\xi(\hat{\pi}) & B & e \\ -B' & 0 & 0 \\ -e' & 0 & 0 \end{bmatrix} \geq 0.$$

This can be viewed as an alternative proof of Theorem 5, that the weak axiom implies uniqueness of equilibrium. It actually proves more, that a local version of the weak axiom implies uniqueness of equilibrium. In fact, since the weak axiom is necessary for uniqueness of equilibrium if the production technology is arbitrary, but the local version of the weak axiom in Theorem 7 is sufficient for uniqueness, at least in the case where the above determinantal expression is strictly positive, we have demonstrated the following result.

Theorem 8. *Suppose that, for every* $\pi \in \mathbb{R}^n_{++}$, $v'D\xi(\pi)v < 0$ *for all* $v \in \mathbb{R}^n$ *such that* $v \neq 0$, $v/\|v\| \neq \pi/\|\pi\|$, *and* $v'\xi(\pi) = 0$. *Then* ξ *satisfies the weak axiom of revealed preference.*

Freixas and Mas-Colell (1983) provide a direct proof of this theorem.

The next result provides us with a simple test for checking whether the weak axiom is satisfied.

Theorem 9. *Suppose that*

$$\det\begin{bmatrix} -\frac{1}{2}D\xi(\pi)_{ii} - \frac{1}{2}D\xi(\pi)'_{ii} & \xi(\pi)_{i1} \\ -\xi(\pi)'_{i1} & 0 \end{bmatrix} > 0$$

for all $i = 2, \ldots, n-1$ *and all* $\pi \in \mathbb{R}^n_{++}$. *Then* ξ *satisfies the weak axiom of revealed preference.*

Proof. Let $\bar{J}$ denote $D\xi(\pi)$ with the last row and column deleted and let $\bar{x}$ denote $\xi(\pi)$ with the last element deleted. If the above inequalities are satisfied, then

$-v'\bar{J}v>0$ for every $v \in \mathbb{R}^{n-1}$ such that $v \neq 0$ and $v'\bar{x}=0$ by Lemma 4. Suppose that $\bar{B}$ is an $(n-1)\times k$ matrix such that $\bar{B}\hat{y}=\bar{x}$ for some $\hat{y}\in\mathbb{R}^k_+$. Then $-v'\bar{J}v>0$ for every $v\in\mathbb{R}^{n-1}$ such that $v\neq 0$ and $v'\bar{B}=0$. Lemma 2 therefore implies that

$$\det\begin{bmatrix}-\bar{J} & \bar{B}\\ -\bar{B}' & 0\end{bmatrix}>0.$$

The desired result now follows directly from Theorem 6. □

Our investigations have produced another result, which is an immediate consequence of Theorem 4 and Theorem 6.

Theorem 10. *If ξ exhibits gross substitutability and $n\leqslant 3$, then ξ satisfies the weak axiom of revealed preference.*

Kehoe and Mas-Colell (1984) provide a direct proof of this theorem, one that dispenses with the assumption of differentiability.

4. A numerical example

In this section we study a numerical example of an economy with four goods that has multiple equilibria. An interesting feature of this example is that the excess demand function ξ exhibits gross substitutability.

To construct this example, we begin by searching for matrices $\bar{J}$ and $\bar{B}$ that satisfy the relevant properties and are such that

$$\det\begin{bmatrix}-\bar{J} & \bar{B}\\ -\bar{B}' & 0\end{bmatrix}<0.$$

Let us normalize quantities of the good so that $\hat{\pi}=e=(1,1,1,1)$. It can easily be verified that this has no effect on the sign of the above determinant because it merely rescales the first $n-1$ rows and $n-1$ columns. The homogeneity assumption implies that $D\xi(\hat{\pi})\hat{\pi}=0$; that is, that the row sums of $D\xi(\hat{\pi})$ must equal zero. Walras's law implies that $\hat{\pi}'D\xi(\hat{\pi})=-\xi(\hat{\pi})'$. The condition that demand equals supply in equilibrium requires that $\hat{\pi}'D\xi(\hat{\pi})=-\hat{y}'B'$ for some $\hat{y}\geqslant 0$. The condition that the activities in B earn zero profit requires that $\hat{\pi}'B=0$; that is, that the column sums of B equal zero.

A pair of matrices that satisfy these conditions are

$$D\xi(\hat{\pi})=\begin{bmatrix}-267 & 43 & 200 & 24\\ 20 & -200 & 80 & 100\\ 2 & 1 & -4 & 1\\ 2 & 1 & 1 & -4\end{bmatrix},\qquad B=\begin{bmatrix}6 & -1\\ -1 & 3\\ -4 & -1\\ -1 & -1\end{bmatrix}.$$

The row sums of $D\xi(\hat{\pi})$ are zero, as are the column sums of B. Furthermore

$$\begin{bmatrix} -267 & 20 & 2 & 2 \\ 43 & -200 & 1 & 1 \\ 200 & 80 & -4 & 1 \\ 24 & 100 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 243 \\ 155 \\ -277 \\ -121 \end{bmatrix} = -\begin{bmatrix} 6 & -1 \\ -1 & 3 \\ -4 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 52 \\ 69 \end{bmatrix}.$$

Although $D\xi(\hat{\pi})$ satisfies the conditions required for gross substitutability, it violates those required for the weak axiom: We can easily see this by deleting the first row and column from $D\xi(\hat{\pi})$ to form $\bar{J}$, deleting the first element from $-D\xi(\hat{\pi})'\hat{\pi}$ to form $\bar{x}$, and computing

$$\det \begin{vmatrix} -\tfrac{1}{2}\bar{J} - \tfrac{1}{2}\bar{J}' & \bar{x} \\ -\bar{x}' & 0 \end{vmatrix}.$$

The answer is $-23\,072\,119$. Lemma 3 and Theorem 7 imply that any demand function ξ consistent with $D\xi(\hat{\pi})$ must violate the weak axiom.

Let us compute index($\hat{\pi}$):

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det \begin{vmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{vmatrix}\right)$$

$$= \operatorname{sgn}\left(\det \begin{vmatrix} 200 & -80 & -100 & -1 & 3 \\ -1 & 4 & -1 & -4 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ 1 & 4 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 & 0 \end{vmatrix}\right) = \operatorname{sgn}(-323) = -1.$$

Theorem 3 implies that any economy (ξ, A) consistent with $D\xi(\hat{\pi})$ and B has multiple equilibria. To construct such an economy is relatively easy: To construct A we merely augment B with disposal activities. To construct ξ we find four Cobb–Douglas consumers whose aggregate excess demand function ξ generates $D\xi(\hat{\pi})$. Suppose that consumer h maximizes $\sum_{i=1}^{4} \alpha_i^h \log x_i$ subject to the budget constraint $\sum_{i=1}^{4} \pi_i x_i = \sum_{i=1}^{4} \pi_i w_i^h$. Then his excess demand for good j is

$$\xi_j^h(\pi) = \alpha_j^h \left(\sum_{i=1}^{4} \pi_i w_i^h \right) / \pi_j - w_j^h.$$

The partial derivatives of the aggregate excess demand function ξ can easily be computed as

$$\frac{\partial \xi_i}{\partial \pi_i}(\pi) = \left(\sum_{h=1}^{4} \alpha_i^h w_i^h \right) / \pi_i - \sum_{h=1}^{4} \alpha_i^h \left(\sum_{j=1}^{4} \pi_j w_j^h \right) / \pi_i^2,$$

$$\frac{\partial \xi_i}{\partial \pi_j}(\pi) = \left(\sum_{h=1}^{4} \alpha_i^h w_j^h \right) / \pi_i, \quad i \neq j.$$

Since we have normalized all prices equal to unity, the problem of parameterizing this economy is reduced to finding sixteen numbers α_i^h and sixteen numbers w_i^h that satisfy $w_i^h \geqslant 0$, $\alpha_i^h \geqslant 0$, $\sum_{i=1}^4 \alpha_i^h = 1$, and

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4 \\ \alpha_3^1 & \alpha_3^2 & \alpha_3^3 & \alpha_3^4 \\ \alpha_4^1 & \alpha_4^2 & \alpha_4^3 & \alpha_4^4 \end{bmatrix} \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ w_1^4 & w_2^4 & w_3^4 & w_4^4 \end{bmatrix} = \begin{bmatrix} * & 43 & 200 & 24 \\ 20 & * & 80 & 100 \\ 2 & 1 & * & 1 \\ 2 & 1 & 1 & * \end{bmatrix},$$

where the elements denoted $*$, $\sum_{h=1}^4 \alpha_i^h w_i^h$, are of no consequence. It greatly facilitates the calculation of these parameters, but is by no means necessary, to choose one of the matrices on lefthand side of the above equation to be diagonal. One pair of matrices, out of an infinite number of possibilities, that satisfy these restrictions is

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4 \\ \alpha_3^1 & \alpha_3^2 & \alpha_3^3 & \alpha_3^4 \\ \alpha_4^1 & \alpha_4^2 & \alpha_4^3 & \alpha_4^4 \end{bmatrix} = \begin{bmatrix} 0.5200 & 0.8600 & 0.5000 & 0.0600 \\ 0.4000 & 0.1000 & 0.2000 & 0.2500 \\ 0.0400 & 0.0200 & 0.2975 & 0.0025 \\ 0.0400 & 0.0200 & 0.0025 & 0.6875 \end{bmatrix},$$

$$\begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ w_1^4 & w_2^4 & w_3^4 & w_4^4 \end{bmatrix} = \begin{bmatrix} 50 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 400 \end{bmatrix}.$$

This economy, and indeed any other whose parameters satisfy the above restrictions, has an equilibrium where $\hat{\pi} = (1, 1, 1, 1)$, $\hat{y} = (0, 0, 0, 0, 52, 69)$, and index$(\hat{\pi}) = -1$. Therefore it has multiple equilibria. Usually this is all we can say if we cannot guarantee that an economy has a unique equilibrium: In general it is an impossible task to compute all of the equilibria of an economy. Kehoe (1984), however, exploits some special features of this example to show that it has exactly two more equilibria, one where $\hat{\pi} = (0.637, 1.000, 0.155, 2.208)$ and $\hat{y} = (0, 0, 0, 0, 42.701, 81.198)$ and another where $\hat{\pi} = (1.100, 1.000, 1.235, 0.665)$ and $\hat{y} = (0, 0, 0, 0, 53.180, 65.148)$. Another choice of parameters that satisfy the above restrictions would, of course result in different equilibria. The essential feature of our example is that any choice of parameters, and indeed of functional forms, of the consumer's demand functions that generates the Jacobian matrix $D\xi(\hat{\pi})$ at $\hat{\pi} = (1, 1, 1, 1)$ results in an economy with multiple equilibria.

5. A numerical investigation

In this section we attempt to answer the question of how perverse is the example presented in the previous section. In particular we ask how common it is for a 4×4

matrix $D\xi(\hat{\pi})$ that satisfies the gross substitutability conditions to violate the conditions required for the weak axiom. In one obvious sense there is nothing at all perverse about the example: The example is a regular economy, and any small perturbation in ξ or A results in another economy with multiple equilibria. Furthermore any small perturbation in ξ results in an excess demand function that satisfies gross substitutability but violates the weak axiom. Even so, numerical investigation seems to indicate that, at least in the case where $n=4$, such excess demand functions are rare.

The focus of our investigation is on 4×4 matrices D whose elements satisfy the properties $d_{ii}<0$, $d_{ij}>0$ for $i\neq j$, and $\sum_{j=1}^{4} d_{ij}=0$. As we have seen in the previous section such a matrix is a Jacobian matrix $D\xi(\hat{\pi})$ evaluated at $\hat{\pi}=(1,1,1,1)$ of an excess demand function that satisfies gross substitutability. The condition that $\sum_{j=1}^{4} d_{ij}=0$ says that the matrix D is completely determined by the twelve off-diagonal elements d_{ij}, $i\neq j$. Furthermore, we can scale these elements so that $0<d_{ij}<1$; The Jacobian matrix in the previous example could have been scaled as

$$\begin{bmatrix} -1.068 & 0.172 & 0.800 & 0.096 \\ 0.080 & -0.800 & 0.320 & 0.400 \\ 0.008 & 0.004 & -0.016 & 0.004 \\ 0.008 & 0.004 & 0.004 & -0.016 \end{bmatrix};$$

it would have given rise to an economy with the same equilibria, except for a rescaling of activity levels, as the previous one.

Let us assume that twelve numbers d_{ij}, $i\neq j$, are independently and identically distributed uniformly on the interval [0, 1]. The question that we ask is what is the probability of finding a matrix D that corresponds to an excess demand function that satisfies gross substitutability but violates the weak axiom. The procedure that we follow can be best understood by considering an example: Suppose we choose twelve numbers at random from the unit interval and get (0.5, 0.4, 0.7, 0.1, 0.5, 0.3, 0.2, 0.8, 0.9, 0.6, 0.0, 0.3). We arrange them in order in the matrix D, filling in the diagonal elements so that row sums are zero:

$$D=\begin{bmatrix} -1.6 & 0.5 & 0.4 & 0.7 \\ 0.1 & -0.9 & 0.5 & 0.3 \\ 0.2 & 0.8 & -1.9 & 0.9 \\ 0.6 & 0.0 & 0.3 & -0.9 \end{bmatrix}.$$

Walras's law says that the value of $\xi(\hat{\pi})$ can be computed as the negative of the column sums:

$$\xi(\hat{\pi})=\begin{bmatrix} 0.7 \\ -0.4 \\ 0.7 \\ -1.0 \end{bmatrix}.$$

To test whether D corresponds to an excess demand function that satisfies the weak axiom we need only check that

$$\det\begin{bmatrix} 1.6 & -0.5 & -0.4 & 0.7 \\ -0.1 & 0.9 & -0.5 & -0.4 \\ -0.2 & -0.8 & 1.9 & 0.7 \\ -0.7 & 0.4 & -0.7 & 0 \end{bmatrix}$$

is positive because we already know that the determinant of this matrix with one row and column deleted, in this case

$$\det\begin{bmatrix} 1.6 & -0.4 & 0.7 \\ -0.2 & 1.9 & 0.7 \\ -0.7 & -0.7 & 0 \end{bmatrix},$$

is positive because it has the right sign pattern. That these conditions are sufficient for the weak axiom to be satisfied locally by any demand function ξ that generates $D\xi(\hat{\pi}) = D$ follows immediately from Theorem 9 and a reordering of rows and columns of D. In this example the weak axiom is satisfied because the crucial determinant is equal to 1.1645.

To estimate the percentage of matrices of the form D, distributed as described above, that correspond to an excess demand function that violates the weak axiom, 250 000 twelve-tuples of random numbers were generated on a computer and the properties of the corresponding D matrices analyzed as above. Unlike the numbers in the above example, each of the elements d_{ij} was stored as a double precision real number, which has about fifteen significant figures. Of these 250 000 examples exactly 250 000 corresponded to demand functions that satisfy the weak axiom and 0 to ones that do not.

This result is startling: We have an example, the one in the previous section, that violates the weak axiom. Furthermore, we know that any small perturbation in this example still violates the weak axiom. In other words, we know that there is a nonempty open set of matrices D in the space of all 4×4 matrices parameterized by the twelve numbers d_{ij}, $i \neq j$, that correspond to violations of the weak axiom. Since an open set cannot have measure zero, it follows that the true answer to the question we are asking cannot be that 0% of the D matrices correspond to violations of the weak axiom. We can, however, put very stringent upper bounds on what this number must be: Let p be the proportion of matrices that correspond to violations of the weak axiom. Suppose that our prior distribution of p is that it is uniformly distributed between 0 and 1. We can calculate the Bayesian $1-\alpha$ probability interval $[0, x]$ by finding the value of x that solves

$$\alpha = \Pr(p \geqslant x \mid n = 0) = \Pr(n = 0 \mid p \geqslant x)\Pr(p \geqslant x)/\Pr(n = 0)$$

$$=\left(\int_x^1 (1-y)^{250000}\,\mathrm{d}y\right)\left(\int_x^1 \mathrm{d}y\right)\Big/\left(\int_0^1 (1-y)^{250000}\,\mathrm{d}y\right)$$

$$=\left(\frac{1}{250\,001}(1-x)^{250001}\right)(1-x)/\left(\frac{1}{250\,001}\right)$$

$$=(1-x)^{250002}.$$

A list of values for α and corresponding upper bounds for p are given below:

α	x
0.05	0.00001198
0.01	0.00001842
0.001	0.00002763
0.00001	0.00004605
1.0×10^{-100}	0.00092060
6.1649×10^{-1092}	0.01

Notice that we can be sure with very high probability that p is a miniscule number.

As another measure of how rare we would now estimate violations of the weak axiom to be we can calculate the mean of the posterior distribution of p. The posterior cumulative distribution is $F(y)=1-(1-y)^{250002}$, as we have calculated above. The posterior density function is its derivative, $250\,002(1-y)^{250001}$. The mean of the posterior distribution is, therefore,

$$E(p)=\int_0^1 250\,000y(1-y)^{250001}\,\mathrm{d}y=\frac{1}{250\,003}.$$

Notice how small this number is compared with the mean of the prior distribution, $\frac{1}{2}$. Notice too how little effect the prior has on any of our calculations because of the large number of observations.

To appreciate how startling this result is let us compare them with analogous results obtained for 250 000 random examples where the numbers d_{ij}, $i\neq j$, were independently and identically distributed uniformly on the interval $[-1,1]$. We are now trying to estimate the proportion of all Jacobian matrices $D\xi(\hat{\pi})$, not just those that satisfy the gross substitutability conditions, that violate the conditions required for the weak axiom. In this case of these 250 000 examples 49 308 corresponded to demand functions that satisfy the weak axiom and 200 692 to ones that do not. Our estimate of the proportion of general Jacobian matrices that violate the weak axiom is 0.802768. Incidentally, the proportion of general Jacobian matrices that satisfy the gross substitutability conditions is $2^{-12}=0.000244$. Of the 250 000 examples 66 actually satisfied the gross substitutability conditions. We would have estimated that $61=(250\,000)(2^{-12})$ would satisfy these conditions. All of these 66 examples satisfied the conditions required for the weak axiom.

6. Concluding remarks

Although the results we have obtained are indeed startling, caution should be used before drawing any conclusions from them. Perhaps they do no more than reminding us, in a very vivid way, that, although a non-empty open set cannot have measure zero, it can be very small. It may be the case that the major reason we get our results is simply that 4 is the next integer after 3: Gross substitutability coupled with a violation of the weak axiom is impossible when $n=3$, rare when $n=4$, but more and more common as n increases. It would be worth investigating this point. It is also worth reminding ourselves that gross substitutability itself is a very restrictive property of demand functions. Nevertheless, our results indicate that perhaps too much emphasis has been put on studying qualitative restrictions that guarantee uniqueness of equilibrium. Perhaps nonuniqueness is common in practice, but our results give some hope that it is not.

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